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ON THE UNIQUENESS OF SOLUTIONS OF VOLTERRA EQUATIONS

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Dedicated to John Nohel on his 65th birthday

ABSTRACT. It is shown that, except for the obvious case of a Lipschitz continuous nonlinearity and a kernel identically zero close to the origin, the uniqueness of the trivial solution $x(t) \equiv 0$ of the equation $x(t) = \int_0^t k(t-s)g(x(s))\,ds$ depends on both a and g.

1. Introduction and statement of results. Consider the equation

$$x(t) = \int_0^t k(t-s)g(x(s))\,ds, \quad t \geq 0, \tag{1}$$

where k is locally integrable and g is continuous with $g(0) = 0$. It is clear that $x(t) \equiv 0$ is a solution of (1), so the question to be answered is whether there are any other, nontrivial, solutions.

This problem is a special case of the problem of uniqueness of the trivial solution of the equation

$$x(t) = \int_0^t k(t,s,x(s))\,ds, \quad t \geq 0.$$

If the trivial solution is unique one says that k is a Kamke function, and this question is relevant in many problems not directly connected with the uniqueness of solutions of Volterra equations. Although there are definite advantages in treating the more general equation, we will here consider onlyconvolution equations of the form (1).

One of the main tools of the analysis is a comparison principle. Although this result can be found in almost every book on Volterra equations, we state it here in the form that we will need.

LEMMA 1. *Assume that*

(i) k *and* K *belong to* $L^1_{\text{loc}}(\mathbf{R}^+;\mathbf{R})$ *and satisfy* $0 \le k(t) \le K(t)$ *for almost every* $t \ge 0$;

(ii) g *and* G *belong to* $C(\mathbf{R};\mathbf{R})$, *satisfy* $0 \le g(s) \le G(s)$ *for* $s \ge 0$, *and* g *or* G *is nondecreasing on* $\mathbf{R}^+$;

(iii) *there exists a function* $x \in C(\mathbf{R}^+;\mathbf{R}^+)$ *and a number* $\tau > 0$ *such that*

$$0 \le x(t) \le \int_0^t k(t-s)g(x(s))\,ds, \quad t \in [0,\tau].$$

Then there exists a number $T \in (0,\tau]$ *and a function* $X \in C(\mathbf{R}^+;\mathbf{R}^+)$ *such that*

$$x(t) \le X(t) = \int_0^t K(t-s)G(X(s))\,ds, \quad t \in [0,T].$$

Moreover, if G *is nondecreasing, then* X *is nondecreasing as well.*

A similar result can, of course, be stated for a more general nonconvolution equation; see, for example, [**4**, Chapters 12 and 13].

Note that this result can be used in two different ways: if one knows that the trivial solution of (1) is unique, then it will be unique also if k and g are replaced by some smaller functions, and, conversely, if there is a nontrivial solution, then there will also be one if k and g are replaced by some larger functions.

Let us first state a well-known result:

PROPOSITION 2. *Assume that*

(i) $k \in L^1_{\text{loc}}(\mathbf{R}^+;\mathbf{R})$;

(ii) $g \in C(\mathbf{R};\mathbf{R})$ *is nondecreasing and satisfies* $g(s) = 0$ *for* $s \le 0$. *Then the trivial solution* $x(t) \equiv 0$ *of* (1) *is unique, provided at least one of the following conditions holds*:

(a) $\int_0^T |k(s)|\,ds = 0$ for some $T > 0$.

(b) $\liminf_{s\downarrow 0} g(s)/s < \infty$.

The main result of this paper is that, if k is in addition nonnegative, then this result cannot be improved in the sense that (a) or (b) cannot be replaced by any weaker conditions. In fact, if k is such that (a) does not hold, then the trivial solution can be unique or nonunique, depending on g. A similar result is true if g does not satisfy (b).

We need the following notation.

$$\mathcal{K} \stackrel{\text{def}}{=} \left\{ k \in L^1(\mathbf{R}^+;\mathbf{R}) \middle| k(t) \geq 0, \quad \int_0^t k(s)\,ds > 0, t \geq 0 \right\},$$
$$\mathcal{G} \stackrel{\text{def}}{=} \left\{ g \in C(\mathbf{R};\mathbf{R}) \middle| g(s) = 0, s \leq 0, \right. \qquad \left. g \text{ is nondecreasing, } \lim_{s\downarrow 0} \frac{g(s)}{s} = \infty \right\}.$$

THEOREM 3. *For every* $k \in \mathcal{K}$, *there exists a* $g \in \mathcal{G}$ *and, for every* $g \in \mathcal{G}$, *there exists a* $k \in \mathcal{K}$ *such that the trivial solution on* (1) *is unique. Conversely, for every* $k \in \mathcal{K}$, *there exists a* $g \in \mathcal{G}$, *and, for every* $g \in \mathcal{G}$, *there exists a* $k \in \mathcal{K}$ *such that equation* (1) *has a continuous solution* x *on some interval* $[0,T]$, $T > 0$, *such that* $x(t) > 0$ *when* $t \in (0,T]$.

This result shows that the conjecture made in [**2**] concerning the existence of nontrivial solutions of (1) with $g(s) = s^p, 0 < p < 1$, is not true.

Thus one sees that, in general, there is a close interplay between the properties of the nonlinearity g and the kernel k. Here we give only one result in this direction. Observe also that the question of uniqueness or nonuniqueness of the trivial solution depends only on the values of k and g in a neighborhood of zero. The following result (as well as the theorems above) could therefore be generalized to take this fact into account.

THEOREM 4. *Assume that* $\alpha > 0$ *and that*

(i) $g \in C(\mathbf{R};\mathbf{R})$ *is nondecreasing,* $g(s) = 0$ *for* $s \leq 0$;

(ii) *the function* $s \mapsto h(s) \stackrel{\text{def}}{=} g(s)/s$ *is nonincreasing on* $(0,\infty)$.

Then the trivial solution $x(t) \equiv 0$ *of the equation*

$$x(t) = \int_0^t (t-s)^{\alpha-1} g(x(s))\, ds, \quad t > 0, \tag{2}$$

is unique if and only if

$$\int_0^1 \frac{1}{s h(s)^{1/\alpha}}\, ds = \infty. \tag{3}$$

Under the additional assumption that $h(s)^p s$ is nonincreasing on some interval $(0, \delta_p)$, this result has been established in [**3**]. When proving that there exists a nontrivial solution of (2) if (3) does not hold, this extra assumption can be removed with the aid of Lemma 1 since one can replace g by a smaller function satisfying the extra assumption such that (3) still does not hold. Another proof is given in [**5**], where the sufficiency for the case $\alpha \geq 1$ is established as well. In this case one can, in fact, drop the assumption (ii), see [**2**].

2. Proof of Theorem 3. Assume that g belongs to $\mathcal{G}$. We shall construct a kernel k such that there exists a nontrivial solution x of (1) on some interval $[0,T]$, where $T > 0$. We may, without loss of generality, assume that the function $h(s) = g(s)/s$ is nonincreasing since we can, by Lemma 1, replace g by a smaller function with this property.

Define the sequence u_n by $u_n = 2^{-n}, n \geq 0$, and let

$$a_n = \frac{u_n}{g(u_{n+1})} = \frac{2}{h(2^{-(n+1)})}.$$

It follows from our assumptions that $\{a_n\}$ is a sequence of positive, nonincreasing numbers tending to 0. Therefore, we get a function $k \in \mathcal{K}$ if we let

$$k(t) = \begin{cases} 2^{n+2}(a_n - a_{n+1}), & t \in (2^{-(n+2)}, 2^{-(n+1)}], \quad n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The motivation for this construction is that we have

$$u_n = \int_0^{2^{-(n+1)}} k(s)\,ds\,g(u_{n+1}), \quad n \geq 0.$$

Hence, if we let V be the set

$$V = \left\{u \in C([0,1];\mathbf{R}) \,|\, u(2^{-n}) \geq u_n,\ u \text{ is nondecreasing}\right\},$$

then we see that the operator taking the function u to the function

$$\int_0^t k(t-s)g(u(s))\,ds, \quad t \in [0,1],$$

maps V into itself. But then it follows that there exists a solution x of equation (1) that belongs to V, see [**4**, Theorem 12.2.5]. This completes the proof in this case.

Next let us consider the case where $k \in \mathcal{K}$ is given. We define the sequence $\{u_n\}$ by

$$u_n = 2^{-n} \int_0^{2^{-(n+1)}} k(s)\,ds, \quad n \geq 0,$$

and we can then construct a function $g \in \mathcal{G}$ such that $g(u_{n+1}) = 2^{-n}$. Thus, we again have

$$U_n = \int_0^{2^{-(n+1)}} k(s)\,ds\,g(u_{n+1}), \quad n \geq 0,$$

and we can complete the proof in exactly the same way as above.

We proceed to consider the cases where we must show that the trivial solution is unique. Assume that x is a nontrivial continuous solution of equation (1). We may, without loss of generality, assume that x is nondecreasing and that $x(t) > 0$ for $t > 0$. This implies that

$$x(t) \leq \int_0^t k(s)\,ds\,g(x(t)), \quad t > 0.$$

If we assume that g is such that the function $s \mapsto s/g(s)$ is strictly increasing on $\mathbf{R}$ and has an inverse q, then we get

$$x(t) \le q\left(\int_0^t k(s)\,ds\right), \quad t \ge 0.$$

Let us choose a sequence $\{a_n\}$, for example, as follows:

$$a_{n+1} = \frac{2a_n}{2+a_n}, \quad a_0 = 1,$$

and then inductively prove that

$$x(t) \le q\left(\int_0^{a_j t} k(s)\,ds\right), \quad 0 \le t \le 1, \quad j \ge 0. \tag{4}$$

If we can do this, then it follows that $x(t) \equiv 0$.

Since x is assumed to be nondecreasing, it follows from (1) that

$$x(t) \le \int_{a_{n+1}t}^t k(s)\,ds\, g(x(t - a_{n+1}t)) + \int_0^{a_{n+1}t} k(s)\,ds\, g(x(t)), \quad 0 \le t \le 1.$$

If we use the induction hypotheses (4) with $j = n-1$ in the first term on the right-hand side, then we get

(5)
$$x(t) - \int_0^{a_n t} k(s)\,ds\, g(x(t)) \le \int_{a_{n+1}t}^t k(s)\,ds\, g\biggl(q\biggl(\int_0^{a_{n+1}t} k(s)\,ds\biggr)\biggr) - \int_{a_{n+1}t}^{a_n t} k(s)\,ds\, g(x(t)), \quad 0 \le t \le 1,$$

where we used the fact that $(1 - a_{n+1})a_{n-1} = a_{n+1}$. Suppose for the moment that

$$g\biggl(q\biggl(\int_0^{a_{n+1}t} k(s)\,ds\biggr)\biggr) \le \frac{\int_{a_{n+1}t}^{a_n t} k(s)\,ds}{\int_{a_{n+1}t}^{t} k(s)\,ds}\, g\biggl(q\biggl(\int_0^{a_n t} k(s)\,ds\biggr)\biggr), \quad 0 \le t \le 1, \quad n \ge 1. \tag{6}$$

Then it follows from (5) that either $x(t) \le \int_0^{a_n t} k(s)\,ds\,g(x(t))$, in which case the induction claim is immediate, or

$$\int_{a_{n+1}t}^{t} k(s)\,ds\,g\left(q\left(\int_0^{a_{n+1}t} k(s)\,ds\right)\right) \ge \int_{a_{n+1}t}^{a_n t} k(s)\,ds\,g(x(t)),$$

in which case the claim follows from (6) provided we assume that g is strictly increasing and k is strictly positive. Thus we see that the critical point is that (6) holds.

Suppose that g is given. We may, without loss of generality, assume that g and the function $s \mapsto s/g(s)$ are strictly increasing, continuously differentiable functions on $(0, \infty)$. Denote the function $g \circ q$ by p. It follows that there exists a strictly positive, continuously differentiable function B on the set $\{(v, w) \,|\, 0 < v < w \le 1\}$ such that

$$p(u) \le \frac{v-u}{w-u}\,p(v) \quad \text{if and only if } u \le B(v, w).$$

Let $K(t) = \int_0^t k(s)\,ds$. We see that we can rewrite (6) as

$$p(K(a_{n+1}t)) \le \frac{K(a_n t) - K(a_{n+1}t)}{K(t) - K(a_{n+1}t)}\,p(K(a_n t)), \tag{7}$$

for $0 \le t \le 1$ and $n \ge 1$.

Let $K(t) = t$ for $a_1 \le t \le 1$. Suppose that we have already constructed K on the interval $[a_m, 1]$ so that K is strictly increasing, positive, and absolutely continuous, and such that the inequality (7) holds for $a_m/a_{n+1} \le t \le 1$ and $1 \le n \le m-1$. For $m = 1$, we have constructed K with the desired properties.

Let

$$b(t) = \frac{K(a_m)(t - a_{m+1}) + B(K(a_m), 1)(a_m - t)}{a_m - a_{m+1}}.$$

Now we can extend the function K to the interval $[a_{m+1}, a_m)$ such that it remains positive, strictly increasing, and absolutely continuous, and such that

$$K(t) \le \min_{1 \le n \le m-1} \left\{ b(t), B\left(K\left(\frac{a_n t}{a_{n+1}}\right), K\left(\frac{t}{a_{n+1}}\right)\right)\right\}, \quad a_{m+1} \le t \le a_m.$$

It is easy to check that this construction can always be performed by an iterative procedure. Thus we see that we can construct an increasing function K such that $K(0) = 0$ and (7) holds for all $0 \leq t \leq 1$ and $n \geq 1$. This completes the proof.

Next let us assume that k is given. We may, without loss of generality, assume that $k(t) \geq 1, 0 \leq t \leq 1$. Again we see that it suffices to construct g such that the inequalities (7) hold for all $0 \leq t \leq 1$ and $n \geq 1$. But then it suffices to construct the function p such that it is nonnegative and strictly increasing on $[0,1]$ and such that

$$p(u) \leq \frac{K\Big(\frac{a_n}{a_{n+1}}u\Big) - K(u)}{K\Big(\frac{u}{a_{n+1}}\Big) - K(u)}\, p\left(K\left(\frac{a_n}{a_{n+1}}u\right)\right), \\ a_{m+1} \leq u \leq a_m, \quad 1 \leq n \leq m-1,$$

for each $m \geq 2$. But it is clear that this can be done inductively, and, by letting $1/h$ be the inverse of the function $s \mapsto sp(s)$, we find a function g with the desired properties. This completes the proof. □

3. Proof of Theorem 4. As we already noted above, the only part of the theorem that we have to prove is that if $0 < \alpha < 1$ and (3) holds, then the trivial solution of equation (2) is unique. We need the fact that not only is this result true if $\alpha \geq 1$, but in this case it suffices to assume that g is nondecreasing. This follows from [**2**] combined with Proposition 2.

If there exists a nontrivial solution of (2), then we may, without loss of generality, assume that x is nondecreasing and $x(t) > 0$ for $t > 0$. Thus the function $g(x(t))$ is nondecreasing as well and we have $g(x(t)) = \mu([0,t]), t > 0$, where μ is some continuous, nonnegative Borel measure. It follows that if we perform an integration by parts on the right-hand side in (2), then we get

$$x(t) = \int_{[0,t]} \frac{1}{\alpha}(t-s)^{\alpha}\mu(ds), \quad t > 0.$$

We apply Hölder's inequality to the right-hand side and obtain

$$x(t) \leq \frac{1}{\alpha}\left(\int_{[0,t]} (t-s)^{\alpha+1}\mu(ds)\right)^{\frac{\alpha}{\alpha+1}} \left(\int_{[0,t]} \mu(ds)\right)^{\frac{1}{\alpha+1}}, \quad t > 0.$$

If we invoke the definition of μ and perform an additional integration by parts, then

$$\frac{x(t)}{h(x(t))^{\frac{1}{\alpha}}} \le c \int_0^t (t-s)^\alpha g(x(s))\,ds, \quad t \ge 0, \tag{8}$$

where $c = (\alpha+1)\alpha^{-(\alpha+1)/\alpha}$. Since h is nonincreasing, it follows that the function $s \mapsto s/h(s)^{1/\alpha}$ is strictly increasing, and, therefore, this function has an increasing and continuous inverse function v. If we let $g_* = g \circ v$ and define y by $y(t) = x(t)/h(x(t))^{1/\alpha}$, then inequality (8) becomes

$$y(t) \le c \int_0^t (t-s)^{(\alpha+1)-1} g_*(y(s))\,ds, \quad t \ge 0.$$

Now $\alpha + 1 > 1$ and g_* is nondecreasing, and, therefore, it suffices to check that

$$\int_0^t \frac{1}{s(g_*(s)/s)^{\frac{1}{\alpha+1}}}\,ds = \infty.$$

But this result follows after a change of integration variable from (3) since the function h is nondecreasing. Thus we know from Lemma 1 that $y(t) \equiv 0$ and, hence, we have $x(t) \equiv 0$ as well. This completes the proof. □

Note added in proof: Assumption (ii) of Theorem 4 can be removed, see [W. Mydlarczyk, "The existence of nontrivial solutions of Volterra equations," Math. Scand., to appear]. See also [W. Okrasiński, "Nontrivial solutions to nonlinear Volterra integral equations," SIAM J. Math. Anal., to appear].

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