JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 19, Number 1, Spring 2007

MAXIMAL REGULARITY AND GLOBAL WELL-POSEDNESS FOR A PHASE FIELD SYSTEM WITH MEMORY

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ABSTRACT. In this paper we obtain global strong well-posedness for a phase field system with memory and relaxing chemical potential in an L_p-setting, employing maximal regularity tools. The global well-posedness result is obtained by an energy estimate, provided that the space dimension n is less than 3.

1. Introduction. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and let $J = [0,T]$, $T > 0$, be an interval. We consider the following system

$$(PFM)\begin{cases} u_t + \dfrac{l}{2}\,\phi_t = \displaystyle\int_{-\infty}^{t} a_1(t-s)\Delta u(s)\,ds & \text{in } J\times\Omega;\\ \tau\phi_t = \displaystyle\int_{-\infty}^{t} a_2(t-s)\Big[\xi^2\Delta\phi + \frac{\phi-\phi^3}{\eta} + u\Big](s)\,ds & \text{in } J\times\Omega;\\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0 & \text{on } J\times\partial\Omega;\\ u(0,x) = u_0(x),\ \ \phi(0,x) = \phi_0(x) & \text{in } \Omega. \end{cases}$$

The phase field system with memory (PFM) was first proposed in [**9**] as a phenomenological model to describe phase transitions in the presence of a slowly relaxing internal variable. Later Novick-Cohen [**6**] obtained a global weak solution of (PFM), by means of the Galerkin method and energy estimates.

Our goal here is to obtain global well-posedness of (PFM) in the strong sense in an L_p-setting. Assuming enough regularity of the kernels, we may apply a recent result in the theory of Volterra equations,

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2000 AMS *Mathematics Subject Classification.* Primary 47D06, Secondary 47A50.
Key words and phrases. Maximal regularity, integro-differential equations, phase field system with memory.
This is part of the author's Ph.D. Thesis, which was supervised by Jan Prüss at the University of Halle.
The research of the author was supported by a grant from the Deutscher Akademischer Austausch Dienst (DAAD).
Received by the editors on Feb. 6, 2005, and in revised form on Dec. 21, 2005.

which was proved in [**11**], to obtain a strong local solution in the framework of Bessel potential spaces. To solve (PFM), we first show the equivalence of it to a semi-linear problem of Volterra type of the form

$$v(t) = \int_0^t b(t-s)\Delta v(s)\,ds + H(v(t)) + f(t), \quad t \in J. \tag{1.1}$$

This fact allows us to prove the local well-posedness of (PFM). Concerning the global well-posedness of (PFM), let us make some considerations. It may seem problematic to consider the current state of the system as dependent on the entire history. To get rid of this problem we consider the model

$$u_t + \frac{l}{2}\,\phi_t = \int_0^t a_1(t-s)\Delta u(s)\,ds + f_1, \quad \text{in} \quad J \times \Omega; \tag{1.2}$$

(1.3)

$$\tau\phi_t = \int_0^t a_2(t-s)[\xi^2\Delta\phi + \frac{\phi-\phi^3}{\eta} + u](s)\,ds + f_2, \quad \text{in} \quad J\times\Omega;$$
$$\mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0, \quad \text{on} \quad J\times\partial\Omega;$$
$$u(0,x) = u_0(x),\ \phi(0,x) = \phi_0(x), \quad \text{in} \quad \Omega,$$

where

$$f_1(t,x) = \int_{-\infty}^0 a_1(t-s)\Delta u(s,x)\,ds, \qquad (t,x)\in J\times\Omega;$$
$$f_2(t,x) = \int_{-\infty}^0 a_2(t-s)\left[\xi^2\Delta\phi + \frac{\phi-\phi^3}{\eta} + u\right](s,x)\,ds, \quad (t,x)\in J\times\Omega$$

are known. According to [**7**, p. XV], we may set $f_1(t,\cdot) = f_2(t,\cdot) = u(t,\cdot) = \phi(t,\cdot) = 0$ whenever $t < 0$. This fact allows us to obtain a-priori estimates, which yield global strong well-posedness of (PFM).

This paper is organized as follows. In Section 2, we consider a linear equation of Volterra type and state a recent result obtained by Zacher [**11**], which is the key to obtain existence and uniqueness of a local solution for (PFM) in the framework of Bessel potential spaces. In Section 3, we establish the equivalence of (PFM) to a semi-linear equation of Volterra type as mentioned before and prove its local well-posedness, by employing maximal L_p-regularity and the contraction mapping principle. Finally, in Section 4, assuming standard conditions of positivity on a_1 and a_2 we obtain *a priori* estimates for u and ϕ, in the case where

$$f_1(t,x) = f_2(t,x) = 0, \quad (t,x) \in (-\infty,0) \times \Omega,$$

which, together with the Gagliardo-Nirenberg inequality, lead to the global existence result (Theorem 4.2) provided $n \leq 3$.

2. Volterra equation. In this section, we would like to present a recent result in the theory of Volterra integral equation obtained by Zacher [**11**, Theorem 3.4], which will be the key to prove local well-posedness. To cite this theorem, as well as other important auxiliary results, it is necessary to recall the definition of sectorial operators, and some of its sub-classes. Furthermore, we will describe the main assumptions on the kernels needed to obtain local and global well-posedness.

Let X be a Banach space, A a closed linear operator in X with dense domain $D(A)$, and $a \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$ a scalar kernel. We consider the Volterra equation

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = f(t), \quad t \geq 0. \tag{2.1}$$

Well-posedness of this problem has been obtained in several important cases, as a general reference we refer to the monograph Prüss [**7**]. Let us begin our study of (2.1) with some considerations on the kernels.

In the sequel we denote by $\hat{f}$ and $\tilde{f}$ the Laplace transform and the Fourier transform of a function f, respectively. The symbol $*$ means the convolution of two functions supported on the half line, i.e., $(a * b)(t) = \int_0^t a(t-s)b(s)\,ds$.

Definition 2.1. Let $a \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$ be of subexponential growth, and suppose $\hat{a}(\lambda) \neq 0$ for all $\mathrm{Re}\,\lambda > 0$. The variable a is called *sectorial* with angle $\theta > 0$ (or merely θ-sectorial) if

$$|\arg \hat{a}(\lambda)| \leq \theta$$

for all $\mathrm{Re}\,\lambda > 0$.

Definition 2.2. Let $a \in L_{1,\text{loc}}(\mathbf{R}_+)$ be of subexponential growth and $k \in \mathbf{N}$. $a(t)$ is called *k-regular*, if there is a constant $c > 0$ such that

$$|\lambda^n \hat{a}^{(n)}(\lambda)| \le c|\hat{a}(\lambda)|$$

for all $\operatorname{Re}\lambda > 0$, and $1 \le n \le k$.

The subsequent class of kernels, which was introduced and used by Zacher [**11**], is appropriate to obtain maximal regularity for the abstract parabolic Volterra equation (2.1) in vector-valued Bessel potential spaces $H_p^\alpha(J;X)$.

Definition 2.3. Let $a \in L_{1,loc}(\mathbf{R}_+)$ be of subexponential growth, and assume $r \in \mathbf{N}$, $\theta_a > 0$, and $\alpha \ge 0$. Then a is said to belong to the class $\mathcal{K}^r(\alpha, \theta_a)$ if

($\mathcal{K}1$) a is r-regular;

($\mathcal{K}2$) a is θ_a-sectorial;

($\mathcal{K}3$) $\limsup_{\mu\to\infty} |\hat{a}(\mu)|\mu^\alpha < \infty$, $\liminf_{\mu\to\infty} |\hat{a}(\mu)|\mu^\alpha > 0$, $\liminf_{\mu\to 0} |\hat{a}(\mu)| > 0$.

Further, $\mathcal{K}^\infty(\alpha, \theta_a) := \{a \in L_{1,\text{loc}}(\mathbf{R}_+) : a \in \mathcal{K}^r(\alpha, \theta_a) \text{ for all } r \in N\}$. The kernel a is called a $\mathcal{K}$-kernel if there exist $r \in \mathbf{N}$, $\theta_a > 0$, and $\alpha \ge 0$, such that $a \in \mathcal{K}^r(\alpha, \theta_a)$.

A typical example of a $\mathcal{K}$-kernel is given by

$$a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\, e^{-\eta t}, \quad t > 0,$$

which belongs to the class $\mathcal{K}^\infty(\alpha, \alpha(\pi/2))$ for every $\alpha > 0$ and $\eta \ge 0$. The $\mathcal{K}$-kernels will be our main assumption in order to obtain local well-posedness. For global well-posedness we will additionally assume the following conditions of positivity on the kernels a_1 and a_2.

(P1) $a_1 \in L_{1,\text{loc}}(\mathbf{R}_+)$, such that

$$\operatorname{Re}\int_0^T a_1 * \psi(t)\overline{\psi(t)}\, dt \ge 0 \quad \text{for all} \quad \psi \in L_2((0,T);\mathbf{C}), \quad \text{and} \quad T > 0.$$

(P2) $a_2 \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$, and there exists $\nu \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$ nonnegative, nonincreasing and such that

$$\int_0^t a_2(t-s)\nu(s)\,ds = 1, \quad \text{for all} \quad t > 0.$$

Observe that condition (P1) means that a_1 is of positive type, while (P2) is a special case of the definition of completely positive type. For this and important properties of all of this type of kernels, we refer to the monograph Prüss [**7**].

Next, we recall some classes of operators.

Definition 2.4. Let X be a complex Banach space, and let A be a closed linear operator in X. We say that A is *sectorial* if $\overline{D(A)} = X$, $\overline{R(A)} = X$, $N(A) = \{0\}$, $(-\infty, 0) \subset \rho(A)$ and

$$|t(t+A)^{-1}| \le M \quad \text{for all} \quad t > 0, \quad \text{and some} \quad M < \infty.$$

We denote the class of sectorial operators in X by $\mathcal{S}(X)$.

It follows from the definition of sectoriality that it makes sense to define the *spectral angle* ϕ_A of $A \in S(X)$ by

$$\phi_A = \inf\Big\{\phi : \ \rho(-A) \supset \Sigma_{\pi-\phi}, \ \sup_{\lambda\in\Sigma_{\pi-\phi}} |\lambda(\lambda+A)^{-1}| < \infty\Big\},$$

where Σ_θ for $\theta \in (0,\pi]$ is defined as the open subset of $\mathbf{C}$ with vertex 0 and opening angle 2θ which is symmetric with respect to the positive half axis $\mathbf{R}_+$.

Remark 2.5. If $A \in S(X)$ with spectral angle $\phi_A < \pi$ and the kernel a is 1-regular and θ-sectorial with $\theta < \pi$, such that the condition of parabolicity $\theta + \phi_A < \pi$ holds, then (2.1) admits a resolvent operator $S \in C((0,+\infty); B(X))$, which is also uniformly bounded in $\mathbf{R}_+$. This follows directly from [**7**, Proposition 3.1 and Theorem 3.1].

A sectorial operator A in X is said to admit *bounded imaginary powers*, if $A^{is} \in \mathcal{B}(X)$ for each $s \in \mathbf{R}$ and there is a constant $C > 0$

such that $|A^{is}| \le C$ for $|s| \le 1$. The class of such operators will be denoted by $\mathcal{BIP}(X)$, and we will call

$$\theta_A = \overline{\lim}_{|s|\to\infty} \frac{1}{|s|} \log |A^{is}|$$

the *power angle* of A. Let Y be another complex Banach space. We recall that a family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called **R**-*bounded*, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbf{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric $\{-1,1\}$-valued random variables ε_j on a probability space $(\Sigma, \mathcal{M}, \mu)$ the inequality

$$\Big| \sum_{j=1}^{N} \varepsilon_j T_j x_j \Big|_{L_p(\Sigma;Y)} \le C \Big| \sum_{j=1}^{N} \varepsilon_j x_j \Big|_{L_p(\Sigma;X)}$$

is valid. The smallest such C is called the $\mathcal{R}$-*bound* of $\mathcal{T}$, we denote it by $\mathcal{R}(\mathcal{T})$. The concept of $\mathcal{R}$-bounded families of operators leads to the notion of $\mathcal{R}$-*sectorial* operators, replacing *bounded* with $\mathcal{R}$-bounded in the definition of sectorial operators.

Definition 2.6. Let X be a complex Banach space, and assume that A is a sectorial operator in X. Then A is called $\mathcal{R}$-*sectorial* if

$$\mathcal{R}_A(0) := \mathcal{R}\{t(t+A)^{-1} : t > 0\} < \infty.$$

The $\mathcal{R}$-*angle* $\phi_A^{\mathcal{R}}$ of A is defined by means of

$$\phi_A^{\mathcal{R}} = \inf\{\theta \in (0,\pi) : \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}\{\lambda(\lambda + A)^{-1} : |\arg \lambda| \le \theta\}.$$

The class of $\mathcal{R}$-sectorial operators will be denoted by $\mathcal{RS}(X)$. The class of operators that admit bounded imaginary powers was introduced by Prüss and Sohr in [**8**]. The class of $\mathcal{R}$-sectorial operators goes back to Clément and Prüss [**3**], where the inclusions

$$\mathcal{BIP}(X) \subset \mathcal{RS}(X) \subset \mathcal{S}(X),$$

and the inequality

$$\phi_A^{\mathcal{R}} \le \theta_A \tag{2.2}$$

were obtained, in the special case, when the space X is such that the Hilbert transform defined by

$$(Hf)(t) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon \le |s| \le 1/\varepsilon} f(t-s)\, \frac{ds}{s}, \quad t \in \mathbf{R},$$

is bounded in $L_p(\mathbf{R}; X)$ for some $p \in (1,\infty)$. The class of spaces with this property will be denoted by $\mathcal{HT}$.

There is a well-known theorem which says that the set of Banach spaces of class $\mathcal{HT}$ coincides with the class of UMD spaces, where UMD stands for *unconditional martingale difference* property. It is further known that $\mathcal{HT}$-spaces are reflexive. Every Hilbert space belongs to the class $\mathcal{HT}$, and if $(\Sigma, \mathcal{M}, \mu)$ is a measure space, $1 < p < \infty$ and $X \in \mathcal{HT}$, then $L_p(\Sigma, \mathcal{M}, \mu; X)$ is an $\mathcal{HT}$-space. For all these results, see the survey article by Burkholder [**1**]. For a detailed study of the mentioned topics, see for instance, [**5**] and also [**4**].

The following result will be of importance below.

Theorem 2.7 (Prüss [**7**, Theorem 8.6]). *Suppose X belongs to the class $\mathcal{HT}$, $p \in (1,\infty)$, and let $a \in L_{1,\text{loc}}(\mathbf{R}_+)$ be of subexponential growth. Assume that a is 1-regular and θ-sectorial, where $\theta < \pi$. Then there is a unique operator $B \in \mathcal{S}(L_p(\mathbf{R}; X))$ such that*

$$(Bf)^{\sim}(\rho) = \frac{1}{\hat{a}(i\rho)}\, \tilde{f}(\rho), \quad \rho \in \mathbf{R}, \quad \tilde{f} \in C_0^\infty(\mathbf{R} \setminus \{0\}; X). \tag{2.3}$$

Moreover, B has the following properties:

(i) *B commutes with the group of translations*;

(ii) *$(\mu + B)^{-1} L_p(\mathbf{R}_+; X) \subset L_p(\mathbf{R}_+; X)$ for each $\mu > 0$, i.e., B is causal*;

(iii) *$B \in \mathcal{BIP}(L_p(\mathbf{R}; X))$, and power angle $\theta_B = \theta_a$, where*

$$\theta_a = \sup\{|\arg \hat{a}(\lambda)| : \operatorname{Re} \lambda > 0\};$$

(iv) $\sigma(B) = \overline{\{1/\hat{a}(i\rho) : \rho \in \mathbf{R} \setminus \{0\}\}}$.

Corollary 2.8 (Prüss [**7**, Corollary 8.1]). *Let the assumptions of Theorem* 2.7 *hold, let B be defined by* (2.3), *and let $\alpha, \beta \geq 0$. Then*

(i) $\overline{\lim}_{\mu\to\infty} |\hat{a}(\mu)|\mu^\alpha < \infty$ *implies* $D(B) \hookrightarrow H_p^\alpha(\mathbf{R}; X)$,

(ii) $\underline{\lim}_{\mu\to\infty} |\hat{a}(\mu)|\mu^\beta > 0$ *and* $\underline{\lim}_{\mu\to 0} |\hat{a}(\mu)| > 0$ *imply* $H_p^\beta(\mathbf{R}; X) \hookrightarrow D(B)$.

The concept of $\mathcal{K}$-kernels is very useful when working with Bessel potential spaces, since it connects the order of the kernels with the order of Bessel potential spaces. The following result due to Zacher [**11**] expresses this fact.

Corollary 2.9. *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, and $J = [0, T]$ or $J = \mathbf{R}_+$. Suppose $a \in \mathcal{K}^1(\alpha, \theta)$ with $\theta < \pi$, and assume in addition $a \in L_1(\mathbf{R}_+)$ in the case $J = \mathbf{R}_+$. Then the restriction $\mathcal{B} := B|_{L_p(J;X)}$ of the operator B constructed in Theorem* 2.7 *to $L_p(J; X)$ is well-defined. The operator $\mathcal{B}$ belongs to the class $\mathcal{BIP}(L_p(J; X))$ with the power angle $\theta_{\mathcal{B}} \leq \theta_B = \theta_a$ and is invertible satisfying $\mathcal{B}^{-1}w = a * w$ for all $w \in L_p(J; X)$. Moreover, $D(\mathcal{B}) = {}_0H_p^\alpha(J; X)$.*

Next, we will state an important result obtained by Zacher [**11**], which gives necessary and sufficient conditions for the existence of a unique solution u of (2.1) in the space

$$H_p^{\alpha+\kappa}(J; X) \cap H_p^\kappa(J; D_A).$$

Here D_A denotes the domain of A equipped with the graph norm.

Theorem 2.10 (Zacher [**11**]). *Let X be a Banach space of class $\mathcal{HT}$, $p \in (1, \infty)$, $J = [0, T]$ or $\mathbf{R}_+$, and let A be an $\mathcal{R}$-sectorial operator in X with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}}$. Suppose that a belongs to $\mathcal{K}^1(\alpha, \theta_a)$ with $\alpha \in (0, 2)$ and that, in addition, $a \in L_1(\mathbf{R}_+)$ in the case $J = \mathbf{R}_+$. Further, let $\kappa \in [0, 1/p)$ and $\alpha + \kappa \notin \{1/p, 1 + 1/p\}$. Assume the parabolicity condition $\theta_a + \phi_A^{\mathcal{R}} < \pi$. Then* (2.1) *has a unique solution in $H_p^{\alpha+\kappa}(J; X) \cap H_p^\kappa(J; D_A)$ if and only if the function f satisfies the subsequent conditions*:

(i) $f \in H_p^{\alpha+\kappa}(J;X)$;

(ii) $f(0) \in D_A(1+(\kappa/\alpha)-(1/p\alpha),p)$, *if* $\alpha+\kappa > 1/p$;

(iii) $\dot f(0) \in D_A(1+(\kappa/\alpha)-(1/\alpha)-(1/p\alpha),p)$, *if* $\alpha+\kappa > 1+1/p$.

Here $D_A(\gamma,p)$ denotes the real interpolation space $(X,D_A)_{\gamma,p}$, for $\gamma \in (0,1)$.

The following result is due to Clément and Prüss [**2**]. It will play an important role in order to obtain *a priori* estimates.

Theorem 2.11. *Let X be a Banach space, $1 \le p < \infty$, $\nu \in L_{1,\mathrm{loc}}(\mathbf{R}_+)$ nonnegative, nonincreasing, and let B_p be defined in $L_p(\mathbf{R}_+;X)$ by*

$$(B_p u)(t) = \frac{d}{dt}\,\nu * u(t), \quad t \ge 0, \quad u \in D(B_p),$$

with domain

$$D(B_p) = \{u \in L_p(\mathbf{R}_+;X) : \nu * u \in {}_0W_p^1(\mathbf{R}_+;X)\}.$$

Then B_p is m-accretive. In particular, if $X = H$ is a Hilbert space, then

$$\int_0^T \langle B_p u(t),\, u(t)\rangle\, |u|_H^{p-2}\, dt \ge 0, \quad T > 0,$$

for each $u \in D(B_p)$.

Remark 2.12. Let $a \in \mathcal{K}^1(\alpha,\theta)$ with $\theta < \pi$. Let B be the operator from Corollary 2.9 associated with a, and assume that the condition (P2) is valid with a in place of a_2. Then, from Theorem 2.11, it follows that $(Bv)(t) = (B_p v)(t) = (d/dt)\nu * v(t)$, for each $v \in D(B) \cap D(B_p)$. In particular, for $p = 2$ and $D(B) = {}_0H_2^\alpha(J, L_2(\Omega))$, it follows that

$$\int_0^T \langle Bv, v\rangle\, dt = \int_0^T \Big\langle \frac{d}{dt}\,\nu * v, v\Big\rangle\, dt \ge 0.$$

3. Local well-posedness. This section is devoted to the local well-posedness of (PFM); for this we will reduce the system (PFM) to a semi-linear equation of Volterra type, such that local well-posedness of it allows us to obtain the same properties in (PFM). Our strategy to solve this semi-linear equation will divide in two parts. Firstly, we solve the linear version of it using maximal regularity tools (Theorem 2.10), and secondly we apply the contraction principle to overcome the nonlinearities. We would like to begin with some definitions.

3.1 Preliminaries. Let $T > 0$ be given and fixed, and let Ω be a smooth bounded domain in $\mathbf{R}^n$. For $0 < \delta \leq T$ and $1 < p < \infty$, we define the spaces

$$\begin{aligned} Z(\delta) &= H_p^{\alpha+\kappa}([0,\delta];X) \cap H_p^{\kappa}([0,\delta];D_A); \\ Z_i(\delta) &= H_p^{1+\alpha_i+\kappa_i}([0,\delta];X) \cap H_p^{\kappa_i}([0,\delta];D_A); \\ \widetilde{X}_i(\delta) &= H_p^{\alpha_i+\kappa_i}([0,\delta];X); \\ X_i(\delta) &= H_p^{1+\alpha_i+\kappa_i}([0,\delta];X), \end{aligned}$$

for $i = 1,2$, where $\alpha,\ \alpha_i > 0$, and $\kappa,\ \kappa_i \geq 0$, and $X := L_p(\Omega)$, and A is a closed linear operator in X with dense domain $D(A)$. The spaces ${}_0Z(\delta)$ and ${}_0Z_i(\delta)$ denote the corresponding spaces $Z(\delta)$ and $Z_i(\delta)$, respectively, with zero trace at $t = 0$. A similar definition holds for ${}_0\tilde{X}_i(\delta)$ and ${}_0X_i(\delta)$. Whenever no confusion may arise, we shall simply write Z, Z_i, etc., respectively ${}_0Z$, ${}_0Z_i$, etc., if $\delta = T$. Furthermore, in case that $\kappa_i \in [0, 1/p)$ and $\alpha_i + \kappa_i \neq 1/p$, we define the natural phase spaces for Z_i by

$$Y_p^i = (X; D_A)_{\gamma_i,p}, \quad \text{with} \quad \gamma_i = 1 + \frac{\kappa_i}{1+\alpha_i} - \frac{1}{p(1+\alpha_i)}, \quad \text{for} \quad i = 1,2;$$

$$\widetilde{Y}_p^i = (X; D_A)_{\varsigma_i,p}, \quad \text{with} \quad \varsigma_i = 1 + \frac{\kappa_i}{1+\alpha_i} - \frac{1}{1+\alpha_i} - \frac{1}{p(1+\alpha_i)}, \quad \text{for} \quad i = 1,2.$$

Next, we would like to recall that, for $1 < p < \infty$ and $n \in \mathbf{N}$, the Bessel potential spaces may be defined as interpolation spaces between the well-known Sobolev spaces W_p^n and L_p, by means of the so-called complex interpolation, i.e.,

$$H_p^{sn} = [L_p, W_p^n]_s, \quad \text{for} \quad s \in (0,1).$$

We may also set $H_p^{sn} = L_p$ if $s = 0$, and $H_p^{sn} = W_p^n$ if $s = 1$. For a general reference concerning these topics, see e.g., [**10**].

Let $J = [0, T]$ be an interval on $\mathbf{R}$. We consider the system

$$u_t + \frac{l}{2}\,\phi_t = a_1 * \Delta u + f_1, \quad \text{in} \quad J \times \Omega; \tag{3.1}$$

$$\begin{gathered} \tau\phi_t = \xi^2 a_2 * \Delta\phi + \frac{1}{\eta}\,a_2 * (\phi - \phi^3) + a_2 * u + f_2, \quad \text{in} \quad J \times \Omega; \\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0, \quad \text{on} \quad J \times \partial\Omega; \\ u(0,x) = u_0(x),\ \phi(0,x) = \phi_0(x), \quad \text{in } \Omega, \end{gathered} \tag{3.2}$$

where f_1 and f_2 are given by

$$f_1(t,x) = \int_{-\infty}^0 a_1(t-s)\Delta u(s,x)\,ds, \qquad (t,x) \in J \times \Omega;$$

$$f_2(t,x) = \int_{-\infty}^0 a_2(t-s)\Big[\xi^2\Delta\phi + \frac{\phi - \phi^3}{\eta} + u\Big](s,x)\,ds, \quad (t,x) \in J \times \Omega.$$

For the discussion of equations (3.1)–(3.2), we will assume that without loss of generality all constants are equal to one. Furthermore, we will also assume that the kernels a_i belong to $\mathcal{K}^1(\alpha_i, \theta_i)$, with $\theta_i \in (0, (\pi/2))$ and $\alpha_i \in (0,1)$ for $i = 1, 2$, and we will set $A = -\Delta$ with Neumann boundary conditions.

If we consider ϕ as known, then equation (3.1) is equivalent to the two problems

$$\text{(I)} \qquad \begin{cases} u_t^* = -\,a_1 * Au^* + f_1 & \text{in } J \times \Omega; \\ u^*(0) = u_0 & \text{in } \Omega, \end{cases}$$

and

$$\text{(II)} \qquad \begin{cases} w_t = -\,a_1 * Aw - (l/2)\phi_t & \text{in } J \times \Omega; \\ w(0) = 0 & \text{in } \Omega, \end{cases}$$

by means of the relation $u = u^* + w$. Observe that Theorem 2.10 gives necessary and sufficient conditions to obtain a strong solution of (I) and also for (II). Indeed, integrating equation (I) over $[0, t]$, we have

$$u^* = -\,1 * a_1 * Au^* + 1 * f_1 + u_0.$$

It is easy to show that $a := 1 * a_1$ is a kernel that belongs to the class $\mathcal{K}^1(1+\alpha_1, \theta_1 + (\pi/2))$. On the other hand, it is well-known that $A = -\Delta$ with Dirichlet- or Neumann- or Robin-boundary conditions belongs to the class $\mathcal{BIP}(X)$ with power angle $\theta_A = 0$. Moreover, from [**3**] it follows that $A \in \mathcal{RS}(X)$, too, with $\mathcal{R}$-angle $\phi_A^{\mathcal{R}} = 0$. Hence, (I) transforms into the equation (2.1), with $f = 1 * f_1 + u_0$. Therefore, we may apply Theorem 2.10. A similar argument holds for (II).

Now we want to have a representation formula of the mild solution of (II). For this, we take $f = -1 * \phi_t$ and $a = 1 * a_1$ in (2.1). On the other hand, since $A \in S(X)$ and spectral angle $\phi_A = 0$, it follows from Remark 2.5 that (2.1) admits a resolvent operator S. Using this fact and the variation of parameters formula, it follows that the mild solution w of equation (II) can be represented as

$$w = \frac{d}{dt}(-S * 1 * \phi_t) = -S * \phi_t. \tag{3.3}$$

Now substituting $u = u^* + w$ in (3.2) and using (3.3) it follows that

$$\phi_t = -a_2 * A\phi + a_2 * (\phi - \phi^3) + a_2 * u^* - a_2 * S * \phi_t + f_2, \quad \text{in} \quad J \times \Omega. \tag{3.4}$$

Defining

$$g(t) = 1 * a_2 * u^* + 1 * f_2 + \phi_0 \quad \text{and} \quad H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t,$$

then (3.4) can be rewritten as

$$\phi = -1 * a_2 * A\phi + H(\phi) + g(t). \tag{3.5}$$

Now we will establish the equivalence between system (3.1)–(3.2) and equation (3.5). To do so, we will first assume that the functions in (3.1)–(3.2) and (3.5) enjoy enough regularity (later, we will make this aspect precise).

We begin assuming that u^* as well as ϕ are known in (I) and (3.5), respectively. Using ϕ in equation (II) we obtain a function w, and by defining a new function $u = u^* + w$, one can show (after an easy computation) that the pair (u, ϕ) is a solution of (3.1)–(3.2). The converse direction is trivial.

We will make precise now the type of regularity which we will give to the solutions.

The regularity that one can await of the solution (u, ϕ) of (3.1)–(3.2) is delivered by Theorem 2.10; therefore, we can assume that (u, ϕ) belongs to $Z_1 \times Z_2$. On the other hand, by applying the contraction mapping principle, we see that the solution ϕ of (3.5) belongs to Z_2, if and only if $H(\phi)+g(t) \in X_2$. From Corollary 2.9 we have that, for each function $u^* \in L_p(J;X)$ (in particular in Z_1), the function $1*a_2*u^*$ is in ${}_0X_2$, hence $g \in X_2$, provided that $u^* \in L_p(J;X)$ and $1 * f_2 + \phi_0 \in X_2$.

From equation (II) and Theorem 2.10, it follows that the solution w of (II) belongs to ${}_0Z_2$. Since $u = u^* + w$ is a solution of (3.1), we have $u \in Z_1$. On the other hand, since $u^* \in Z_1$ and $w \in Z_2$, we have to impose a compatibility condition between the spaces Z_1 and Z_2. In fact, the Sobolev embedding $Z_2 \hookrightarrow Z_1$ is an admissible condition, which is equivalent to

$$\alpha_2 - \alpha_1 \geq \kappa_1 - \kappa_2 \quad \text{and} \quad \kappa_2 \geq \kappa_1. \tag{3.6}$$

The following auxiliary results are needed to estimate the nonlinear term $H(\phi)$ of equation (3.5) in X_2. For this purpose we begin with an estimate of product of functions in Bessel potential spaces.

Lemma 3.1. *Let $0 \leq \kappa < 1$, $\alpha > 0$, $n \in \mathbf{N}$. Suppose that $p > (n/3) + (2/3\alpha)$. Then there is a constant $C > 0$ and an $\varepsilon > 0$ such that*

$$|u\,v\,w|_{H_p^{\kappa+\varepsilon}(L_p)} \leq C|u|_Z\,|v|_Z\,|w|_Z \tag{3.7}$$

is valid for every $u, v, w \in Z$.

Proof. Let $\rho_i > 1$ for $i = 1, \dots, 4$, such that

$$1 = \frac{1}{\rho_1} + \frac{2}{\rho_3} = \frac{1}{\rho_2} + \frac{2}{\rho_4},$$

in particular ρ_3 and ρ_4 must be greater than 2. Let $\varepsilon > 0$ be such that $0 < \kappa + \varepsilon < 1$; then, from the characterization of $H_p^{\kappa+\varepsilon}$ via differences, see [**10**], and with the help of Hölder's inequality, it follows that

$$|uvw|_{H_p^{\kappa+\varepsilon}(L_p)} \leq C|u|_{H^{\kappa+\varepsilon}_{p\rho_1}(L_{p\rho_2})}|v|_{H^{\kappa+\varepsilon}_{p\rho_3}(L_{p\rho_4})}|w|_{H^{\kappa+\varepsilon}_{p\rho_3}(L_{p\rho_4})}. \tag{3.8}$$

Observe that (3.8) is valid for $\kappa = \varepsilon = 0$, too.

On the other hand, the mixed derivative theorem yields

$$Z \hookrightarrow H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}).$$

Then, for completion of the proof, we have to check the validity of Sobolev embeddings

$$H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}) \hookrightarrow H_{p\rho_1}^{\kappa+\varepsilon}(L_{p\rho_2})$$

and

$$H_p^{(1-\theta)\alpha+\kappa}(H_p^{2\theta}) \hookrightarrow H_{p\rho_3}^{\kappa+\varepsilon}(L_{p\rho_4}).$$

Is easy to verify that the first embedding is valid for some $\theta \in (0,1)$, provided

(3.9)

$$p \geq \frac{\alpha n}{2(\alpha-\varepsilon)}\left(1-\frac{1}{\rho_2}\right)+\frac{1}{\alpha-\varepsilon}\left(1-\frac{1}{\rho_1}\right) = \frac{\alpha n}{2(\alpha-\varepsilon)}\left(\frac{2}{\rho_4}\right)+\frac{1}{\alpha-\varepsilon}\left(\frac{2}{\rho_3}\right)$$

and the second one is valid for some $\theta \in (0,1)$, provided

$$p \geq \frac{\alpha n}{2\,(\alpha-\varepsilon)}\left(1-\frac{1}{\rho_4}\right)+\frac{1}{\alpha-\varepsilon}\left(1-\frac{1}{\rho_3}\right). \tag{3.10}$$

Taking $\rho_3 = \rho_4 = 3$, (3.9) and (3.10) are equivalent to

$$p \geq \frac{\alpha n}{3(\alpha-\varepsilon)} + \frac{2}{3(\alpha-\varepsilon)}.$$

Then the claim follows from the strict inequality

$$\frac{\alpha n}{3(\alpha-\varepsilon)} + \frac{2}{3(\alpha-\varepsilon)} > \frac{n}{3} + \frac{2}{3\alpha},$$

since $\varepsilon > 0$. □

Lemma 3.2. *Let X be a Banach space of class $\mathcal{HT}$, and let $J = [0,T]$, $T > 0$. Further let $b \in \mathcal{K}^1(\beta,\theta)$, $\beta > 1$, $\theta < \pi$. Assume that the constants $\kappa \geq 0$ and $\varepsilon \in (0,1)$ are given, and suppose further* *that* $1 < \beta + \kappa < 2$. *Then, for all* $u \in H_p^{\kappa+\varepsilon}(J;X)$, *there is a constant* $c(T) > 0$, *such that*

$$|b * u|_{{}_0H_p^{\beta+\kappa}(J;X)} \le c(T)|u|_{H_p^{\kappa+\varepsilon}(J;X)}. \tag{3.11}$$

Moreover, $c(T) \to 0$ *as* $T \to 0$.

Proof. We begin by recalling the notion of fractional derivatives. Let $\alpha > 0$. The fractional derivative of order α for all functions $f \in {}_0H_p^{\alpha}(J;X)$ is defined by

$$D_t^{\alpha} f(t) = \frac{d^m}{dt^m} \int_0^t g_{m-\alpha}(t-s) f(s)\, ds,$$

where $m = [\alpha] \in \mathbf{N}$, and $g_{\alpha}(t) := (t^{\alpha-1}/\Gamma(\alpha))$.

Observe that by Corollary 2.9 the operator D_t^{α} coincides with the operator given there. Moreover, it defines an isometric isomorphism from ${}_0H_p^{\alpha}(J;X)$ to $L_p(J;X)$. On the other hand, since $f \in {}_0H_p^{\alpha}(J;X)$, it follows that

$$|g_{\varepsilon} * f|_{{}_0H_p^{\alpha}(J;X)} \le c(T)|f|_{{}_0H_p^{\alpha}(J;X)}, \tag{3.12}$$

where $c(T) > 0$ and $c(T) \to 0$ as $T \to 0$. Indeed, observing that the operators D_t^{α} and $g_{\varepsilon} * \cdot$ commute in ${}_0H_p^{\alpha}(J;X)$, we have

$$|g_{\varepsilon} * f|_{{}_0H_p^{\alpha}(J;X)} = |{}_t^{\alpha}(g_{\varepsilon} * f)|_{L_p(J;X)} = |g_{\varepsilon} * D_t^{\alpha} f|_{L_p(J;X)}.$$

Using this and Young's inequality, the claim follows with $c(T) := |g_{\varepsilon}|_{L_1(J)}$.

Now, since $b * g_{\varepsilon}$ and $(d/dt) b * g_{\varepsilon}$ are of order $t^{\beta+\varepsilon}$ and $t^{\beta+\varepsilon-1}$, respectively, it follows that the operator $D_t^{\varepsilon}(b * \cdot) : H_p^{\kappa+\varepsilon}(J;X) \to {}_0H_p^{\beta+\kappa}(J;X)$ is well-defined, linear and bounded. On the other hand, since $\varepsilon < 1$ and the identity $g_{\varepsilon} * D_t^{\varepsilon} = I$ is valid in ${}_0H_p^{\varepsilon}(J;X)$, we obtain

$$|b * u|_{{}_0H_p^{\beta+\kappa}(J;X)} = |g_{\varepsilon} * D_t^{\varepsilon}(b * u)|_{{}_0H_p^{\beta+\kappa}(J;X)}. \tag{3.13}$$

Therefore, (3.11) follows from (3.12) and (3.13) with $\alpha = \beta + \kappa$, since the operator $D_t^{\varepsilon}(b * \cdot)$ is bounded in $H_p^{\kappa+\varepsilon}(J;X)$. □

We can now estimate $H(\phi)$ in X_2.

Corollary 3.3. *Let* $\alpha_1, \alpha_2 \in (0,1)$ *and* $\kappa_1, \kappa_2 \in [0, 1/p)$ *such that the compatibility condition* (3.6) *holds. Let* $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$, *with* $\theta_i < \pi/2$, *for* $i = 1, 2$, *and let* S *be the operator given in* (3.3). *Suppose that* $p > (n/3) + (2/3(\alpha_2 + 1))$. *Then the map* $H : Z_2 \to {}_0X_2$, *defined as*

$$H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t$$

is continuous and bounded in Z_2. *Moreover, there is a constant* $K(T) > 0$, *with* $K(T) \to 0$ *as* $T \to 0$, *such that*

$$|H(v)|_{{}_0X_2} \le K(T) \cdot \left[|v|_{Z_2}^3 + |v|_{Z_2} + |v - v(0)|_{{}_0X_2} \right]. \tag{3.14}$$

is valid for every $v \in Z_2$.

Proof. Let $v \in Z_2$, then $1 * v_t \in {}_0X_2$. From Lemma 3.2 with $b = 1 * a_2$ and $\beta = 1 + \alpha_2$, it follows that there is a constant $c(T) > 0$, such that

$$|1 * a_2 * S * v_t|_{{}_0X_2} \le c(T) |S * v_t|_{H_p^{\kappa_2 + \varepsilon}(L_p)}. \tag{3.15}$$

On the other hand, from the embedding $Z_2 \hookrightarrow H_p^{\kappa_2+\varepsilon}$, $\varepsilon < \alpha_2$, and maximal regularity of equation (II), we obtain the existence of a constant $C > 0$, such that

$$|S * v_t|_{H_p^{\kappa_2+\varepsilon}(L_p)} \le |S * v_t|_{Z_2} \le C \cdot |1 * v_t|_{{}_0X_2} = C \cdot |v - v(0)|_{{}_0X_2}. \tag{3.16}$$

Therefore, from (3.15) and (3.16), there exists a constant $K(T) > 0$ with

$$|1 * a_2 * S * v_t|_{{}_0X_2} \le K(T) |v - v(0)|_{{}_0X_2}. \tag{3.17}$$

Finally, Lemma 3.2 yields

$$|1 * a_2 * (v - v^3)|_{{}_0X_2} \le c(T) \left(|v|_{H_p^{\kappa_2+\varepsilon}(L_p)} + |v^3|_{H_p^{\kappa_2+\varepsilon}(L_p)} \right). \tag{3.18}$$

Hence, using the embedding $Z_2 \hookrightarrow H_p^{\kappa_2+\varepsilon}(L_p)$, $\varepsilon < \alpha_2$, and Lemma 3.1, the proof is complete. □

3.2 Contraction mapping principle. In this section we solve the equation

$$\phi = -1 * a_2 * A\phi + H(\phi) + g(t), \quad t \in J, \tag{3.19}$$

in Z_2, where the nonlinearity $H(\phi)$ and the function $g(t)$ are defined by

$$H(\phi) = 1 * a_2 * (\phi - \phi^3) - 1 * a_2 * S * \phi_t, \quad t \in J, \tag{3.20}$$

and

$$g(t) = 1 * a_2 * u^* + 1 * f_2 + \phi_0, \qquad t \in J. \tag{3.21}$$

We begin with the linear version of (3.19), that is,

$$v^* = -1 * a_2 * Av^* + g(t), \quad t \in J. \tag{3.22}$$

Theorem 2.10 allows us to define an operator $\mathcal{L}$ in Z_2 by

$$\mathcal{L}v = v + 1 * a_2 * Av, \quad \text{for all} \quad v \in Z_2,$$

which is an isomorphism between Z_2 and the space

$$\mathbf{E} := \Big\{ g \in X_2 : g(0) \in Y_p^2, \text{ and } g_t(0) \in \widetilde{Y}_p^2, \text{ if } \alpha_2 + \kappa_2 > \frac{1}{p} \Big\}.$$

Observe that a function g defined by (3.21) belongs to $\mathbf{E}$, if and only if

(i) $u^* \in L_p(J; X)$ and $f_2 \in \widetilde{X}_2$,

(ii) $\phi_0 \in Y_p^2$,

(iii) $f_2(0) \in \widetilde{Y}_p^2$, if $\alpha_2 + \kappa_2 > 1/p$.

On the other hand, from Corollary 3.3, it follows $H(w) \in X_2$, for each $w \in Z_2$. Furthermore, it is easy to check that $H(w) \in \mathbf{E}$ too, actually $H(w)(0) = d/(dt)H(w)(t)|_{t=0} = 0$. Now, let $v^* \in Z_2$ denote the solution of $\mathcal{L}v^* = g$, and assume that in equation (3.19) $\phi \in Z_2$ is known. By defining $v = \phi - v^*$, equation (3.19) is equivalent to the fix point problem

$$v = \mathcal{L}^{-1} H(v + v^*) =: \mathcal{T} v \quad \text{in} \quad {}_0Z_2.$$

Here is the result concerning the solution of equation (3.19).

Theorem 3.4. *Let $\alpha_i \in (0,1)$, $0 < \theta_i < \pi/2$, $\kappa_i \in [0,1/p)$ for $p > 1$, and let $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ for $i = 1,2$. Suppose that $p > (n/3) + (2/3(\alpha_2+1))$, $\alpha_i + \kappa_i \neq 1/p$, $i = 1,2$, and that condition* (3.6) *holds. Then for some $0 < \delta \leq T$, equation* (3.19) *has a unique local solution in $Z_2(\delta)$, if*

(i) $u^* \in L_p(J;X)$ *and* $f_2 \in \widetilde{X}_2$,

(ii) $\phi_0 \in Y_p^2$,

(iii) $f_2(0) \in \widetilde{Y}_p^2$, *if* $\alpha_2 + \kappa_2 > 1/p$,

are fulfilled.

Proof. Assume that the conditions (i)–(iii) are fulfilled. Defining g by (3.21), it follows that $g \in \mathbf{E}$, and from Theorem 2.10 there is a unique solution v^* in Z_2 of equation

$$\mathcal{L}v^* = g.$$

Since $H(w) \in \mathbf{E}$, for each $w \in Z_2$ we have that equation (3.19) is equivalent to a fixed point problem. Consider the ball $\mathcal{B}_r(0) \subset {}_0Z_2(\delta)$, where $r > 0$ is fixed, and define $\mathcal{T} : \mathbf{B}_r(0) \subset {}_0Z_2(\delta) \to {}_0Z_2(\delta)$ by $\mathcal{T}v = \mathcal{L}^{-1}H(v^* + v)$. Furthermore, let $b := 1 * a_2$. We first show that $\mathcal{T}$ is a contraction by using Lemma 3.1 and Corollary 3.3.

$$\begin{aligned}
|\mathcal{T}v - \mathcal{T}w|_{{}_0Z_2(\delta)} &\leq |\mathcal{L}^{-1}||H(v^*+v) - H(v^*+w)|_{{}_0X_2(\delta)} \\
&\leq C|b*(v-w)[(v^*+w)^2 \\
&\qquad + (v^*+v)(v^*+w) + (v^*+v)^2]|_{{}_0X_2(\delta)} \\
&\quad + C|b*S*(v_t-w_t)|_{{}_0X_2(\delta)} + C|b*(v-w)|_{{}_0X_2(\delta)} \\
&\leq CK(\delta)|v-w|_{{}_0Z_2(\delta)} \\
&\quad \times \left[2|v^*|_{Z_2(\delta)} + |w|_{{}_0Z_2(\delta)} + |v|_{{}_0Z_2(\delta)}\right]^2 \\
&\quad + C|b*S*(v_t-w_t)|_{{}_0X_2(\delta)} + CK(\delta)|v-w|_{{}_0Z_2(\delta)}
\end{aligned}$$

Then, using the same argument as in the proof of Corollary 3.3, it follows that

$$\begin{aligned}
|\mathcal{T}v - \mathcal{T}w|_{{}_0Z_2(\delta)} &\leq CK(\delta)|v-w|_{{}_0Z_2(\delta)}[4(|v^*|_{Z_2(\delta)} + r)^2 + C_1] \\
&\leq \frac{1}{2}|v-w|_{{}_0Z_2(\delta)},
\end{aligned} \tag{3.23}$$

since $K(\delta) \to 0$ as $\delta \to 0$.

To show that $\mathcal{T}\mathbf{B}_r(0) \subset \mathbf{B}_r(0)$, in a similar way we obtain that

$$
\begin{aligned}
|\mathcal{T}v|_{{}_0Z_2(\delta)} &\leq |\mathcal{L}^{-1}||H(v^* + v)|_{X_2(\delta)} \\
&\leq CK(\delta)[|v^* + v|_{Z_2(\delta)} + |v^* + v|^3_{Z_2(\delta)} + |v^* \\
&\quad + v - v^*(0)|_{{}_0X_2(\delta)}] \\
&\leq 0CK(\delta)[|v^*|_{Z_2(\delta)} + 2r + (|v^*|_{Z_2(\delta)} + r)^3 \\
&\quad + |v^* - v^*(0)|_{{}_0X_2(\delta)}] \\
&< r,
\end{aligned}
\tag{3.24}
$$

provided $\delta > 0$ is small enough. Note that $|v^*|_{Z_2(\delta)} \to 0$ as $\delta \to 0$, since v^* is a fixed function.

Hence, the contraction mapping principle yields a unique fixed point $v \in \mathbf{B}_r(0)$ of $\mathcal{T}$ and, therefore, $\phi = v^* + v$ is the unique strong solution of (3.19) in $[0, \delta]$. □

Concerning continuation of the solution ϕ, observe that by Theorem 3.4, there exist a $\delta > 0$ and a unique solution $\phi = v + v^*$ of (3.19) in $Z_2(\delta)$. On the other hand, from the embedding $Z_2(\delta) \hookrightarrow C^1([0,\delta]; \widetilde{Y}_p^2) \cap C([0,\delta]; Y_p^2)$ we have $\phi(\delta) \in Y_p^2$ and $\phi_t(\delta) \in \widetilde{Y}_p^2$. This fact allows us to continue the solution. Indeed, let $\mathcal{T}$ be the map defined in the proof of Theorem 3.4, and let $v \in {}_0Z_2(\delta)$ be its unique fix point. For $\eta > 0$, consider the space

$$\mathcal{M}_v := \{\psi \in_0 Z_2(\delta + \eta) : \psi|_{[0,\delta]} = v\}.$$

The set $\mathcal{M}_v$ is not empty and with the metric induced by $Z_2(\delta + \eta)$, we have that $(\mathcal{M}_v, d)$ is a complete metric space, where

$$d(f, g) := |f - g|_{Z_2(\delta+\eta)}, \quad \text{for all} \quad f, g \in \mathcal{M}_v.$$

Now we can apply the contraction mapping principle to $\mathcal{T}$ in $\mathcal{M}_v$. From (3.23) and (3.24), it is easy to show that $\mathcal{T}$ has a unique fixed point $\psi \in \mathcal{M}_v$, for some $\delta_1 \in (\delta, \delta + \eta)$, provided $\eta > 0$ is chosen sufficiently small. Hence, the function $\phi := v^* + \psi$ is the unique solution of (3.5) in $Z_2(\delta_1)$. A successive application of this argument yields a solution ϕ on a maximal time interval $[0, t_{max})$, which is characterized by the two equivalent conditions

$$
\begin{cases}
\lim_{\delta \to t_{\max}} |\phi(\delta)|_{Y_p^2} & \text{does not exist;} \\
\lim_{\delta \to t_{\max}} |\phi_t(\delta)|_{\widetilde{Y}_p^2} & \text{does not exist, if } \alpha_2 > 1/p,
\end{cases}
$$

and

$$|\phi|_{Z_2(t_{\max})} = \infty.$$

As we already proved in subsection 3.1, (3.19) and the system (1.2)–(1.3) are equivalent. Therefore, we obtain the following result.

Theorem 3.5. *Let $\alpha_i \in (0,1)$, $0 < \theta_i < \pi/2$, $\kappa_i \in [0,1/p)$ for $p > 1$, and let $a_i \in \mathcal{K}^1(\alpha_i,\theta_i)$ for $i = 1,2$. Suppose that $p > (n/3)+(2/3(\alpha_2+1))$, $\alpha_i+\kappa_i \neq 1/p$ for $i = 1,2$, and that condition (3.6) holds. Then for some $0 < \delta < t_{\max}$ the system (3.1)–(3.2) has a unique solution $(u,\phi) \in Z_1(\delta) \times Z_2(\delta)$, provided that the data are subject to the following conditions.*

(i) $f_1 \in \widetilde{X}_1$ *and* $f_2 \in \widetilde{X}_2$,

(ii) $u_0 \in Y_p^1$ *and* $\phi_0 \in Y_p^2$,

(iii) $f_i(0) \in \widetilde{Y}_p^i$, *if* $\alpha_i + \kappa_i > 1/p$ *for* $i = 1,2$.

4. Global well-posedness. In this section we want to solve the nonlinear system

$$\text{(PFM)}\quad \begin{cases} u_t + l/2\phi_t = \displaystyle\int_{-\infty}^t a_1(t-s)\Delta u(s)\,ds & \text{in } J\times\Omega;\\ \tau\phi_t = \displaystyle\int_{-\infty}^t a_2(t-s)\left[\xi^2\Delta\phi + \frac{\phi-\phi^3}{\eta} + u\right]ds & \text{in } J\times\Omega;\\ \mathbf{n}\cdot\nabla u = \mathbf{n}\cdot\nabla\phi = 0 & \text{on } J\times\partial\Omega;\\ u(0,x) = u_0(x),\ \phi(0,x) = \phi_0(x) & \text{in } \Omega, \end{cases}$$

globally in time in the setting established in previous sections. As we already explained in the introduction, we may set

$$u(t,x) = \phi(t,x) = 0,\ (t,x) \in (-\infty,0)\times\Omega. \tag{4.1}$$

For the sake of simplicity, we also set $\kappa_2 = 0$. In case $\kappa_2 \neq 0$, the global existence result remains true, but the calculation is more lengthy. Observe that, from (3.6), it follows that $\alpha_2 \geq \alpha_1$ if $\kappa_2 = 0$.

We now begin the discussion concerning global existence of (PFM). From (4.1) and the definition of operator B in Corollary 2.9, which is associated with kernel a_2, (PFM) can be written as follows

$$u_t + \phi_t = \int_0^t a_1(t-s)\Delta u(s)\,ds, \quad \text{in} \quad J\times\Omega; \tag{4.2}$$

$$\begin{aligned} B\phi_t &= \Delta\phi + \phi - \phi^3 + u, && \text{in} \quad J\times\Omega; \\ \mathbf{n}\cdot\nabla u &= \mathbf{n}\cdot\nabla\phi = 0, && \text{on} \quad J\times\partial\Omega; \\ u(0,x) &= u_0(x),\ \phi(0,x) = \phi_0(x), && \text{in} \quad \Omega. \end{aligned} \tag{4.3}$$

The next result gives an *a priori* estimate in the case that the kernels a_1 and a_2 satisfy the conditions (P1) and (P2), respectively.

Lemma 4.1. *Let $(u,\phi)\in Z_1(\delta)\times Z_2(\delta)$ be the solution of* (4.2)–(4.3), *for $p\geq 2$. Assume that the conditions* (P1) *and* (P2) *are fulfilled. Then there is constant $M>0$, independent of δ, such that, the inequality*

$$\begin{aligned} -M \leq \sup_{0<\delta<t_{\max}} &\Big\{ |u(\delta)|^2_{L_2(\Omega)} + |\phi(\delta)|^2_{H^1_2(\Omega)} + \frac{1}{2}\,|\phi(\delta)|^4_{L_4(\Omega)} - |\phi(\delta)|^2_{L_2(\Omega)} \Big\} \\ &+ 2\int_0^{t_{\max}} \langle a_1 * \nabla u, \nabla u\rangle\,dt + 2\int_0^{t_{\max}} \langle B\phi_t, \phi_t\rangle\,dt \\ &\leq 2\left[|u_0|^2_{L_2(\Omega)} + |\phi_0|^2_{H^1_2(\Omega)} + |\phi_0|^4_{L_4(\Omega)} \right] \end{aligned}$$

holds.

Proof. We multiply (4.2) by u and (4.3) by ϕ_t, add the result and integrate by parts, to obtain

$$\begin{aligned} \frac{1}{2}\,\partial_t &\Big\{ \int_\Omega |u|^2\,dx + \int_\Omega |\nabla\phi|^2\,dx + \frac{1}{2}\int_\Omega |\phi|^4\,dx - \int_\Omega |\phi|^2\,dx \Big\} \\ &+ \langle a_1 * \nabla u, \nabla u\rangle + \langle B\phi_t, \phi_t\rangle = 0 \end{aligned} \tag{4.4}$$

Integrating (4.4) over $(0,\delta)$, $\delta < t_{\max}$, it is easy to verify the inequality

$$\begin{aligned} |u|^2_{L_2(\Omega)} + |\phi|^2_{H^1_2(\Omega)} + \frac{1}{2}\,|\phi|^4_{L_4(\Omega)} - |\phi|^2_{L_2(\Omega)} + 2\int_0^\delta \langle B\phi_t, \phi_t\rangle\,ds \\ + 2\int_0^\delta \langle a_1 * \nabla u, \nabla u\rangle\,ds \\ \leq 2\left[|u_0|^2_{L_2(\Omega)} + |\phi_0|^2_{H^1_2(\Omega)} + |\phi_0|^4_{L_4(\Omega)} \right]. \end{aligned} \tag{4.5}$$

Note that the term $\int_0^\delta \langle B\phi_t, \phi_t\rangle\, ds$ is positive since B is accretive. On the other hand, the parabola $x^4 - 2x^2$ is bounded from below by -1. Therefore, taking the supremum over $(0, t_{\max})$ in (4.5), the proof is completed. □

Now we can state our main result of this section.

Theorem 4.2. *Let $\alpha_i \in (0,1)$, $0 < \theta_i < \pi/2$, and let $a_i \in \mathcal{K}^1(\alpha_i, \theta_i)$ for $i = 1, 2$. Suppose that $p \geq 2$ and $n \leq 3$, $\alpha_i \neq 1/p$ for $i = 1, 2$, and the condition $\alpha_2 \geq \alpha_1$ holds. If*

(i) *the conditions* (P1) *and* (P2) *are fulfilled, and*

(ii) $u_0 \in Y_p^1$ *and* $\phi_0 \in Y_p^2$,

then the system (4.2)–(4.3) *has a unique global solution* $(u, \phi) \in Z_1 \times Z_2$.

Proof. Let $0 < \delta < t_{\max}$, and let $(u, \phi) \in Z_1(\delta) \times Z_2(\delta)$ be the unique local solutions of (4.2)–(4.3), given by Theorem 3.5. From Lemma 4.1 it follows that

$$\phi \in L_\infty([0, t_{\max}); L_6(\Omega)). \tag{4.6}$$

If $\varrho \in (1/4, 1/3)$, then the inequality

$$-\frac{n}{3p} \leq \varrho\left(2 - \frac{n}{p}\right) - \frac{n(1-\varrho)}{6}$$

is valid for $p \geq 2$ and $n \leq 3$. Therefore, by the Gagliardo-Nirenberg inequality, it follows that there is a constant $C := C(\Omega) > 0$, such that

$$|\phi|_{L_{3p}(\Omega)} \leq C|\phi|^{\varrho}_{H_p^2(\Omega)} |\phi|^{1-\varrho}_{L_6(\Omega)}. \tag{4.7}$$

Furthermore, from (4.6) and (4.7), we obtain

$$|\phi^3|_{L_p(L_p)} \leq C_0|\phi|^{3\varrho}_{L_{3\varrho p}(H_p^2)} \leq C_0|\phi|^{3\varrho}_{L_p(H_p^2)} \leq C_0|\phi|^{3\varrho}_{Z_2(\delta)}. \tag{4.8}$$

On the other hand, by maximal L_p-regularity, there is a constant $M := M(T) > 0$, such that

$$|u|_{Z_1(\delta)} + |\phi|_{Z_2(\delta)} \leq M\left(1 + |\phi^3|_{L_p([0,\delta];X)}\right).$$

Hence, (4.8) yields

$$|\phi|_{Z_2(\delta)} \leq M\left(1 + |\phi|^{3\varrho}_{Z_2(\delta)}\right)$$

with a different constant M, which is independent of $\delta < t_{max}$. Therefore,

$$|\phi|_{Z_2(t_{\max})} < \infty.$$

This in turn yields the boundedness of $u \in Z_1(t_{\max})$. Hence the global existence of (4.2)–(4.3) follows. □

REFERENCES

1. D.L. Burkholder, *Martingales and Fourier analysis in Banach spaces*, Lecture Notes in Math., vol. 1206, Springer-Verlag, New York, 1986, pp. 61–108.

2. Ph. Clément and J. Prüss, *An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces*, Lecture Notes in Pure and Appl. Math., vol. 215, Dekker, New York, 2001, pp. 67–87.

3. ———, *Completely positive measures and Feller semigroups*, Math. Ann. **287** (1990), 73–105.

4. R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni, *New thoughts on old results of R.T. Seeley*, Math. Ann. **328** (2004), 545–583.

5. R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Memoirs Amer. Math. Soc., vol. 166, 2003, viii+144 pp.

6. A. Novick-Cohen, *A phase-field system with memory*: *Global existence*, J. Integral Equations Appl. **14** (2002), 73–107.

7. J. Prüss, *Evolutionary integral equations and applications*, Birkhäuser Verlag, Basel, 1993.

8. J. Prüss and H. Sohr, *On operators with bounded imaginary powers in Banach spaces*, Math. Z. **203** (1990), 429–452.

9. H.G. Rotstein, S. Brandon, A. Novick-Cohen and A. Nepomnyashchy, *Phase field equations with memory*: *hyperbolic case*, SIAM J. Appl. Math., **62** (2001), 264–282.

10. H. Triebel, *Theory of function spaces* II, Birkhäuser Verlag, Basel, 1992.

11. R. Zacher, *Maximal regularity of type L_p for abstract parabolic Volterra equations*, J. Evol. Equ. **5** (2005), 79–103.

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