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MAXIMAL REGULARITY OF LINEAR VECTOR-VALUED PARABOLIC VOLTERRA EQUATIONS

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In Honor of Professor John A. Nohel
on the Occasion of his 65th Birthday

ABSTRACT. Maximal regularity in C^α-spaces of linear Volterra equations in a Banach space X of the form

$$(*)\qquad u(t) = f(t) + \int_0^t a(t-\tau)Au(\tau)\,d\tau, \quad t \geq 0,$$

is studied. The conditions which ensure maximal regularity involve a parabolicity condition for $(*)$, but also some regularity conditions on the kernel $a(t)$. As an illustration of the results, examples from the theory of viscoelasticity and heat conduction in materials with memory are discussed.

1. Introduction. Let X be a Banach space, A a closed linear operator in X with dense domain $D(A)$, $a \in L^1_{\text{loc}}(\mathbf{R}_+)$, and $f \in C(J;X)$, where $J = [0,T]$. We consider the following vector-valued Volterra equation of scalar type:

$$(1)\qquad u(t) = f(t) + \int_0^t a(t-\tau)Au(\tau)\,d\tau, \quad t \in J.$$

In the sequel $*$ will be used for the convolution of two functions on the halfline. Recall that $u \in C(J;X)$ is called a *mild solution* of (1) if $a * u \in C(J;X_A)$ and

$$(2)\qquad u(t) = f(t) + A(a*u)(t), \quad t \in J,$$

holds; here X_A denotes the Banach space $D(A)$ equipped with the graph norm of A. A *strong solution* of (1) is a function $u \in C(J;X_A)$ such that (1) holds on J.

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In this note we are concerned with *maximal regularity of type* C^α for (1); by this we mean the property that, for any $f \in C_0^\alpha(J;X)$, $\alpha \in (0,1)$, there is a unique mild solution $u \in C_0^\alpha(J;X)$ of (1). This property has turned out to be very useful in the study of linear and also nonlinear Cauchy problems of parabolic type. For Volterra equations of the form (1) it has been obtained by Da Prato and Iannelli [**5**], Da Prato, Iannelli and Sinestrari [**6**], and it has been applied successfully to nonlinear Volterra equations of parabolic type. We are going to describe their main result on maximal regularity of the type C^α for (1) briefly.

Suppose a is Laplace transformable (i.e, $\int_0^\infty |a(t)|e^{-\omega t}\,dt < \infty$ for some $\omega \geq 0$), and suppose the Laplace transform $\hat{a}(\lambda)$ of $a(t)$ admits holomorphic extension to some sector

$$\Sigma_\theta = \{\lambda \in \mathbf{C} : |\arg(\lambda - \omega)| < \theta\},$$

with $\theta > \pi/2$, and satisfies $|\lambda^\gamma \hat{a}(\lambda)| \leq C$ on Σ_θ for some $\gamma > 0$. Suppose, moreover, that $\hat{a}(\lambda) \in \Sigma_\phi \subset \rho(A)$ on Σ_θ, where $\rho(A)$ denotes the resolvent set of A, and that the estimate

$$|(I - \mu A)^{-1}| \leq M, \quad \mu \in \Sigma_\phi, \tag{3}$$

is satisfied for some constant $M \geq 1$. Then (1) possesses the maximal regularity property of type C^α for any $\alpha \in (0,1)$ and each interval $J = [0,T]$.

As an example where this result applies, we mention the case where $a(t)$ is completely monotonic on $(0,\infty)$ and A generates a bounded analytic semigroup in X.

Here we show by quite different methods that the assumptions of Da Prato and Iannelli mentioned above can be relaxed to some extent. We will only require the parabolicity condition (3) with $\theta = \pi/2$ and a certain regularity of the kernel $a(t)$, namely $a(t)$ is assumed to be 2-regular, see Section 4. For example, if $a(t)$ is nonnegative, nonincreasing and convex, then $a(t)$ is 1-regular, and if, in addition, $-\dot{a}(t)$ is convex, $a(t)$ is 2-regular. This will follow from the Shea-Wainger estimates for such kernels; cp. Shea and Wainger [**10**] and also Carr and Hannsgen [**2**].

It should also be mentioned that the results of this paper admit extensions to certain Volterra equations with operator-valued kernels by means of perturbation arguments. Assume (1) satisfies the assumptions of Theorem 3 in Section 4; in particular, (1) admits a resolvent and has the maximal regularity property of type C^α. Then

$$u(t) = f(t) + \int_0^t a(t-s)\left[Au(s) + \int_0^s dB(\tau)u(s-\tau)\right] ds, \quad t \in J,$$

enjoys the same properties, provided $B : J \to \mathcal{B}(X_A, X)$ is of bounded variation and $B(0) = B(0+)$. The proof of this has been given in Prüss [**7**] for the case $a(t) = 1$; it carries over quite directly to the general case; however, we shall not go into this here.

Our plan for this note is as follows. In Section 2 we prepare some Laplace transform inversion results which are the basis of our study of maximal regularity. Section 3 contains the definition of k-regular and k-monotone kernels, and it will be shown that k-monotonicity of a kernel $a(t)$ implies its $(k-1)$-regularity; also kernels admitting holomorphic extensions to a sector Σ_θ, $\theta > \pi/2$, as described above are k-regular for any $k \geq 0$. Section 4 contains the main results. Somewhat simplified, we show that boundedness of $S(t)$, $tS'(t)$, $t^2S''(t)$ on finite intervals implies maximal regularity of type C^α; these conditions, in turn, follow from 2-regularity of $a(t)$ and from the parabolic assumption, i.e., (3). Some examples and applications to viscoelasticity are discussed in Section 5.

2. Laplace transforms of vector Lipschitz functions. Let us recall the following extension of a theorem of Widder [**11**] to the vector-valued case which is due to Arendt [**1**].

PROPOSITION 1. *A function $g \in C^\infty((0,\infty);X)$ has the representation*

$$g(\lambda) = \lambda \int_0^\infty e^{-\lambda t} f(t)\, dt, \quad \lambda > 0, \tag{4}$$

for some function $f \in C(\mathbf{R}_+;X)$ with $f(0) = 0$ and such that

$$|f(t) - f(s)| \leq M|t-s|, \quad \textit{for all} \quad t, s \geq 0,$$

if and only if

$$|\lambda^{n+1} g^{(n)}(\lambda)| \leq Mn!, \quad \textit{for all} \quad \lambda > 0,\ n \in \mathbf{N}_0. \tag{5}$$

Note that (5) involves only real values of λ, but all derivatives of g. However, expanding $\hat{g}(\lambda)$ into Taylor series it is not difficult to show that (5) implies already that $g(\lambda)$ is holomorphic in the open right half plane $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ (complexify X in the usual way if X is originally a real Banach space) and that

$$(5') \qquad |(\operatorname{Re}\lambda)^{n+1} g^{(n)}(\lambda)| \le Mn!, \quad \text{for all} \quad \operatorname{Re}\lambda > 0,\ n \in \mathbf{N}_0$$

is satisfied. Verification of (5) for applications, e.g., to Volterra equations, is, in general, difficult; however, it is possible in some cases, cf. Prüss [**8**].

For the case of parabolic Volterra equations we typically encounter a decay of $g(\lambda)$ of the type

$$|g(\lambda)| \le M/|\lambda|, \quad \operatorname{Re}\lambda > 0. \tag{6}$$

We are not able to prove that (6) implies (5); however, we have

PROPOSITION 2. *Suppose $g : \mathbf{C}_+ \to X$ is holomorphic and satisfies* (6) *as well as*

$$|g'(\lambda)| \le M/|\lambda|^2, \quad \operatorname{Re}\lambda > 0. \tag{7}$$

Then (5) *holds and there is a Lipschitz function $f \in C(\mathbf{R}_+; X)$, with $f(0) = 0$ representing $g(\lambda)$ by* (4).

PROOF. Obviously, (5) holds for $n = 0, 1$ by (6) and (7). For $n > 1$ we use the Cauchy integral representation

$$g'(\lambda) = 1/(2\pi i) \int_{\epsilon - i\infty}^{\epsilon + i\infty} g'(z)(\lambda - z)^{-1}\, dz, \quad \lambda > \epsilon.$$

Differentiating $(n-1)$-times this formula yields

$$g^{(n)}(\lambda) = (-1)^{(n-1)}(n-1)!(2\pi i)^{-1} \int_{\epsilon - i\infty}^{\epsilon + i\infty} g'(z)(\lambda - z)^{-n}\, dz, \quad \lambda > \epsilon.$$

By means of estimate (7) we obtain

$$\begin{aligned}|g^{(n)}(\lambda)| &\le (n-1)!(M/2\pi)\int_{-\infty}^{\infty}|\epsilon+i\rho|^{-2}|\lambda-\epsilon-i\rho|^{-n}\,d\rho\\ &\le (n-1)!(M/2\pi)(\lambda-\epsilon)^{-n}\int_{-\infty}^{\infty}(\epsilon^2+\rho^2)^{-1}\,d\rho\\ &\le (n-1)!(M/2\pi)(\lambda-\epsilon)^{-n}(\pi/\epsilon).\end{aligned}$$

Choosing $\epsilon=\lambda/n$ the latter gives

$$|g^{(n)}(\lambda)| \le n!(M/2)\lambda^{-(n+1)}(1-1/n)^{-n} \le n!(2M)\lambda^{-(n+1)},$$

thereby proving (5) for all $n\in\mathbf{N}_0$. □

If only (6) is satisfied we can prove the following weaker result which shall, nevertheless, be useful as well.

PROPOSITION 3. *Suppose $g:\mathbf{C}_+\to X$ is holomorphic and satisfies*

$$|g(\lambda)|\le M/|\lambda|,\quad \operatorname{Re}\lambda>0. \tag{6}$$

Then there is an $f\in C(\mathbf{R}_+;X)$ with $f(0)=0$ such that

$$g(\lambda)=\lambda\hat{f}(\lambda),\quad \operatorname{Re}\lambda>0; \tag{4}$$

moreover, there is an $L>0$ such that

$$|f(t)-f(s)|\le L(t-s)[1+\log(t/(t-s))],\quad \text{for all}\quad t>s\ge 0. \tag{8}$$

PROOF. We define

$$f(t)=(2\pi i)^{-1}\int_{\Gamma_{\epsilon,r}} g(\lambda)e^{\lambda t}\,d\lambda/\lambda,\quad t\ge 0, \tag{9}$$

where $\Gamma_{\epsilon,r}$ denotes the contour $\epsilon+i(-\infty,r)$, $\epsilon+re^{i[-\pi/2,\pi/2]}$, $\epsilon+i(r,\infty)$, where $\epsilon,r>0$. Observe that the integral is absolutely convergent, by virtue of (6). Clearly, the definition of $f(t)$ is independent of $\epsilon,r>0$, and contracting the contour in the right half plane it follows that

$f(0)=0$, by Cauchy's theorem. Fix $t>s\geq 0$ and estimate by means of (6) to the result

$$|f(t)-f(s)| \leq (M/2\pi)\int_{\Gamma_{\epsilon,r}} |e^{\lambda t}-e^{\lambda s}||d\lambda|/|\lambda|^2.$$

Letting $\epsilon\to 0$, this yields

$$\begin{aligned}|f(t)-f(s)| &\leq (M/\pi)\left[\int_r^\infty |e^{i\rho t}-e^{i\rho s}|\,d\rho/\rho^2 \right.\\ &\qquad\qquad \left. + \int_0^{\pi/2} |e^{rte^{i\phi}}-e^{rse^{i\phi}}|\,d\phi/r\right]\\ &\leq (M/\pi)\left[2\int_r^\infty |\sin(\rho(t-s)/2)|\,d\rho/\rho^2 + e^{rt}(t-s)\pi/2\right]\\ &\leq (M/2)(t-s)\left[2\pi^{-1}\int_{r(t-s)/2}^\infty |\sin\tau|\,d\tau/\tau^2 + e^{rt}\right].\end{aligned}$$

Choosing $r=2/t$ we obtain

$$\begin{aligned}|f(t)-f(s)| &\leq (M/2)(t-s)\left[\pi^{-1}\int_{1-s/t}^\infty |\sin\tau|\,d\tau/\tau^2 + e^2\right]\\ &\leq L(t-s)(1+\log(t/(t-s))],\end{aligned}$$

what was to be proved.

Finally, we have, by means of Cauchy's theorem and Fubini's theorem,

$$\begin{aligned}\lambda\hat{f}(\lambda) &= \lambda\int_0^\infty e^{-\lambda t}f(t)\,dt = \lambda(2\pi i)^{-1}\int_0^\infty\int_{\Gamma\epsilon,r} e^{-\lambda t}e^{\mu t}g(\mu)\,d\mu/\mu\,dt\\ &= (2\pi i)^{-1}\int_{\Gamma_{\epsilon,r}} g(\mu)\lambda\left(\int_0^\infty e^{-\lambda t}e^{\mu t}\,dt\right)d\mu/\mu\\ &= (2\pi i)^{-1}\int_{\Gamma_{\epsilon,r}} g(\mu)\lambda(\lambda-\mu)^{-1}\,d\mu/\mu\\ &= (2\pi i)^{-1}\int_{\Gamma_{\epsilon,r}} g(\mu)(1+\mu/(\lambda-\mu))\,d\mu/\mu\\ &= (2\pi i)^{-1}\int_{\Gamma_{\epsilon,r}} g(\mu)(\lambda-\mu)^{-1}\,d\mu\\ &= g(\lambda) \quad \text{for all} \quad \operatorname{Re}\lambda > r+\epsilon;\end{aligned}$$

hence, (4) holds on $\mathbf{C}_+$ by holomorphy. □

Combining Propositions 2 and 3 we obtain the following result on inversion of the vector-valued Laplace transform

THEOREM 1. *Suppose $g : \mathbf{C}_+ \to X$ is holomorphic and satisfies*

$$|g^{(n)}(\lambda)| \le M|\lambda|^{-(n+1)} \quad \textit{for} \quad \operatorname{Re}\lambda > 0,\ 0 \le n \le k+1,$$

where $k \ge 0$. Then there is a function $f \in C^k((0,\infty);X)$ such that $g(\lambda) = \hat{f}(\lambda)$ for $\operatorname{Re}\lambda > 0$. Moreover, f satisfies

$$|t^n f^{(n)}(t)| \le C \quad \textit{for} \quad t > 0,\ 0 \le n \le k,$$

and

$$|t^{k+1} f^{(k)}(t) - s^{k+1} f^{(k)}(s)| \le C|t-s|[1+\log(t/(t-s))], \quad 0 \le s < t < \infty.$$

PROOF. For $n \le k+1$ we define $g_n(\lambda) = \lambda^n g^{(n)}(\lambda)$; then, for $n \le k$, $g_n(\lambda)$ satisfies the assumptions of Proposition 2. Hence, there are functions $f_n \in C(\mathbf{R}_+;X)$ with $f_n(0) = 0$ such that $g_n(\lambda) = \lambda \hat{f}_n(\lambda)$ and there is an $L > 0$ such that

$$|f_n(t) - f_n(s)| \le L|t-s| \quad \text{for all} \quad t, s \in \mathbf{R}_+,\ 0 \le n \le k.$$

Proposition 3 yields $f_{k+1} \in C(\mathbf{R}_+, X)$ with $f_{k+1}(0) = 0$ such that $g_{k+1}(\lambda) = \lambda \hat{f}_{k+1}(\lambda)$ and

$$|f_{k+1}(t) - f_{k+1}(s)| \le L(t-s)(1 + \log(t/(t-s))), \quad 0 \le s < t < \infty.$$

Since

$$g_n'(\lambda) = n\lambda^{n-1} g^{(n)}(\lambda) + \lambda^n g^{(n+1)}(\lambda) = (n g_n(\lambda) + g_{n+1}(\lambda))/\lambda$$

for all $\operatorname{Re}\lambda > 0$ and $0 \le n \le k$, we obtain

$$\hat{f}_n'(\lambda) = ((n-1)\hat{f}_n(\lambda) + \hat{f}_{n+1}(\lambda))/\lambda,$$

which implies

$$-t f_n(t) = (n-1)\int_0^t f_n(\tau)\,d\tau + \int_0^t f_{n+1}(\tau)\,d\tau, \quad t \ge 0,\ 0 \le n \le k,$$

by uniqueness of the Laplace transform. This identity shows $f_n \in C^1((0,\infty);X)$ and

$$-tf_n'(t) = nf_n(t) + f_{n+1}(t), \quad t>0,\ 0\le n\le k.$$

Let $f(t) = f_0'(t)$; then $|f(t)| = |f_1(t)|/t \le L$; hence,

$$\hat{f}(\lambda) = \lambda \hat{f}_0(\lambda) = g_0(\lambda) = g(\lambda), \quad \operatorname{Re}\lambda > 0,$$

and $f_n(t) = (-1)^n(t^n f(t))^{(n-1)}$ as is easily seen by induction. The assertion now follows from the properties of $f_n(t)$. □

It would be interesting to know whether the logarithmic factor in (8) can be removed even in the one-dimensional case $X = \mathbf{C}$.

3. k-regular kernels. Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ is of subexponential growth, i.e.,

$$\int_0^\infty |a(t)|e^{-\epsilon t}\,dt < \infty \quad \text{for each} \quad \epsilon > 0. \tag{10}$$

The following class of kernels will be of central importance in the next section.

Definition 1. Let $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ be of subexponential growth and $k \in \mathbf{N}$. $a(t)$ is called *k-regular* if there is a constant $c > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \le cn!|\hat{a}(\lambda)| \quad \text{for} \quad \operatorname{Re}\lambda > 0 \quad \text{and} \quad 0 \le n \le k. \tag{11}$$

Observe that any k-regular kernel ($k \ge 1$) has the property that $\hat{a}(\lambda)$ has no zeros in the open right half plane. Convolutions of k-regular kernels are again k-regular; in particular, integrals of such kernels again have this property. On the other hand, sums, in general, do not have this property. It is not difficult to verify that (11) is equivalent to

$$|(\lambda^n \hat{a}(\lambda))^{(n)}| \le c'n!|\hat{a}(\lambda)|, \quad \text{for} \quad \operatorname{Re}\lambda > 0, \quad 1 \le n \le k, \tag{11'}$$

as well as to

$$(11'') \qquad \begin{aligned} &\hat{a}(\lambda) \neq 0, \quad \text{for} \quad \operatorname{Re}\lambda > 0, \\ |\lambda^n(\log \hat{a}(\lambda))^{(n)}| &\leq c'' n!, \quad \text{for} \quad \operatorname{Re}\lambda > 0, \quad 1 \leq n \leq k. \end{aligned}$$

It has been shown in Clèment and Prüss [**4**] that every completely monotonic $a \in L^1_{\text{loc}}(\mathbf{R}_+)$ is k-regular for any $k \in \mathbf{N}$; in fact, $c = 1$ in (11) will do. If $a(t)$ is real-valued and 1-regular, then $|\arg \hat{a}(\lambda)| \leq \pi c/2$ for $\operatorname{Re}\lambda > 0$; use the identity

$$\arg \hat{a}(re^{i\phi}) = \operatorname{Im} \int_0^{\phi} [\hat{a}'(re^{it})/\hat{a}(re^{it})] ire^{it}\, dt$$

for a proof. The converse of this is not true, as the example $a(t) = 1$ for $t \in (0, 1]$, $a(t) = 0$ for $t > 1$ shows; in fact, then $\hat{a}(\lambda) = (1 - e^{-\lambda})/\lambda$. Hence, $-\lambda\hat{a}'(\lambda)/\hat{a}(\lambda) = 1 - \lambda e^{-\lambda}/(1 - e^{-\lambda})$, which is not bounded in the right half plane.

LEMMA 1. *Suppose the function $g : \mathbf{C}_+ \to \mathbf{C}$ is holomorphic and satisfies $g(\lambda) \neq 0$ and $|\arg g(\lambda)| \leq \Theta$ for* $\operatorname{Re}\lambda > 0$. *Then, for each $n \in \mathbf{N}$, there is a constant $c_n > 0$ such that*

$$(12) \qquad (\operatorname{Re}\lambda)^n |g^{(n)}(\lambda)| \leq c_n(\Theta/\pi)|g(\lambda)| \quad \textit{for} \quad \operatorname{Re}\lambda > 0.$$

The constants c_n depend only on n.

PROOF. Let $u(\lambda) = \arg g(\lambda)$; the Poisson formula for the half plane and its analytic completion yield with some constant α

$$\log g(\lambda) = \alpha + (2\pi)^{-1} \int_{-\infty}^{\infty} [(1 - i\rho\lambda)/(\lambda - i\rho)] u(i\rho)\, d\rho/(1 + \rho^2);$$

hence, differentiation gives

$$(\log g(\lambda))^{(n)} = (n!/2\pi) \int_{-\infty}^{\infty} (\lambda - i\rho)^{-(n+1)} u(i\rho)\, d\rho, \quad \operatorname{Re}\lambda > 0.$$

A direct estimation leads to

$$|(\log g(\lambda))^{(n)}| \le (n!/\pi)(\operatorname{Re}\lambda)^{-n}\Theta \int_{-\infty}^{\infty} (1+\rho^2)^{-(n+1)/2}\, d\rho$$

from which (12) follows. □

Let $\Sigma_\phi = \{z \in \mathbf{C} : |\arg z| < \phi\}$. With this notation Lemma 1 yields a large class of kernels $a(t)$ which are k-regular for any $k \in \mathbf{N}$, including the class used by Da Prato and Iannelli [**5**] mentioned in the introduction.

PROPOSITION 4. *Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ is such that $\hat{a}(\lambda)$ admits analytic extension to the sector Σ_ϕ, where $\phi > \pi/2$, such that $\hat{a}(\lambda) \neq 0$ and $|\arg \hat{a}(\lambda)| \le \theta$ for all $\lambda \in \Sigma_\phi$. Then a is k-regular for every $k \in \mathbf{N}$.*

PROOF. We let $g : \mathbf{C}_+ \to \mathbf{C}$ be defined by $g(z) = \hat{a}(z^p)$, where $p = 2\phi/\pi$. Then the assumptions of Lemma 1 are fulfilled; hence, (12) holds. This implies, with $\hat{a}(\lambda) = g(\lambda^\alpha)$, $\alpha = 1/p$,

$$\lambda^n \hat{a}^{(n)}(\lambda) = \sum_{k=1}^{n} b_k^n g^{(k)}(\lambda^\alpha)\lambda^{\alpha k}, \quad \operatorname{Re}\lambda > 0,\ n \in \mathbf{N},$$

for some constants b_k^n. Therefore

$$\begin{aligned} |\lambda^n \hat{a}^{(n)}(\lambda)| &\le \sum_{k=1}^{n} |b_k^n||\lambda^{\alpha k} g^{(k)}(\lambda^\alpha)| \\ &\le (\theta/\pi)|\hat{a}(\lambda)| \sum_{k=1}^{n} c_k |b_k^n| (|\lambda^\alpha|/\operatorname{Re}\lambda^\alpha)^k \le C(n)|\hat{a}(\lambda)|, \end{aligned}$$

for $\operatorname{Re}\lambda > 0$, since $\operatorname{Re}\lambda^\alpha = |\lambda|^\alpha \cos(\alpha \arg\lambda) \ge |\lambda|^\alpha \cos(\alpha\pi/2)$ and $\alpha < 1$. □

We have seen above that kernels of positive type are, in general, not even 1-regular. However, we can prove that nonnegative, nonincreasing, convex kernels do have this property. More generally, let us introduce

Definition 2. Let $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$, $k \geq 2$. Then a is called *k-monotone* if $a \in C^{k-2}(0,\infty)$, $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0$, $0 \leq n \leq k-2$, and $(-1)^{k-2}a^{(k-2)}(t)$ is nonincreasing and convex.

Thus, by this definition, a 2-monotone kernel is nonnegative, nonincreasing and convex. We have the following

Proposition 5. *Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ is k-monotone for some $k \geq 2$. Then a is $(k-1)$-regular.*

Proof. Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ is k-monotone for some $k \geq 2$. Since 2-monotone kernels are of positive type, Lemma 1 yields, with $\lambda = \sigma + i\rho$, $\sigma > 0$, $\rho \in \mathbf{R}$,

$$\sigma^n |\hat{a}^{(n)}(\lambda)| \leq C_n |\hat{a}(\lambda)|, \quad \operatorname{Re}\lambda > 0,\ n \in \mathbf{N}.$$

Therefore, it is sufficient to prove

$$|\rho|^n |\hat{a}^{(n)}(\lambda)| \leq C_n |\hat{a}(\lambda)|, \quad \operatorname{Re}\lambda > 0,\ 1 \leq n \leq k-1.$$

Let $a_\sigma(t) = a(t)e^{-\sigma t}$, $t, \sigma > 0$; then a_σ is, again, k-monotone. So it remains to show

$$\rho^n |\hat{a}^{(n)}_\sigma(i\rho)| \leq C_n |\hat{a}_\sigma(i\rho)|, \quad \sigma, \rho > 0,\ 1 \leq n \leq k-1,$$

since $a(t)$ is real-valued. The constants C_n will be independent of the particular function a and of σ as well, and so we drop the index σ in the sequel and assume also that $a \in L^1(\mathbf{R}_+)$. To simplify further, we recall the Shea-Wainger estimate for 2-monotone functions (cf. Shea-Wainger [**10**]),

$$|\hat{a}(i\rho)| \geq (2^{-3/2}) \int_0^{1/\rho} a(\tau)\, d\tau, \quad \rho > 0.$$

Thus, if we can prove the inequality

$$|((i\rho)^{k-1}\hat{a}^{(k-1)}(i\rho))| \leq C \int_0^{1/\rho} a(\tau)\, d\tau, \quad \rho > 0, \tag{13}$$

for a given k-monotone kernel $a \in L^1(\mathbf{R}_+)$, the result will follow.

After these preparations we are going to prove (13). It is not difficult to prove the following identity inductively via integration by parts:

(14)

$$\int_0^t a(\tau)\,d\tau = \sum_{j=0}^{k-1}(-1)^j a^{(j)}(t)t^{j+1}/(j+1)! + \int_0^t (-1)^k \tau^k/k!\,da^{(k-1)}(\tau),$$
$$t>0.$$

In particular, $a^{(j)}(t)t^{j+1} \to 0$ as $t \to 0$ for each $j \le k-1$, and the integral on the right-hand side of this equation is absolutely convergent. Similarly, one also obtains the formula

(15)

$$\int_t^\infty (-1)^k \tau^{k-1}/(k-1)!\,da^{(k-1)}(\tau) = \sum_{j=0}^{k-1}(-1)^j a^{(j)}(t)t^j/j!, \quad t>0;$$

in particular, the integral on the left-hand side converges absolutely, and $a^{(j)}(t)t^j \to 0$ as $t \to \infty$ for each $j \le k-1$. Also, an integration by parts yields

$$\int_0^\infty da^{(k-1)}(t)\left(\sum_{j=0}^{k-1}(-i\rho t)^j/j! - e^{-i\rho t}\right) = (i\rho)^k \hat{a}(i\rho), \quad \rho \in \mathbf{R};$$

hence,

$$-((i\rho)^k\hat{a}(i\rho))^{(k-1)} = \int_0^\infty da^{(k-1)}(t)(-1)^k t^{k-1}(1-e^{-i\rho t}), \quad \rho>0.$$

Since

$$|1-e^{-ix}| \le 2, \quad \text{for} \quad |x|\ge 1, \quad \text{and} \quad |1-e^{-ix}| \le |x| \quad \text{for} \quad |x| \le 1,$$

we obtain, by (14), (15) and k-monotonicity of $a(t)$,

$$
\begin{aligned}
|[(i\rho)^k \hat{a}(i\rho)]^{(k-1)}| &\le \rho \int_0^{1/\rho} da^{(k-1)}(t)(-t)^k + 2\int_{1/\rho}^{\infty} da^{(k-1)}(-1)^k t^{k-1} \\
&\le k!\rho \int_0^{1/\rho} a(\tau)\,d\tau \\
&\quad + 2\sum_{j=0}^{k-1} (-1)^j a^{(j)}(1/\rho)\rho^{-j}(k-1)!/j! \\
&\le 3k!\rho \int_0^{1/\rho} a(\tau)\,d\tau.
\end{aligned}
$$

From this inequality (13) follows easily by induction. □

4. Resolvents and maximal regularity. Recall that a family $\{S(t)\}_{t\ge 0} \subset \mathcal{B}(X)$, the space of bounded linear operators in a Banach space X, is called a *resolvent* for (1) if the following conditions are satisfied:

(S1) $S(t)$ is strongly continuous on $\mathbf{R}_+$, i.e., for each $x \in X$, the function $t \mapsto S(t)x$ is continuous on $\mathbf{R}_+$.

(S2) $S(t)$ commutes with A, i.e., $S(t)D(A) \subset D(A)$ for all $t \ge 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \ge 0$.

(S3) The *resolvent equation* is satisfied, i.e., for each $x \in D(A)$,

$$S(t)x = x + \int_0^t a(t-\tau)AS(\tau)x\,d\tau, \quad t \ge 0. \tag{16}$$

Once a resolvent of (1) is known to exist, the mild solutions $u(t)$ are given by the *variation of parameters formula*

$$u(t) = d/dt\left(\int_0^t S(t-\tau)f(\tau)\,d\tau\right), \quad t \in J; \tag{17}$$

in particular, $S * f \in C^1(J;X)$ if a mild solution of (1) exists. Maximal regularity of type C^α will be based on (17); note that, in case S is differentiable, (17) can be rewritten as

$$u(t) = f(t) + \int_0^t S'(t-\tau)f(\tau)\,d\tau, \quad t \in J. \tag{18}$$

In general, a resolvent of (1) need not be differentiable, and even if it is, $S'(t)$ will have a strong singularity at $t = 0$ unless A is bounded, even if $a(t)$ is smooth. However, as in the case of an analytic semigroup $S(t) = e^{At}$, a strong singularity of $S(t)$ can be compensated by Hölder-continuity of the inhomogeneity $f(t)$.

Definition 3. Suppose $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ satisfies $\int_0^\infty |a(t)|e^{-\omega t}\,dt < \infty$ for all $\omega \geq \omega_0$. Equation (1) is called *parabolic* if there is an $\omega \geq \omega_0$ such that $1/\hat{a}(\lambda) \in \rho(A)$ for all $\mathrm{Re}\,\lambda > \omega$ and

$$|(I - \hat{a}(\lambda)A)^{-1}| \leq M, \quad \mathrm{Re}\,\lambda > \omega, \tag{19}$$

holds for some constant $M \geq 1$.

Observe that Definition 3 is a natural extension of parabolic Cauchy problems, i.e., $a(t) \equiv 1$ and A generates an analytic C_0-semigroup.

It can be shown that every parabolic Volterra equation (1) admits an L^p-resolvent $S \in L^p_{\mathrm{loc}}(\mathbf{R}_+; \mathcal{B}(X))$ which satisfies (S2) and (S3) almost everywhere. However, it is not known whether this L^p-resolvent also has the strong continuity property (S1), unless the kernel $a(t)$ has extra smoothness properties. Theorem 1 yields the following result.

Theorem 2. *Let X be a Banach space, A a closed linear densely defined operator in X, $a \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$, and assume*

(H1) $a_\omega(t) = a(t)e^{-\omega t}$ *is k-regular for some $k \in \mathbf{N}$, $\omega \in \mathbf{R}$.*

(H2) *Equation* (1) *is parabolic.*

Then there is a resolvent $S \in C^{k-1}((0,\infty); \mathcal{B}(X))$ for (1); *moreover, for each $T > 0$ there is a constant $M_T > 0$ such that*

$$|t^n S^{(n)}(t)| \leq M_T \quad \text{for all} \quad t \in [0,T], \quad n \leq k-1, \tag{20}$$

and

$$|t^k S^{(k-1)}(t) - s^k S^{(k-1)}(s)| \leq M_T(t-s)[1 + \log(t/(t-s))], \quad 0 \leq s < t \leq T. \tag{21}$$

PROOF. Since we are only interested in the local behavior of the Volterra equation (1), we may assume that $\omega = 0$ in (H1) as well as in the definition of parabolicity. Define

$$H(\lambda) = (I - \hat{a}(\lambda)A)^{-1}/\lambda, \quad \operatorname{Re}\lambda > 0;$$

by (H2) we then have $|H(\lambda)| \le M/|\lambda|$ for $\operatorname{Re}\lambda > 0$. By (H1) and by

$$H'(\lambda) = -H(\lambda)/\lambda + (\hat{a}'(\lambda)/\hat{a}(\lambda))\hat{a}(\lambda)A(I - \hat{a}(\lambda)A)^{-1}H(\lambda),$$

we obtain $|H'(\lambda)| \le M_1/|\lambda|^2$, $\operatorname{Re}\lambda > 0$, for some constant M_1. Proceeding this way it follows inductively that there is a constant $C > 0$ such that

$$|H^{(n)}(\lambda)| \le C|\lambda|^{-(n+1)} \quad \text{for} \quad \operatorname{Re}\lambda > 0,\ n \le k.$$

Thus, Theorem 1 yields $S \in C^{k-1}((0,\infty);\mathcal{B}(X))$ satisfying (20) and (21), such that

$$H(\lambda) = \hat{S}(\lambda) \quad \text{for} \quad \operatorname{Re}\lambda > 0$$

holds. It remains to show that $S(t)$ is the resolvent for (1). For this purpose let $x \in D(A)$. Then the identity

$$H(\lambda)x = x/\lambda + \hat{a}(\lambda)H(\lambda)Ax$$

implies

$$S(t)x = x + (a * S)(t)Ax, \quad t \ge 0;$$

hence, $S(t)x$ is continuous on $\mathbf{R}_+$ for each $x \in D(A)$ since $S(t)$ is bounded on each interval $J = [0,T]$, and so $S(t)$ satisfies (S1) by the Banach-Steinhaus Theorem. Since $H(\lambda)$ commutes with A, $S(t)$ does as well; therefore, (S2) and (S3) are also verified and $S(t)$ is the resolvent for (1). $\square$

Having established the existence of the resolvent for (1) with the properties (20) and (21), we now prove the maximal regularity result for (1). The following standard notation will be employed: for $\alpha \in (0,1)$ and $J = [0,T]$, we let

$$C^{\alpha}(J;X) = \{u \in C(J;X); |u|_{\alpha} := \sup\{|u(t) - u(s)|(t-s)^{-\alpha} : 0 \le s < t \le T\} < \infty\}$$

and

$$C_0^\alpha(J;X) = \{u \in C^\alpha(J;X)\ ;\ u(0) = 0\}.$$

THEOREM 3. *Suppose $S \in C^1((0,\infty);X)$ is a resolvent for* (1)*, such that $|S(t)| + |tS'(t)| \le M_T$ for $t \in J = [0,T]$, and*

$$|t^2S'(t) - s^2S'(s)| \le M_T(t-s)[1+\log(t/(t-s))], \quad 0 \le s < t \le T. \tag{22}$$

Let $\alpha \in (0,1)$. Then, for every $f \in C^\alpha(J;X)$, the function $u(t)$ given by (18) *is well-defined and a mild solution of* (1)*. Moreover, if $f \in C_0^\alpha(J;X)$, then $u \in C_0^\alpha(J;X)$, i.e.,* (1) *has the maximal regularity property of type C^α.*

PROOF. Let $f \in C^\alpha(J;X)$; rewrite (18) according to

$$u(t) = S(t)f(0) + S(t)(f(t)-f(0)) + \int_0^t S'(t-\tau)(f(\tau)-f(t))\,d\tau, \quad t \in J. \tag{23}$$

(23) shows that $u(t)$ is well-defined on J and that in the sequel we may assume $f(0) = 0$. Once we know that $u(t)$ is also continuous on J, it follows by standard arguments involving the resolvent equation that u is a mild solution of (1). Thus, it remains to show $u \in C_0^\alpha(J;X)$. For this purpose let $t, s \in J$, $h = t - s > 0$ and use (23) to rewrite $u(t) - u(s)$ as

$$\begin{aligned} u(t) - u(s) &= (S(t) - S(s))f(s) + S(h)(f(t) - f(s)) \\ &\quad + \int_s^t S'(t-\tau)(f(\tau) - f(t))\,d\tau \\ &\quad + \int_0^s (S'(\tau+h) - S'(\tau))(f(s-\tau) - f(s))\,d\tau \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

The single terms I_j are estimated by

$$|I_1| \le \left|\int_s^t S'(\tau)\,d\tau\right| |f(s)| \le M_T \log(t/s)|f|_\alpha s^\alpha \le M_T|f|_\alpha h^\alpha/\alpha,$$

where we used the elementary inequality

$$\log(1+\rho) \le \rho^\alpha/\alpha \quad \text{for all} \quad \rho > 0, \quad \alpha \in (0,1),$$

as well as $f(0) = 0$. Then

$$\begin{aligned}
|I_2| &\le |S(h)||f(t) - f(s)| \le M_T |f|_\alpha h^\alpha, \\
|I_3| &\le \int_s^t |S'(t-\tau)||f(\tau) - f(t)|\, d\tau \le M_T |f|_\alpha \int_s^t (t-\tau)^{(\alpha-1)}\, d\tau \\
&= M_T |f|_\alpha h^\alpha/\alpha.
\end{aligned}$$

To estimate I_4 observe that

$$|S'(t+h) - S'(t)| \le 3M_T (h/t)(t+h)^{-1}(1 + \log(1 + t/h)), \quad t, h > 0,$$

which follows from (22). Hence

$$\begin{aligned}
|I_4| &\le \int_0^s |S'(\tau + h) - S'(\tau)||f(s-\tau) - f(s)|\, d\tau \\
&= 3M_T h |f|_\alpha \int_0^s \tau^{\alpha-1}(h+\tau)^{-1}(1 + \log(1 + \tau/h))\, d\tau \\
&= 3M_T h^\alpha |f|_\alpha \int_0^{s/h} \tau^{\alpha-1}(1+\tau)^{-1}(1 + \log(1+\tau))\, d\tau \\
&\le 3M_T h^\alpha |f|_\alpha \int_0^\infty \tau^{\alpha-1}(1+\tau)^{-1}(1 + \log(1+\tau))\, d\tau \\
&= 3M_T h^\alpha |f|_\alpha C_\alpha.
\end{aligned}$$

Collecting terms, we obtain

$$|u|_\alpha \le M_T |f|_\alpha [2/\alpha + 1 + 3C_\alpha];$$

in particular, $u \in C^\alpha(J; X)$; $u(0) = 0$ is trivial from (23) since $f(0) = 0$. $\square$

5. Examples and applications. Combining Theorems 2 and 3, we obtain the following corollary which extends the results of Da Prato and Iannelli [**5**] and Da Prato, Iannelli and Sinestrari [**6**].

COROLLARY 1. *Suppose A generates a uniformly bounded analytic semigroup in X, and let $a(t)$ be nonnegative, nonincreasing, and convex. Then* (1) *admits a resolvent $S(t)$ which, in addition, belongs to $C^{\alpha}_{\mathrm{loc}}((0,\infty);\mathcal{B}(X))$ for every $\alpha \in (0,1)$. If, in addition, $-\dot{a}(t)$ is also convex, then $S(t)$ is of class $C^{1+\alpha}_{\mathrm{loc}}((0,\infty);\mathcal{B}(X))$ for every $\alpha \in (0,1)$, and* (1) *has the maximal regularity property of type C^{α}.*

It is also interesting to compare Corollary 1 with the result on existence of the resolvent for (1) obtained by Clèment and Nohel [**3**]. They prove existence of $S(t)$ assuming that A generates a bounded C_0-semigroup and the kernel $a(t)$ is completely positive. Even in the parabolic case when the semigroup is analytic, their result is of a different nature than Corollary 1 since completely positive kernels need not be 1-regular, but also, for every $k \geq 2$, there are k-monotone kernels which are not completely positive.

For the proof of Corollary 1, we only have to observe that, under these assumptions, equation (1) is parabolic. This follows from the facts that $\operatorname{Re}\hat{a}(\lambda) > 0$ for all $\operatorname{Re}\lambda > 0$ and that $\rho(A) \supset \mathbf{C}_+$ and $|(I-\mu A)^{-1}| \leq M$ holds for $\operatorname{Re}\mu > 0$. Application of Proposition 5 and Theorems 2 and 3 then yield the result.

EXAMPLE 1. We want to apply Corollary 1 to the problem

$$\begin{cases} u(t,x) = \int_0^t a(t-\tau)\Delta u(\tau,x)\,d\tau + f(t,x), & t \in J,\ x \in \Omega, \\ u(t,x) = 0, \quad t \in J,\ x \in \partial\Omega, \end{cases} \tag{24}$$

where $\Omega \subset \mathbf{R}^N$ is an open domain with compact smooth boundary, and Δ denotes the Laplacian on $\mathbf{R}^N$. It is well-known that the realization A_p (or A_0) of Δ with Dirichlet boundary conditions generates a bounded analytic C_0-semigroup in $L^p(\Omega)$, $1 \leq p < \infty$, (or in $C_0(\Omega)$). Thus, if $a(t) \geq 0$ is nonincreasing and convex, then (24) admits a resolvent; if $-\dot{a}(t)$ is also convex, then (24) has the maximal regularity property for $L^p(\Omega)$ and for $C_0(\Omega)$. Then, for any $f \in C_0^{\alpha}(J;L^p(\Omega))$, there is a unique mild solution u of (24), which means $u \in C_0^{\alpha}(J;L^p(\Omega))$ and $a * u \in C_0^{\alpha}(J;W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega))$ since $D(A_p) = W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$, $1 < p < \infty$. If, in addition, $f = a * g$ with $g \in C_0^{\alpha}(J;L^p(\Omega))$, then even $u \in C_0^{\alpha}(J;W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. □

As another application of Theorems 2 and 3, we consider kernels of the form

$$a(t) = a_0 + \int_0^t a_1(\tau)\, d\tau, \quad t > 0, \tag{25}$$

where $a_0 \geq 0$, $a_1(t) \geq 0$ is nonincreasing and convex. The operator A is assumed to generate a C_0-semigroup which has an analytic extension to $\mathbf{C}_+$ and is bounded on each sector Σ_θ, $\theta < \pi/2$. This implies $\sigma(A) \subset (-\infty, 0]$, as well as the estimate

$$|(I - \mu A)^{-1}| \leq M_\phi, \quad \text{for} \quad \mu \in \Sigma_\phi,$$

for all $\phi < \pi$. This is still not enough to ensure the parabolicity of (1); however, in addition, we need, in case $a_0 = 0$,

$$|\arg \hat{a}_1(\lambda)| \leq \theta < \pi/2, \quad \operatorname{Re} \lambda > 0. \tag{26}$$

The following corollary extends results of Carr and Hannsgen [**2**] and of Prüss [**8**] in the parabolic case.

Corollary 2. *Suppose A generates a C_0-semigroup, analytic on $\mathbf{C}_+$, which is uniformly bounded on each sector Σ_θ, $\theta < \pi/2$, let $a(t)$ be of the form* (25) *and assume that the parabolicity condition* (26) *holds in case $a_0 = 0$. Then* (1) *admits a resolvent $S(t)$ which, in addition, belongs to $C^\alpha_{\text{loc}}((0,\infty); \mathcal{B}(X))$, for every $\alpha \in (0,1)$. If $-\dot{a}_1(t)$ is also convex, then $S \in C^{1+\alpha}_{\text{loc}}((0,\infty); \mathcal{B}(X))$ for every $\alpha \in (0,1)$, and* (1) *possesses the maximal regularity property of type C^α.*

Proof. It remains to show that $a(t)$ is 1-regular (respectively, 2-regular). We have

$$\begin{aligned} -\lambda \hat{a}'(\lambda)/\hat{a}(\lambda) &= (a_0/\lambda - \lambda \hat{a}_1'(\lambda)/(a_0/\lambda + \hat{a}_1(\lambda)) \\ &= g(\lambda) + (1 - g(\lambda))(-\lambda \hat{a}_1'(\lambda)/\hat{a}_1(\lambda)), \end{aligned}$$

where

$$g(\lambda) = a_0(a_0 + \lambda \hat{a}_1(\lambda))^{-1}.$$

Since $a_1(t)$ is of positive type and $-\operatorname{Im} \lambda \cdot \operatorname{Im} \hat{a}_1(\lambda) \geq 0$ for $\operatorname{Re} \lambda > 0$, we obtain $|g(\lambda)| \leq 1$; therefore, $a(t)$ is 1-regular since $a_1(t)$ has this property, according to Proposition 5. The proof of 2-regularity is similar, taking into account that $a_1(t)$ is 2-regular if $-\dot{a}_1(t)$ is convex. $\square$

EXAMPLE 2. Consider a domain $\Omega \subset \mathbf{R}^N$ with compact smooth boundary $\partial\Omega$ which is filled by a linear viscoelastic incompressible fluid. The velocity field $u(t,x)$ of this fluid is then governed by

(27)

$$\begin{cases} u_t(t,x) = a_0\Delta u(t,x) + \int_0^t a_1(t-\tau)\Delta u(\tau,x)\,d\tau - \operatorname{grad} p(t,x) \\ \qquad\qquad\qquad\qquad + g(t,x), \quad t \in J,\ x \in \Omega, \\ \operatorname{div} u(t,x) = 0, \quad t \in J,\ x \in \Omega, \\ u(t,x) = 0, \quad t \in J,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), \quad x \in \Omega \end{cases}$$

where $p(t,x)$ denotes the hydrostatic pressure, $u_0(x)$ the initial velocity field and $g(t,x)$ an external force field. The kernel $a_1(t)$ is always nonnegative, nonincreasing and of positive type, and $a_0 \geq 0$. For more on the physical background of (27) we refer to Renardy, Hrusa and Nohel [**9**].

To apply Corollary 2, we put $X = L^p_\sigma(\Omega;\mathbf{R}^N)$, where the subscript σ means divergence-free and let $A_p = P\Delta$ denote the Stokes operator, where P denotes the Helmholtz-projection in $L^p(\Omega;\mathbf{R}^N)$ which projects onto X along the gradient fields. Applying P to (27) and integrating with respect to time, (27) is equivalent to (1) with $f(t) = u_0 + \int_0^t g(\tau)\,d\tau$. It is known that, for $1 < p < \infty$, A_p generates a C_0-semigroup in X which admits analytic extension to $\mathbf{C}_+$ and is uniformly bounded on each sector Σ_θ, $\theta < \pi/2$; for $p = 2$, this is a consequence of the fact that A_2 is negative semi-definite. Thus, if the parabolic condition (26) holds and $a_1(t)$ is also convex, a resolvent $S(t)$ for (1) exists. Consequently, for every $u_0 \in X$, $g \in C(J;X)$, there is a mild solution $u(t)$ for (1) which satisfies $u \in C(J;X)$, $a * u \in C(J;D(A_p))$. If, in addition, $-\dot{a}_1(t)$ is convex, $u_0 \in D(A_p)$ and $g \in C^\alpha(J;X)$, then $u \in C^1(J;X)$ and $a_0 u + a_1 * u \in C(J;D(A_p))$; hence, even $u \in C(J;D(A_p))$ if $a_0 > 0$. If even $g = a_1 * h$ (in case $a_0 = 0$), then one also has $u \in C(J;D(A_p))$, i.e., u is a strong solution of (27).

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