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VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH ACCRETIVE OPERATORS AND NON-AUTONOMOUS PERTURBATIONS

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ABSTRACT. This paper is devoted to study a class of nonlinear scalar Volterra equations in general Banach spaces, with an m-accretive leading operator and a nonautonomous perturbation. We shall consider both the case of Lipschitz perturbations and the case of dissipative perturbations. We prove the existence of a generalized solution and discuss some useful estimates for it.

1. Introduction. The type of Volterra equations studied in this paper is the nonlinear evolution equation

(1.1)
$$\begin{cases} \frac{d}{dt}\left(k_0(u(t)-x) + \int_0^t k_1(t-s)(u(s)-x)\,ds\right) + G(u(t)) = F(t,u(t)), \\ t \in (0,\infty), \quad u(0+) = x, \end{cases}$$

in a real Banach space X. Here, $k_0 \geq 0$ is a constant and k_1 is a real, nonnegative function that satisfy Hypothesis 1a) below, G is an accretive operator in X, see Hypothesis 1b), and we shall consider the operator $F(t,u)$ as a nonlinear, nonautonomous perturbation of the operator G, see Hypothesis 1c) for details.

Since the early 1970s, the case where $F(t,u) = f(t)$ has been under consideration; this problem has an interest also in our setting, and it shall be further discussed in Section 2.1. The next step in the literature was to consider functional perturbations of such a problem, compare [**4, 8**].

In this paper, on the contrary, we consider perturbation operators acting on X, but we can allow such operators to be nonautonomous. The study of (1.1) with the operator $F(t,u)$ is based on the results for

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the inhomogeneous problem $F = f(t)$ and a fixed point argument; this should justify the appellative of "perturbation term" given to $F(t, u)$.

In order to state the main result of the paper, we shall introduce the main assumptions on the coefficients of (1.1). A comprehensive explanation of the notation employed in the paper will be given in Section 3.

Hypothesis 1. a) *The kernel* $k(t) = k_0 + \int_0^t k_1(s)\,ds$ *is a Bernstein function associated to a kernel* $a(t)$

$$k_0 a(t) + \int_0^t k_1(t-s)a(s)\,ds = 1, \quad t \in (0, \infty); \tag{1.2}$$

b) $G(x)$ *is an operator in* X, *with domain* $D(G) \subset X$, *and there exists* $\omega \geq 0$ *such that* $G + \omega I$ *is m-accretive in* X.

c) *The perturbation term* F *maps* $\mathbf{R}_+ \times X$ *into* X; *it is uniformly continuous on bounded sets of* $\mathbf{R}_+ \times X$ *and for each* $t \in \mathbf{R}_+ = [0, \infty)$, $F(t, \cdot)$ *is m-dissipative on* X.

We shall also need the generalized domain $\hat{D}(G)$: this is a suitable subset of X which contains $D(G)$, defined in terms of Yosida approximations of G; see Definition 3.1 below.

Our main result provides the existence of a generalized solution for problem (1.1).

Theorem 1.1. *Assume* X *is a real Banach space, and let Hypothesis* 1 *be satisfied. Then, for any* $x \in \overline{D(G)}$, *there exists a unique generalized solution to the abstract nonlinear Volterra equation* (1.1).

The paper is organized as follows. In Section 2, we shall discuss how the results provided here are related with those already known in the literature. Our notation, and some preliminary results about the coefficients of (1.1), are given in Section 3. In particular, in subsection 3.3 we discuss some properties of the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left(k_0 u(t) + \int_0^t k_1(t-s)u(s)\,ds\right).$$

Although there exists a large literature about this subject, we obtain a representation of the Yosida approximations $L_\mu = L(I + (1/\mu)L)^{-1}$ which may deepen the understanding of the relation with the associated completely monotonic kernel. Finally, the remaining sections are devoted to the study of (1.1), first in the case $F(t,x) = f(t)$, then in the case of a Lipschitz nonlinearity, and the last section provides the proof of Theorem 1.1.

2. Nonlinear equations with accretive operators. The equation that we consider in this paper is a nonautonomous perturbation of the inhomogeneous problem

(2.1)

$$\begin{cases} \dfrac{d}{dt}\left(k_0(u(t)-x) + \displaystyle\int_0^t k_1(t-s)(u(s)-x)\,ds\right) + G(u(t)) = f(t), \\ t \in (0,\infty), \quad u(0+) = x. \end{cases}$$

There is a wide literature concerning such equations, motivated also by their relevance in applications. Actually, Volterra integro-differential equations of convolution type with completely monotone kernel arise naturally in several fields, as heat conduction in materials with memory and in the theory of thermo-viscoelasticity; see for instance the monograph of Prüss [**12**] and the references therein.

2.1 The case of a perturbation independent of u. We start by considering the simpler case where the perturbation on the right-hand side of (1.1) is independent of u. This case shall provide us with the estimates that we need in order to study the general case of equation (1.1), compare also [**9**]. Therefore, in this section we are concerned with the equation (2.1).

In order to define a generalized solution to (2.1), we shall consider an approximate equation, where the operator L is replaced by its Yosida approximation $L_\mu = L(I + (1/\mu)L)^{-1}$, $\mu > 0$. Let u_μ be the solution of the following equation

$$L_\mu[u_\mu(\cdot) - x](t) + G(u_\mu(t)) = f(t), \quad t \in (0,\infty). \tag{2.2}$$

In the next theorem, we establish the existence of a generalized solution of (2.1).

Theorem 2.1. *Assume that Hypotheses* 1 a)–1 b) *are satisfied, and let* $x \in \overline{D(G)}$ *and* $f \in C(\mathbf{R}_+; X)$*. Then, for every* $\mu > 0$*, equation* (2.2) *has a unique solution* $u_\mu \in C(\mathbf{R}_+; X)$*.*

As $\mu \to \infty$*, there exists a function* $u = U(x, f)$ *with* $u \in C(\mathbf{R}_+; X)$ *such that* $u_\mu \to u$ *in* $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; X)$*.*

The function $u = U(x, f)$, that exists according to Theorem 2.1, is said to be the generalized solution for problem (2.1).

Let us discuss briefly our setting as compared to that of Gripenberg [**8**]. The results in that paper distinguish the cases $k_0 = 0$ and $k_0 > 0$. In the latter case, the quoted result fully describes the case $\omega = 0$ (ω is the type of the operator G). In general, however, we may write $G(u) = \widetilde{G}(u) - \omega u$, $\widetilde{G}$ is an m-accretive operator of negative type, and ωu is a linear perturbation, so that this case may be as well treated by means of Theorem 3 of that paper.

In Section 4, we shall discuss the case $k_0 = 0$ in full detail. Here, actually, the results in [**8**] do not suffice and a refinement of the estimates for the solution is necessary. We collect in Theorem 4.7 the relevant estimates that we obtain in our setting. In case $k_0 = 0$ and G an m-accretive operator on X, similar results were already proved in [**3**], see also formula (4.16) here.

Remark 2.1. Using the estimates in [**3**], Gripenberg et al. [**9**] solved the problem of existence of a strong solution for (2.1). In our setting, the extension of this result does not seem straightforward, since one of the relevant estimates couldn't be proved with our techniques; see Remark 4.3 for more details. We hope to return to this problem in a subsequent paper.

2.2 The case of a Lipschitz perturbation. Now we return to the nonlinear problem (1.1). Before we discuss the case of dissipative operators, that is the object of Theorem 1.1, we shall consider the case of a Lipschitz perturbation. We say that $u(\cdot)$ is a generalized solution of (1.1) if $u = U(x, F(\cdot, u))$.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *be fulfilled, and assume that the nonlinear term F satisfies*

$$t \longmapsto F(t,\xi) \in C(\mathbf{R}_+; X), \quad \text{for all} \quad \xi \in X, \tag{2.3}$$

and there exists a function $\eta \in L^\infty_{\mathrm{loc}}(\mathbf{R}_+)$ such that, for any $t \in \mathbf{R}_+$

$$\|F(t,\xi_1) - F(t,\xi_2)\| \le \eta(t)\|\xi_1 - \xi_2\|, \quad \text{for all} \quad \xi_1, \xi_2 \in X. \tag{2.4}$$

Then there exists a unique generalized solution to equation (1.1)

$$\begin{cases} L[u(\cdot) - x](t) + G(u(t)) = F(t, u(t)), \\ t \in (0,\infty), \quad u(0+) = x. \end{cases}$$

As we mentioned in the previous section, Theorem 3 in [**8**] is concerned with the existence of a generalized solution to (1.1). As before, cases $k_0 = 0$ and $k_0 > 0$ are treated separately and, again, the second case, $k_0 > 0$, is fully described by Gripenberg. Instead, in case $k_0 = 0$, the Lipschitz perturbation term in Theorem 2.2 is not contained in the assumption of [**8**, Theorem 3], that is,

$$\|F(v_1) - F(v_2)\|_{L^1(0,t;X)} \le \int_0^t \eta(s)\|v_1 - v_2\|_{L^1(0,s;X)}\, ds, \quad t \in \mathbf{R}_+.$$

2.3 The case of a non-autonomous dissipative perturbation. In the last section we finish the proof of the main result stated in Theorem 1.1. We are concerned here with the case of a continuous and m-dissipative operator $F(t,u)$, see Hypothesis 1c). Since this term is nonautonomous, it is not possible to include it into G and to apply the previous theorems, also if we suppose that $-G + F$ is m-dissipative.

The techniques applied in this part, although very different from those employed in the previous sections, are usually applied in the theory of dissipative systems; in particular, we refer to the proof of [**6**, Theorem 7.13].

Remark 2.2. In the literature, it is often assumed that G is a multi-valued accretive operator in X, while in this paper this is not allowed. Our choice is mainly motivated by the quest for simplicity of notation.

Actually, it should be noticed that the extension of Theorems 2.1 and 2.2 to cover this setting is straightforward, and, in that case, they match the results in [**8**]. On the other hand, the "multi-valued version" of Theorem 1.1 requires more attention. A sufficient condition to prove the result is the following assumption: there exists (at least) one element $y \in D(G)$ such that $\sup_{z\in G(y)} \|z\| < +\infty$.

In any case, this problem does not affect much the central substance of this paper, while it is not relevant in the applications to stochastic differential equations, see [**1**].

3. Notation and preliminary results. We shall denote the norm in the Banach space X by $\|\cdot\|$.

3.1 Properties of accretive operators. For the sake of completeness, we recall some properties of accretive operators from the book of Da Prato [**5**].

An operator G on X is said to be accretive if, for any $x, y \in D(G)$ and for all $\lambda > 0$: $\|x - y\| \le \|x - y + \lambda(G(x) - G(y))\|$; moreover, an operator F on X is said to be dissipative if $-F$ is accretive. We denote by $\Lambda_{mc}(X)$ the space of accretive operators G on X such that $\text{Range}\,(I + \lambda G) = X$; such operators are called m-accretive.

We also denote by $\widetilde{\Lambda}_{mc}(X)$ the space of operators G on X such that $G+\omega I$ belongs to $\Lambda_{mc}(X)$ for a suitable real number ω. If $G \in \widetilde{\Lambda}_{mc}(X)$ we set $\omega_G = \inf\{\omega \in \mathbf{R} : G + \omega I \in \Lambda_{mc}(X)\}$; then we say that ω_G is the type of G; if $\omega_G < 0$ we say that G is of negative type.

As stated in the introduction, we assume that the operator G belongs to $\widetilde{\Lambda}_{mc}(X)$ and we denote by $\omega = \omega_G \ge 0$ the type of G. If G is of negative type, then we choose $\omega = 0$.

The resolvent operator J_μ, associated with $\widetilde{G} = G + \omega I$, is defined by

$$J_\mu = \left(I + \frac{1}{\mu}\,\widetilde{G})\right)^{-1}, \quad \mu > 0.$$

We have that J_μ satisfies the following properties:

$$\|J_\mu(x) - J_\mu(y)\| \le \|x - y\|, \quad \text{for all} \quad x, y \in X,$$

and

$$\lim_{\mu\to\infty} J_\mu(x) = x, \quad \text{for all} \quad x \in \overline{D(G)}.$$

We also introduce the Yosida approximations G_α, $\alpha > 0$, of $\widetilde{G}$ by setting

$$G_\alpha(x) = \widetilde{G}(J_\alpha(x)) = \alpha(x - J_\alpha(x)), \quad x \in X.$$

We remark that $G_\alpha(x)$ is a Lipschitz continuous mapping, and it holds that $\|G_\alpha(x)\| \le \|\widetilde{G}(x)\|$ for any $x \in D(G)$.

Definition 3.1. We denote the generalized domain $\widehat{D}(G)$ the set $\{x \in X : \sup_{\alpha>0} \|G_\alpha(x)\| < +\infty\}$.

We have $D(G) \subseteq \widehat{D}(G) \subseteq \overline{D(G)}$. If X is not reflexive, then it is possible that $D(G) \subsetneq \widehat{D}(G)$.

3.2 Properties of the scalar kernel. A function $f : (0,\infty) \to \mathbf{R}$ is called *completely monotonic* if f belongs to $C^\infty(0,\infty)$ and

$$(-1)^n \frac{d^n}{dx^n} f(x) \ge 0, \quad x > 0, \quad n = 0, 1, 2, \dots .$$

Below we list some properties of completely monotonic functions.

Remark 3.1. Assume that $f : (0,\infty) \to \mathbf{R}$ is completely monotonic; then

i. if $f(x_0) = 0$ for some $x_0 > 0$ then f is identically zero;

ii. f has an analytic extension to $\{z \in \mathbf{C} : \Re(z) > 0\}$;

iii. if $f(0+) = +\infty$, then $(-1)^n \frac{d^n}{dx^n} f(0+) = +\infty$ for $n = 1, 2, \dots$;

iv. $(-1)^n \frac{d^n}{dx^n} f(+\infty) = 0$ for $n = 1, 2, \dots$.

For an exhaustive introduction to completely monotonic functions, as well as a proof of these properties, we refer to [**10, 12**] or the introduction in [**11**].

A C^∞ function $\varphi : \mathbf{R}_+ \to \mathbf{R}$ is called a *Bernstein function* if $\varphi(t) \ge 0$ for $t > 0$ and φ' is completely monotonic.

Remark 3.2. The kernel $a : (0,\infty) \to \mathbf{R}$ is completely monotonic, $a \in L^1_{\text{loc}}(0,\infty)$. Moreover $k_0 = 0$ implies $a(0+) = +\infty$.

We consider in Table 1 some examples of Bernstein functions k and the corresponding completely monotone functions a.

TABLE 1. $\alpha \in (0,1)$ and $E_1(x) = \int_x^\infty e^{-t}\,dt/t$.

$k(t)$	$a(t)$
1	1
$1+t$	e^{-t}
$\int_0^t E_1(s)\,ds$	$\int_0^\infty e^{-t}t^{\rho-1}\,(d\rho/\Gamma(\rho))$
$\int_0^t (s^{-\alpha}/\Gamma(1-\alpha))\,ds$	$t^{\alpha-1}/\Gamma(\alpha)$

Let us consider the family of functions $s_\mu(t)$, $t \geq 0$, $\mu \in \mathbf{R}$, where s_μ is the solution of the scalar Volterra equation

$$s_\mu(t) + \mu \int_0^t s_\mu(t-\vartheta)a(\vartheta)\,d\vartheta = 1, \quad t > 0. \tag{3.1}$$

Under Hypothesis 1a), it follows that $s_\mu(t)$ is positive and nonincreasing with respect to $t > 0$, for every $\mu > 0$.

Table 2 contains examples of scalar resolvent functions for various completely monotonic functions.

TABLE 2. $\mathcal{E}_\alpha(x) = \sum_{k=0}^\infty ((-x)^k/\Gamma(\alpha k+1))$ is known as Mittag-Leffler's function; as before, $\alpha \in (0,1)$.

$a(t)$	$s(t;\mu)$
1 e^{-t}	$e^{-\mu t}$ $(1+\mu)^{-1}[1+\mu e^{-(1+\mu)t}]$
$\int_0^\infty e^{-t}t^{\rho-1}\,d\rho/\Gamma(\rho)$	$1 - \int_0^\infty \mu e^{-\mu\rho}[\int_0^t e^{-\tau}\tau^{\rho-1}\,d\tau]\,d\rho/\Gamma(\rho)$
$t^{\alpha-1}/\Gamma(\alpha)$	$\mathcal{E}_\alpha(\mu t^\alpha)$

Proposition 3.2. *For any* $\mu \in \mathbf{R}$:

$$\frac{d}{d\mu}\, s_\mu(t) \le 0 \quad \textit{for all} \quad t > 0.$$

Proof. For a proof we refer to [**12**, p. 98], noticing that the case $\mu < 0$ can be treated similarly to the case $\mu > 0$. $\square$

Notice that the above proposition not only implies that $s\mu(t) \le 1$ for any $\mu > 0$, but also that $s_\mu(t) \ge 1$ for any $\mu < 0$.

Let us denote by r_μ the solution to the integral equation

$$r_\mu(t) + \mu \int_0^t r_\mu(t-s)a(s)\,ds = \mu a(t). \tag{3.2}$$

By [**12**, Lemma 4.1], since a is completely monotonic, we know that, for any $\mu > 0$, r_μ belongs to $L^1(\mathbf{R}_+) \cap C(0,\infty)$, it is completely monotonic, $0 \le r_\mu(t) \le \mu a(t)$ for all $t \in \mathbf{R}_+$, and

$$\int_0^\infty r_\mu(s)\,ds = \hat{r}_\mu(0) = \frac{\mu\hat{a}(0)}{1+\mu\hat{a}(0)} \le 1.$$

Moreover, if $\mu < 0$, then r_μ belongs to $L^1_{loc}(\mathbf{R}_+) \cap C(0,\infty)$ and $r_\mu(t) \le \mu a(t) < 0$ for all $t \in \mathbf{R}_+$, compare also [**7**].

The relation between s_μ and r_μ is clarified in the following statement.

Proposition 3.3. *It holds that*

$$s_\mu(t) = \left(1 - \int_0^t r_\mu(\tau)\,d\tau\right), \quad t > 0. \tag{3.3}$$

We shall summarize, in the next proposition, some results about the limit behavior of r_μ and s_μ as $\mu \to \infty$.

Proposition 3.4. *The following relation holds between s_μ and the function k:*

$$\mu s_\mu(t) = (r_\mu * k_1)(t) + k_0 r_\mu(t). \tag{3.4}$$

Moreover,

$$\mu \int_0^t s_\mu(\tau)\, d\tau \longrightarrow k(t)$$

for almost every $t > 0$.

Proof. The proof is given in [**8**, Lemma 3.1]; let us briefly sketch it in our notation. Taking the convolution of (3.2) with $k_1(\cdot)$ and recalling that from (1.2) it follows that $k_0 a(t) + (k_1 * a)(t) = 1$, we obtain

$$(r_\mu * k_1)(t) + \mu\big((r_\mu * k_1) * a\big)(t) = \mu(a * k_1)(t) = \mu(1 - k_0 a(t)).$$

On the other hand, again from (3.2) and (3.1) it follows that

$$(\mu s_\mu - k_0 r_\mu)(t) + \mu\big((\mu s_\mu - k_0 r_\mu) * a\big)(t) = \mu(1 - k_0 a(t)),$$

and comparing this expression with the previous one, we prove (3.4).

The second part of the proof follows in a straightforward manner by using Laplace transform methods. $\square$

In the following, we discuss a Gronwall-type lemma that will allow us to prove estimates for the solution of a Volterra equation.

Lemma 3.5 (A generalized Gronwall-type Lemma). *Let v be a continuous, nonnegative function which satisfies the estimate*

$$v(t) \le s_\lambda(t) x + \frac{1}{\lambda} f(t) + \frac{\omega}{\lambda} v(t) + r_\lambda * v(t), \quad t \in \mathbf{R}_+, \tag{3.5}$$

where $\lambda > \omega$, while $s_\lambda(t)$ and $r_\lambda(t)$ are defined in (3.1) *and* (3.2)*, respectively. Then*

$$v(t) \le \frac{d}{dt}\left(\frac{\omega_\lambda}{\omega} \left(x + \frac{1}{\lambda} f + a * f \right) * s_{-\omega_\lambda} \right)(t), \tag{3.6}$$

where $s_{-\omega_\lambda}(t)$ *is defined as in* (3.1) *with* $\omega_\lambda = (\lambda\omega)/(\lambda-\omega)$.

Remark 3.3. In case $f \equiv 0$, we obtain from the above lemma the following estimate:

$$v(t) \le \frac{\omega_\lambda}{\omega}\, x\, s_{-\omega_\lambda}(t). \tag{3.7}$$

If we consider, instead, the case $\omega = 0$, then estimate (3.6) becomes

$$v(t) \le x + \frac{1}{\lambda}\, f(t) + (a*f)(t). \tag{3.8}$$

Proof. If we take the convolution with a of both sides of (3.5), we have

$$(a*v)(t) \le (a*s_\lambda)(t)x + \frac{1}{\lambda}\,(a*f)(t) + \frac{\omega}{\lambda}\,(a*v)(t) + (a*r_\lambda*v)(t).$$

Using the very definition of r_λ in the above expression, we get

$$(a*v)(t) \le (a*s_\lambda)(t)x + \frac{1}{\lambda}\,(a*f)(t) + \frac{\omega}{\lambda}\,(a*v)(t) + (a*v)(t) - \frac{1}{\lambda}\,(r_\lambda*v)(t),$$

that we read

$$(r_\lambda*v)(t) \le \lambda(a*s_\lambda)(t)x + (a*f)(t) + \omega(a*v)(t). \tag{3.9}$$

Now we substitute what we have found in (3.5) to get

$$v(t) \le s_\lambda(t)x + \frac{1}{\lambda}\, f(t) + \frac{\omega}{\lambda}\, v(t) + \lambda(a*s_\lambda)(t)x + (a*f)(t) + \omega(a*v)(t),$$

and the definition of s_λ implies

$$\begin{aligned} \frac{\lambda-\omega}{\lambda}\, v(t) &\le x + \frac{1}{\lambda}\, f(t) + (a*f)(t) + \omega(a*v)(t) \\ v(t) &\le \frac{\lambda}{\lambda-\omega}\left(x + \frac{1}{\lambda}\, f(t) + (a*f)(t)\right) + \omega_\lambda(a*v)(t). \end{aligned}$$

Now we conclude, since we can apply [**10**, Lemma 9.8.2], with $g(t) = (\lambda/\lambda-\omega)(x + (1/\lambda)f(t) + (a*f)(t))$. $\square$

3.3 Volterra operators. In this section we shall discuss some properties of the linear Volterra operator

$$Lu(t) = \frac{d}{dt}\left[k_0 u(t) + \int_0^t k_1(t-s)u(s)\,ds\right], \quad t > 0, \tag{3.10}$$

with domain

$$D(L) = \left\{ f \in L^1(\mathbf{R}_+; X) \mid k_0 f + (k_1 * f) \in W_0^{1,1}(\mathbf{R}_+; X)\right\}.$$

The operator L is m-accretive in $L^1(\mathbf{R}_+; X)$ and densely defined; notice that by [**10**, Proposition 3.2], the same holds on $L^p(\mathbf{R}_+; X)$ for any $1 \le p < \infty$, but we shall not use this extension.

There is a natural representation of its inverse operator L^{-1} in terms of the kernel a.

Lemma 3.6. *Given the operator L defined in* (3.10)*, the operator L^{-1} is defined by*

$$L^{-1}v(t) = \int_0^t a(t-s)v(s)\,ds, \quad t \in \mathbf{R}_+. \tag{3.11}$$

Proof. Let us prove one implication, say, that $L(a * v)(t) = v(t)$, the other being similar. We start from (1.2), taking the convolution of both sides with $v(t)$, to get

$$k_0(a * v)(t) + (k_1 * a * v)(t) = (1 * v)(t). \tag{3.12}$$

Next, observe that the definition of L implies

$$L(a * v)(t) = \frac{d}{dt}\,[k_0(a * v)(t) + (k_1 * (a * v))(t)]\,;$$

if we substitute what we have found in (3.12) and use the identity $(d/dt)(1 * f)(t) = f(t)$, we obtain (3.11). $\square$

We now proceed to analyze the operator $L_\mu = L(I + (1/\mu)L)^{-1}$.

Lemma 3.7. *The operator* $L_\mu = L(I + (1/\mu)L)^{-1}$ *is given by*

$$L_\mu v(t) = \mu\bigg(v(t) - \int_0^t v(t-s) r_\mu(s)\,ds \bigg), \tag{3.13}$$

where r_μ *is a solution to* (3.2).

Proof. Let $y = L_\mu v$; then

$$\bigg(I + \frac{1}{\mu}\,L\bigg)L^{-1}y = v \Longrightarrow L^{-1}y + \frac{1}{\mu}\,y = v \Longrightarrow a * y + \frac{1}{\mu}\,y = v.$$

If we take convolution with r_μ, recalling (3.2), we get

$$a * y = r_\mu * v \Longrightarrow \mu(r_\mu * v) + y = \mu v. \qquad \square$$

Remark 3.4. We shall use (3.13) in this equivalent form:

$$L_\mu v(t) = \mu\,\frac{d}{dt}\,(v * s_\mu)(t), \quad t \in \mathbf{R}_+. \tag{3.14}$$

3.4 Some estimates on convolution operators. Let α be a positive real number, $\alpha \in (0,1)$, and a a completely monotonic function on $\mathbf{R}_+$ and $a \in L^1_{loc}(0,\infty)$. We define a measure $\rho([0,s]) = \alpha + \int_0^s a(\sigma)\,d\sigma$. The following lemmas treat the estimates on the convolution powers of a and ρ, respectively.

Lemma 3.8. *Let a satisfy Hypothesis* 1a); *then, for each* $T > 0$ *and for any constant* $C > 0$,

$$C^n \|a^{*n}\|_{L^1(0,T)} \longrightarrow 0.$$

More precisely, we have

$$\sum_{n=0}^{\infty} C^n \|a^{*n}\|_{L^1(0,T)} < \infty. \tag{3.15}$$

Proof. Let $C > 0$ be fixed, and define the operator $\mathcal{A} : L^1(0,T) \to L^1(0,T)$ as

$$\mathcal{A}v(t) = C(a * v)(t), \quad t \in (0,T).$$

$\mathcal{A}$ is a linear bounded operator from $L^1(0,T)$ into itself. We claim that the spectral radius $\sigma(\mathcal{A})$ is 0. Then it will follow, from the formula

$$\sigma(\mathcal{A}) = \lim_{n\to\infty} \|\mathcal{A}^n\|_{\mathcal{L}(L^1(0,T))}^{1/n}$$

(here $\|\cdot\|_{\mathcal{L}(L^1(0,T))}$ is the norm of operators on $L^1(0,T)$), that

$$C\|a^{*n}\|_{L^1(0,T)}^{1/n} \le \|\mathcal{A}^n\|_{\mathcal{L}(L^1(0,T))}^{1/n} \longrightarrow 0.$$

In particular, from the *root test* for the convergence of series, we have

$$\sum_{n=0}^{\infty} C^n \|a^{*n}\|_{L^1(0,T)} < \infty.$$

It remains to show that $\sigma(\mathcal{A}) = f0$. From the definition of spectral radius, it is sufficient to show that, for any $\alpha > 0$ and any function $u \in L^1(0,T)$, the following problem has a solution $v \in L^1(0,T)$:

$$u(t) = C(a * v)(t) + C\alpha v(t).$$

But, since a is a completely monotonic kernel, we have

$$v(t) = \frac{1}{\alpha C}\big(u(t) - (r_{1/\alpha} * u)(t)\big),$$

and this shows the lemma. □

Next, we state a useful generalization of the previous lemma.

Lemma 3.9. *Let ρ be a completely positive measure on* $\mathbf{R}$*, defined by*

$$\rho([0,t]) = \alpha + \int_0^t a(s)\,ds,$$

where $\alpha \in (0,1)$ *and* a *satisfies Hypothesis* 1a)*. Let us define*

$$\rho^{*i}([0,t]) = \int_0^t \rho([0,t-\sigma])\,\rho^{*(i-1)}(d\sigma).$$

Then we have that

$$\sum_{n=0}^{\infty} \rho^{*i}([0,t]) < +\infty.$$

Proof. By direct calculations it follows that

$$\rho^{*n}([0,t]) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \|a^{*k}\|_{L^1(0,t)},$$

so we have

$$\begin{aligned}\sum_{n=0}^{\infty} \rho^{*n}([0,t]) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \|a^{*k}\|_{L^1(0,t)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\,\|a^{*k}\|_{L^1(0,t)} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\,\alpha^{n-k}.\end{aligned}$$

Now

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\,\alpha^{n-k} = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)\alpha^n.$$

But

$$(n+k)(n+k-1)\cdots(n+1)\alpha^n = \frac{d^k}{d\alpha^k}\,\alpha^{n+k},$$

then

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\,\alpha^{n-k} = \frac{d^k}{d\alpha^k} \sum_{n=0}^{\infty} \alpha^{n+k} = \frac{d^k}{d\alpha^k}\,\frac{\alpha^k}{1-\alpha}.$$

Since

$$\frac{\alpha^k}{1-\alpha} = \frac{1}{1-\alpha} - (1+\alpha+\cdots+\alpha^{k-1}),$$

we have

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!}\, \alpha^{n-k} = \frac{d^k}{d\alpha^k}\, \frac{1}{1-\alpha} = \frac{k!}{(1-\alpha)^{k+1}}.$$

Hence we have

$$\begin{aligned}\sum_{n=0}^{\infty} \rho^{*n}([0,t]) &= \sum_{k=0}^{\infty} \frac{1}{k!}\, \|a^{*k}\|_{L^1(0,t)}\, \frac{k!}{(1-\alpha)^{k+1}} \\ &= \frac{1}{1-\alpha} \sum_{k=0}^{\infty} \frac{\|a^{*k}\|_{L^1(0,t)}}{(1-\alpha)^k} < \infty,\end{aligned}$$

where the last series converges thanks to estimate (3.15) in Lemma 3.8.
□

4. Construction of the approximate solution. In this section, we shall prove the results stated in Theorem 2.1. As explained in subsection 2.1, we shall only be concerned with the case $k_0 = 0$. We first consider the approximate equation:

$$L_\mu(u_\mu(\cdot) - x)(t) + G(u_\mu(t)) = f(t), \quad t > 0. \tag{4.1}$$

Applying J_μ to both sides of (4.1), we get that this is equivalent to the following

$$u_\mu(t) = J_\mu\bigg(\frac{\omega}{\mu}\, u_\mu(t) + \frac{1}{\mu}\, f(t) + s_\mu(t)x + \int_0^t u_\mu(t-s) r_\mu(s)\, ds\bigg). \tag{4.2}$$

Lemma 4.1. *Let $\mu > \omega$; then for each $T > 0$ there exists a unique solution u_μ to (4.1) in $C([0,T];X)$.*

Proof. For fixed $f \in C(\mathbf{R}_+;X)$ and $x \in X$, we define the mapping

$$\mathcal{K}(v)(t) = J_\mu\bigg(\frac{\omega}{\mu}\, v(t) + \frac{1}{\mu}\, f(t) + s_\mu(t)x + \int_0^t v(t-s) r_\mu(s)\, ds\bigg), \quad t > 0.$$

It is easy to show that $\mathcal{K}$ maps $C([0,T];X)$ into itself; moreover, we can bound the norm of $\mathcal{K}$ by

$$\|\mathcal{K}(v_2)-\mathcal{K}(v_1)\|(t)\le\frac{\omega}{\mu}\,\|(v_2-v_1)(t)\|+(r_\mu*\|v_2-v_1\|)(t)$$

(recall that J_μ is non-expansive). Let us introduce the measure ρ on $\mathbf{R}$ by

$$\rho([0,t])=\frac{\omega}{\mu}+\int_0^t r_\mu(s)\,ds.$$

Then ρ is a completely positive measure; moreover,

$$\|\mathcal{K}^i(v_2)-\mathcal{K}^i(v_1)\|_{L^\infty(0,T)}\le\|v_2-v_1\|_{L^\infty(0,T)}\rho^{*i}([0,T]),$$

where $\rho^{*i}([0,t])=\int_0^t\rho^{*(i-1)}([0,t-s])\,\rho(ds)$.

It holds that

$$\rho^{*n}([0,t])=\sum_{k=0}^n\binom{n}{k}\left(\frac{\omega}{\mu}\right)^{n-k}\|r_\mu^{*k}\|_{L^1(0,T)},$$

and from Lemma 3.9 this goes to zero. □

Let us denote $U(x,f,\mu)$ the solution to (4.1) constructed in Lemma 4.1. Before we establish the convergence of $U(x,f,\mu)$, we proceed to study *a priori* estimates.

Lemma 4.2. *Let* $u_1=U(x_1,f_1,\mu)$ *and* $u_2=U(x_2,f_2,\mu)$ *be two solutions to* (4.1); *then it holds that*

$$\|u_2(t)-u_1(t)\|\le\frac{\omega_\mu}{\omega}\,\|x_2-x_1\|s_{-\omega_\mu}(t)+\frac{\omega_\mu}{\omega}\,\frac{d}{dt}\bigg(\bigg(\frac{1}{\mu}\,\|f_2(\cdot)-f_1(\cdot)\| + (a*\|f_2(\cdot)-f_1(\cdot)\|)\bigg)*s_{-\omega_\mu}\bigg)(t).$$

Proof. It follows from (4.2) and the fact that J_μ is nonexpansive that

$$\|u_2(t)-u_1(t)\|\le s_\mu(t)\|x_2-x_1\|+\frac{1}{\mu}\,\|f_2(t)-f_1(t)\| + \frac{\omega}{\mu}\,\|u_2(t)-u_1(t)\|+\big(r_\mu*\|u_2(\cdot)-u_1(\cdot)\|\big)(t),$$

so we have from Lemma 3.5 that, for every $\mu > \omega$,

$$\|u_2(t)-u_1(t)\| \le \frac{\omega_\mu}{\omega}\,\|x_2-x_1\| s_{-\omega_\mu}(t) + \frac{\omega_\mu}{\omega}\,\frac{d}{dt}\Bigg(\bigg(\frac{1}{\mu}\,\|f_2(\cdot)-f_1(\cdot)\| + (a * \|f_2(\cdot)-f_1(\cdot)\|)\bigg) * s_{-\omega_\mu}\Bigg)(t)$$

where $s_{-\omega_\mu}$ is defined in (3.1) with $\omega_\mu = (\mu\omega/\mu - \omega)$. □

Lemma 4.3. *Assume further that*

$$f(t) \in BV_{\mathrm{loc}}(\mathbf{R}_+; X), \tag{4.4}$$

and let the assumptions of Theorem 2.1 *be satisfied. Then the solution* u_μ *belongs to* $BV_{\mathrm{loc}}(\mathbf{R}_+)$ *and it holds that*

$$\begin{aligned}
&\mathrm{var}\,(\|u_\mu(\cdot) - x\|; [t_1, t_2]) \\
&\quad\le \bigg(\|x\| + \frac{1}{\omega}\,\|G_\mu(x)\| + \frac{1}{\omega}\,\|f(0+)\|\bigg)(s_{-\omega_\mu}(t_2) - s_{-\omega_\mu}(t_1)) \\
&\qquad - \frac{1}{\omega}\int_0^T r_{-\omega_\mu}(s)\,\mathrm{var}\,(f; [\,\max\{0, t_1 - s\}, t_2 - s])\,ds \\
&\qquad + \frac{1}{\mu}\,\mathrm{var}\,(f; [t_1, t_2]).
\end{aligned} \tag{4.5}$$

Proof. For any $h > 0$, from (4.2) it holds that

$$\begin{aligned}
u_\mu(t+h) - u_\mu(t) = J_\mu\bigg(&s_\mu(t+h)x + \frac{\omega}{\mu}\,u_\mu(t+h) + \frac{1}{\mu}\,f(t+h) \\
&+ \int_0^{t+h} u_\mu(t+h-\tau) r_\mu(\tau)\,d\tau\bigg) \\
- J_\mu\bigg(&s_\mu(t)x + \frac{\omega}{\mu}\,u_\mu(t) + \frac{1}{\mu}\,f(t) \\
&+ \int_0^t u_\mu(t-\tau) r_\mu(\tau)\,d\tau\bigg).
\end{aligned}$$

Taking the norm, since J_μ is nonexpansive, we get

$$\begin{aligned}\|u_\mu(t+h)-u_\mu(t)\| \le \frac{\omega}{\mu}\,\|u_\mu(t+h)-u_\mu(t)\| &+ \frac{1}{\mu}\,\|f(t+h)-f(t)\| \\ &+ \int_0^t \|u_\mu(t+h-\tau)-u_\mu(t-\tau)\| r_\mu(\tau)\,d\tau \\ &+ \int_t^{t+h} \|u_\mu(t+h-\tau)-x\| r_\mu(\tau)\,d\tau.\end{aligned}$$

Thanks to Lemma 3.5 we obtain the estimate

$$\|u_\mu(t+h)-u_\mu(t)\| \le \frac{1}{\mu}\,q(\mu,h,t) + (q(\mu,h,\cdot)*a)(t), \tag{4.6}$$

where we set

$$\begin{aligned} &q(\mu,h,t) \\ &\quad= \mu\int_t^{t+h} \|u_\mu(t+h-\tau)-x\| r_\mu(\tau)\,d\tau + \|f(t+h)-f(t)\| \\ &\qquad - \mu\int_0^t \left(\int_s^{s+h} \|u_\mu(s+h-\tau)-x\| r_\mu(\tau)\,d\tau\right) r_{-\omega_\mu}(t-s)\,ds \\ &\qquad - \int_0^t \|f(s+h)-f(s)\| r_{-\omega_\mu}(t-s)\,ds \\ &\quad= \mu\int_t^{t+h} \|u_\mu(t+h-\tau)-x\| \left(r_\mu(\tau) - \int_0^t r_\mu(\tau-s) r_{-\omega_\mu}(s)\,ds\right) d\tau \\ &\qquad + \|f(t+h)-f(t)\| - \int_0^t \|f(s+h)-f(s)\| r_{-\omega_\mu}(t-s)\,ds.\end{aligned}$$

Let us consider next the convolution term which appears in (4.6):

$$\begin{aligned} &(q(\mu,h,\cdot)*a)(t) \\ &\quad= \int_0^t d\vartheta \Bigg[\mu a(t-\vartheta) \int_0^h \|u_\mu(h-\tau)-x\| \\ &\qquad\qquad \times \left(r_\mu(\vartheta+\tau) - \int_0^\vartheta r_\mu(\vartheta+\tau-s) r_{-\omega_\mu}(s)\,ds\right) d\tau\Bigg] \\ &\qquad + \int_0^t \|f(s+h)-f(s)\| a(t-s)\,ds\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_0^t \left[\int_0^\vartheta \|f(s+h) - f(s)\| r_{-\omega_\mu}(\vartheta - s)\, ds \right] a(t-\vartheta)\, d\vartheta \\
&= \mu \int_t^{t+h} d\tau \Bigg[\|u_\mu(t+h-\tau) - x\| \\
&\quad \times \int_0^t a(\vartheta) \left(r_\mu(\tau - \vartheta) - \int_0^{t-\vartheta} r_\mu(\tau - \vartheta - s) r_{-\omega_\mu}(s)\, ds \right) d\vartheta \Bigg] \\
&\quad + \int_0^t \|f(s+h) - f(s)\| \\
&\qquad\qquad \times \left(a(t-s) - \int_0^{t-s} a(t-s-\vartheta) r_{-\omega_\mu}(\vartheta - s)\, d\vartheta \right) ds.
\end{aligned}
$$

Finally, since $r_{-\omega_\mu} \leq 0$, we obtain the following bound

$$
\begin{aligned}
&\|u_\mu(t+h) - u_\mu(t)\| \\
&\leq \left(\sup_{t \in (0,h)} \|u_\mu(t) - x\| \right) \int_t^{t+h} d\tau \Bigg[\left(r_\mu(\tau) + \mu \int_0^t r_\mu(\tau - \vartheta) a(\vartheta)\, d\vartheta \right) \\
&\quad - \left(\int_0^t r_\mu(\tau - s) r_{-\omega_\mu}(s)\, ds + \mu \int_0^t \int_0^{t-\vartheta} r_\mu(\tau - \vartheta - s) r_{-\omega_\mu}(s)\, ds\, a(\vartheta)\, d\vartheta \right) \Bigg] \\
&\quad + \frac{1}{\mu} \left(\|f(t+h) - f(t)\| - \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds \right) \\
&\quad - \frac{1}{\omega_\mu} \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds \\
&\leq \left(\sup_{t \in (0,h)} \|u_\mu(t) - x\| \right) \int_t^{t+h} d\tau \left[\left(r_\mu(\tau) + \mu \int_0^t r_\mu(\tau - \vartheta) a(\vartheta)\, d\vartheta \right) \right. \\
&\qquad \left. - \left(\int_0^t r_{-\omega_\mu}(s) \left(r_\mu(\tau - s) + \mu \int_0^{t-s} r_\mu(\tau - s - \vartheta)\, a(\vartheta)\, d\vartheta \right) ds \right) \right] \\
&\quad + \frac{1}{\mu} \|f(t+h) - f(t)\| - \frac{1}{\omega} \int_0^t \|f(s+h) - f(s)\| r_{-\omega_\mu}(t-s)\, ds
\end{aligned}
$$

Now, we divide the interval $[t_1, t_2]$ in $(N+1)$ intervals of length h, and we compute the variation of $U(x, f, \mu)$ along this partition to get

(4.8)
$$\begin{aligned}
&\sum_{k=0}^{N} \|u_\mu((k+1)h) - u_\mu(kh)\| \\
&\le \Big(\sup_{t\in(0,h)} \|u_\mu(t) - x\|\Big) + \sum_{k=0}^{N} \int_{kh}^{(k+1)h} d\tau \bigg[\Big(r_\mu(\tau) + \mu \int_0^{kh} r_\mu(\tau-\vartheta) a(\vartheta)\, d\vartheta\Big) \\
&\quad - \Big(\int_0^{kh} r_{-\omega_\mu}(s) \Big(r_\mu(\tau - s) + \mu \int_0^{kh-s} r_\mu(\tau - s - \vartheta)\, a(\vartheta)\, d\vartheta\Big)\, ds\Big)\bigg] \\
&\quad + \frac{1}{\mu} \sum_{k=0}^{N} \|f((k+1)h) - f(kh)\| \\
&\quad - \frac{1}{\omega} \sum_{k=0}^{N} \int_0^{kh} \|f((k+1)h - s) - f(kh - s)\| r_{-\omega_\mu}(s)\, ds.
\end{aligned}$$

Now we estimate the expression $\sup_{t\in(0,h)} \|u_\mu(t) - x\|$. Subtracting to both sides of (4.2) $J_\mu(x)$ we have:

$$\begin{aligned}
&u_\mu(t) - J_\mu(x) \\
&\quad = J_\mu\Big(s_\mu(t)x + \frac{\omega}{\mu}\, u_\mu(t) + \frac{1}{\mu}\, f(t) + \int_0^t u_\mu(t-s) r_\mu(s)\, ds\Big) - J_\mu(x),
\end{aligned}$$

then

$$\begin{aligned}
\|u_\mu(t) - J_\mu(x)\| &\le \frac{\omega}{\mu}\, \|u_\mu(t) - x\| \\
&\quad + \frac{\omega}{\mu}\, \|x\| + \frac{1}{\mu}\, \|f(t)\| + \int_0^t \|u_\mu(t-s) - x\| r_\mu(s)\, ds
\end{aligned}$$

and, since $J_\mu(x) - x = (1/\mu) G_\mu(x)$, we have

$$\begin{aligned}
\|u_\mu(t) - x\| &\le \frac{\omega}{\mu}\, \|u_\mu(t) - x\| + \frac{\omega}{\mu}\, \|x\| + \frac{1}{\mu}\, \|f(t)\| + \frac{1}{\mu}\, \|G_\mu(x)\| \\
&\quad + \int_0^t \|u_\mu(t-s) - x\| r_\mu(s)\, ds.
\end{aligned}$$

Using Lemma 3.5, we obtain

$$\begin{aligned}
\|u_\mu(t) - x\| &\le \frac{\omega_\mu}{\omega} \left(\omega\|x\| + \|G_\mu(x)\|\right) \left[\frac{1}{\mu}\, s_{-\omega_\mu}(t) + (a * s_{-\omega_\mu})(t)\right] \\
&\quad + \frac{\omega_\mu}{\omega}\, \frac{d}{dt}\left(\left(\frac{1}{\mu}\, \|f(\cdot)\| + (a * \|f(\cdot)\|)\right) * s_{-\omega_\mu}\right)(t) \\
&= \frac{\omega_\mu}{\omega} \left(\omega\|x\| + \|G_\mu(x)\|\right) \left[\frac{1}{\mu} + \frac{1}{\omega}\left(s_{-\omega_\mu}(t) - 1\right)\right] \\
&\quad + \frac{\omega_\mu}{\omega}\, \frac{d}{dt}\left(\left(\frac{1}{\mu}\, \|f(\cdot)\| + (a * \|f(\cdot)\|)\right) * s_{-\omega_\mu}\right)(t),
\end{aligned}$$

therefore

$$\begin{aligned}
\sup_{t\in(0,h)} \|u_\mu(t) - x\| &\le \frac{\omega_\mu}{\omega} \left(\omega\|x\| + \|G_\mu(x)\|\right) \left[\frac{1}{\mu} + \frac{1}{\omega}\left(s_{-\omega_\mu}(h) - 1\right)\right] \\
&\quad + \frac{\omega_\mu}{\omega} \sup_{t\in(0,h)} \|f(t)\| \left(\frac{1}{\mu} + \int_0^h a(s)\, ds \right. \\
&\qquad \left. - \frac{1}{\mu}\int_0^h r_{-\omega_\mu}(s)\, ds - \int_0^h \left(a * r_{-\omega_\mu}(s)\right) ds\right)
\end{aligned}$$

so

$$\begin{aligned}
\sup_{t\in(0,h)} \|u_\mu(t) - x\| &\le \frac{\omega_\mu}{\omega}\left(\frac{1}{\mu} + \frac{1}{\omega}\left(s_{-\omega_\mu}(h) - 1\right)\right) \\
&\qquad \left[\omega\|x\| + \|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\|\right].
\end{aligned} \tag{4.9}$$

In case $\omega = 0$, the above estimate simplifies to

$$\sup_{t\in(0,h)} \|u_\mu(t) - x\| \le \left(\frac{1}{\mu} + \int_0^h a(s)\, ds\right)\left[\|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\|\right].$$

Letting $h \to 0$, the right-hand side of (4.8) becomes

$$
\begin{aligned}
&\frac{1}{\mu-\omega}\,(\omega\|x\| + \|G_\mu(x)\| + \|f(0+)\|) \\
&\quad \times \int_{t_1}^{t_2} d\tau \left[\left(r_\mu(\tau) + \mu \int_0^\tau r_\mu(\tau-\vartheta) a(\vartheta)\, d\vartheta \right) \right. \\
&\quad\quad \left. - \left(\int_0^\tau r_{-\omega_\mu}(s) \left(r_\mu(\tau-s) + \mu \int_0^{\tau-s} r_\mu(\tau-s-\vartheta)\, a(\vartheta)\, d\vartheta \right) ds \right) \right] \\
&\quad + \frac{1}{\mu}\,\mathrm{var}\,(f;[t_1,t_2]) - \frac{1}{\omega} \int_{t_1}^{t_2} r_{-\omega_\mu}(s)\,\mathrm{var}\,(f;[\max\{0,t_1-s\},t_2-s])\, ds \\
&= \frac{\mu}{\mu-\omega}\,(\omega\|x\| + \|G_\mu(x)\| + \|f(0+)\|) \\
&\quad \times \int_{t_1}^{t_2} d\tau \left[a(\tau) - \left(\int_0^\tau r_{-\omega_\mu}(s) a(\tau-s)\, ds \right) \right] \\
&\quad + \frac{1}{\mu}\,\mathrm{var}(f;[t_1,t_2]) - \frac{1}{\omega} \int_{t_1}^{t_2} r_{-\omega_\mu}(s)\,\mathrm{var}(f;[\max\{0,t_1-s\},t_2-s])\, ds \\
&= \left(\|x\| + \frac{1}{\omega}\,\|G_\mu(x)\| + \frac{1}{\omega}\,\|f(0+)\| \right) \left(- \int_{t_1}^{t_2} r_{-\omega_\mu}(\tau)\, d\tau \right) \\
&\quad + \frac{1}{\mu}\,\mathrm{var}\,(f;[t_1,t_2]) - \frac{1}{\omega} \int_{t_1}^{t_2} r_{-\omega_\mu}(s)\,\mathrm{var}\,(f;[\max\{0,t_1-s\},t_2-s])\, ds \\
&= \left(\|x\| + \frac{1}{\omega}\,\|G_\mu(x)\| + \frac{1}{\omega}\,\|f(0+)\| \right) (s_{-\omega_\mu}(t_2) - s_{-\omega_\mu}(t_1)) \\
&\quad + \frac{1}{\mu}\,\mathrm{var}\,(f;[t_1,t_2]) - \frac{1}{\omega} \int_{t_1}^{t_2} r_{-\omega_\mu}(s)\,\mathrm{var}\,(f;[\max\{0,t_1-s\},t_2-s])\, ds.
\end{aligned}
$$

Therefore, the thesis follows:

$$
\begin{aligned}
&\mathrm{var}(\|u_\mu(\cdot) - x\|;[t_1,t_2]) \\
&\le \left(\|x\| + \frac{1}{\omega}\,\|G_\mu(x)\| \right) (s_{-\omega_\mu}(t_2) - s_{-\omega_\mu}(t_1)) \\
&\quad + \frac{1}{\mu}\,\mathrm{var}\,(f;[t_1,t_2]) - \frac{1}{\omega} \int_0^T r_{-\omega_\mu}(s)\,\mathrm{var}\,(f;[\max\{0,t_1-s\},t_2-s])\, ds.
\end{aligned}
$$

□

Corollary 4.4. *Let the assumptions of Theorem* 2.1 *and* (4.4) *be satisfied. Then it follows from* (4.9) *that*

$$\|U(x,f,\mu)(0+)-x\| \le \frac{1}{\mu-\omega}\Big(\omega\|x\|+\|G_\mu(x)\|+\|f(0+)\|\Big). \tag{4.10}$$

Remark 4.1. We note that, for any $x \in \overline{D(G)}$, the estimate in (4.10) holds and it converges to 0 as $\mu \uparrow \infty$.

In order to prepare the relevant material for the next proof, we recall the following result that is proved in [**8**, Lemma 3.4].

Proposition 4.5. *Assume that* $b \in L^1_{\rm loc}(\mathbf{R}_+)$ *and* $v \in BV_{\rm loc}(\mathbf{R}_+;X)$. *Then the function* $t \mapsto \int_0^t b(t-s)v(s)\,ds$ *is locally absolutely continuous and differentiable almost everywhere on* $\mathbf{R}_+$. *Moreover,*

$$\begin{aligned} \int_0^T \left\| \frac{d}{dt}\int_0^t b(t-s)v(s)\,ds\right\| dt \\ \le \left(\int_0^T |b(t)|\,dt\right)[\|v(0+)\|+\mathrm{var}\,(v;[0,T])]. \end{aligned} \tag{4.11}$$

Lemma 4.6. *Under the additional assumptions* $f \in BV_{\rm loc}(\mathbf{R}_+,X)$, $x \in \widehat{D}(G)$, *we have*

$$\lim_{\mu\to\infty} U(x,f,\mu) \stackrel{\rm def}{=} U(x,f) \tag{4.12}$$

exists in $L^1_{\rm loc}(\mathbf{R}_+;X)$.

Remark 4.2. In the proof of the lemma we shall use the assumption $k_0 = 0$; for the case $k_0 > 0$ a different proof is given in [**8**, Lemma 3.6].

Proof. Using (4.1) with λ and μ, we obtain

$$L_\lambda(u_\mu - x)(t) + G(u_\mu(t)) - f(t) = L_\lambda(u_\mu - x)(t) - L_\mu(u_\mu - x)(t).$$

Setting

$$p(\lambda,\mu,t) = L_\lambda(u_\mu - x)(t) - L_\mu(u_\mu - x)(t), \tag{4.13}$$

and using formula (3.13), we get

$$p(\lambda,\mu,t) = \lambda[u_\mu(t) - (u_\mu * r_\lambda)(t) - s_\lambda(t)x] - f(t) + G(u_\mu(t));$$

hence, u_μ satisfies the equation

$$u_\mu(t) = J_\lambda\left(s_\lambda(t)x + \frac{1}{\lambda}\,f(t) + \frac{1}{\lambda}\,p(\lambda,\mu,t) + \frac{1}{\lambda}\,\omega u_\mu(t) + (r_\lambda * u_\mu)(t)\right).$$

Since J_λ is nonexpansive, this equation combined with (4.2) implies

$$\begin{aligned}\|u_\lambda(t) - u_\mu(t)\| \le \frac{1}{\lambda}\,&\|p(\lambda,\mu,t)\|\\ &+ \frac{1}{\lambda}\,\omega\|u_\lambda(t) - u_\mu(t)\| + \big(r_\lambda * \|u_\lambda(\cdot) - u_\mu(\cdot)\|\big)(t).\end{aligned}$$

Using Lemma 3.5 we obtain

$$\|u_\lambda(t) - u_\mu(t)\| \le \frac{\omega_\lambda}{\omega}\,\frac{d}{dt}\left(\left(\frac{1}{\lambda}\,\|p(\lambda,\mu,\cdot)\| + a * \|p(\lambda,\mu,\cdot)\|\right) * s_{-\omega_\lambda}\right)(t),$$

that in another form we can write

$$\|u_\lambda(t) - u_\mu(t)\| \le \frac{1}{\lambda-\omega}\,\|p(\lambda,\mu,t)\| - \frac{\lambda}{\omega(\lambda-\omega)}\,(\|p(\lambda,\mu,\cdot)\| * r_{-\omega_\lambda})(t).$$

We now proceed to prove that $p(\lambda,\mu,\cdot)$ converges to 0 as $\lambda,\mu \to \infty$ in $L^1_{\mathrm{loc}}(\mathbf{R}_+;X)$. Recall that $p(\lambda,\mu,\cdot)$ is defined by (4.13); then, by (3.14),

$$p(\lambda,\mu,t) = \frac{d}{dt}\int_0^t (u_\mu(\tau) - x)(\lambda s_\lambda(t-\tau) - \mu s_\mu(t-\tau))\,d\tau.$$

By formula (4.11), we obtain

$$\int_0^T \|p(\lambda,\mu,t)\|\,dt \le \operatorname{var}\left(\|u_\mu(\cdot) - x\|;[0,T]\right)\int_0^T |\lambda s_\lambda(t) - \mu s_\mu(t)|\,dt.$$

Since the variation of $\|u_\mu(\cdot) - x\|$ is bounded by a constant for μ large enough, compare Remark 4.1, and the integral tends to 0 by Proposition 3.4, we have the thesis. □

We conclude the preparatory material for the proof of Theorem 2.1 with the following theorem, where we collect some useful estimates for the solution of problem (2.1).

Theorem 4.7. *Let $x_i \in \widehat{D}(G)$ and $f_i \in C(\mathbf{R}_+; X)$ for $i = 1, 2$, and let $u_i = U(x_i, f_i)$ be the generalized solutions of equation defined in Theorem 2.1. Then we have, for each $t > 0$ and $h > 0$,*

$$\|u_2(t) - u_1(t)\| \le \|x_2 - x_1\| s_{-\omega}(t) - \frac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t); \tag{4.14}$$

$$\sup_{t\in(0,h)} \|u(t) - x\| \le (\omega(s_{-\omega}(h) - 1)) \left(\omega\|x\| + \sup_{\mu>0} \|G_\mu(x)\| + \sup_{t\in(0,h)} \|f(t)\|\right). \tag{4.15}$$

Proof. Notice first that (4.15) was already proved in Lemma 4.3, see formula (4.9).

For the proof of (4.14), let for $i = 1, 2$, $u_i(\mu; \cdot) = U(x_i, f_i, \mu)$. Now, observe that

$$\|u_2(t) - u_1(t)\| \le \|u_2(t) - u_2(\mu; t)\| + \|u_1(t) - u_1(\mu; t)\| + \|u_2(\mu; t) - u_1(\mu; t)\|.$$

Since (4.3) holds for any $\mu > 0$, while $u_i(\mu; t) \to u_i(t)$ for $i = 1, 2$ and for any $t > 0$, it follows from the previous estimate that

$$\|u_2(t) - u_1(t)\| \le \liminf_{\mu\to\infty} \|u_2(\mu; t) - u_1(\mu; t)\|.$$

It remains to evaluate the right-hand side of the previous estimate where we get, using (4.3):

$$\liminf_{\mu\to\infty} \|u_2(\mu; t) - u_1(\mu; t)\| \le \|x_2 - x_1\| s_{-\omega}(t) + \frac{d}{dt}\Big(\big(a * \|f_2(\cdot) - f_1(\cdot)\|\big) * s_{-\omega}\Big)(t).$$

Now since

$$\frac{d}{dt}\Big(\big(a * \|f_2(\cdot) - f_1(\cdot)\|\big) * s_{-\omega}\Big)(t) = -\frac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t)$$

we finally obtain

$$\|u_2(t) - u_1(t)\| \le \|x_2 - x_1\| s_{-\omega}(t) - \frac{1}{\omega}\Big(r_{-\omega} * \|f_2(\cdot) - f_1(\cdot)\|\Big)(t). \qquad \square$$

We are in a position to conclude the proof of Theorem 2.1. Under the additional assumptions $f \in BV_{\mathrm{loc}}(\mathbf{R}_+, X)$, $x \in \widehat{D}(G)$, we obtain the convergence of $U(x, f, \mu)$ toward $U(x, f)$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; X)$ and the continuity of the limit function via an Ascoli-Arzelà theorem, by invoking the equicontinuity of the functions $U(x, f, \mu)$ that follows from Lemma 4.3. Then it follows from Remark 4.1 and Corollary 4.4 that $U(x, f) \in BV_{\mathrm{loc}}(\mathbf{R}_+; X)$ and

$$U(x, f)(0+) = x.$$

Now it follows from Theorem 4.7 that $U(x, f, \mu)$ converges to $U(x, f)$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+; X)$ to a continuous function also in the case that f and x satisfy the assumptions of Theorem 2.1.

Remark 4.3. From the proof of the Lemma 4.3, compare (4.7), we obtain, for $\omega = 0$, the estimate

$$\|u(t+h) - u(t)\| \le \int_0^t \|f(t+h-s) - f(t-s)\| a(s)\, ds + \Big(\sup_{\mu > 0} \|G_\mu(x)\| + \sup_{s \in (0,h)} \|f(s)\|\Big) \int_t^{t+h} a(s)\, ds,$$

for each $t > 0$ and $h > 0$. This is the same formula that is proved in [**3**]. Using this result, Gripenberg et al. [**9**] were able to prove the existence of a strong solution for (2.1).

A similar estimate, up to now, does not seem to hold for $\omega \neq 0$; we hope to return to this problem in a future work.

5. Lipschitz nonlinearity. In this section, we shall prove the results stated in Theorem 2.2.

Let us define the mapping $\mathcal{H} : C(\mathbf{R}_+; X) \to C(\mathbf{R}_+; X)$

$$\mathcal{H}(u) = U(x, F(\cdot, u)); \tag{5.1}$$

then a generalized solution to equation (1.1) is a function u such that

$$u = \mathcal{H}(u).$$

We can achieve the existence of the solution from a fixed point theorem if we prove that some iterate of $\mathcal{H}$ is contractive. For this purpose we need the following lemma.

Lemma 5.1. *For each $T > 0$ there exists $k \geq 1$ such that the k-iterate of $\mathcal{H}$ is a contraction*:

$$\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \leq \varepsilon \|u - v\|_{L^\infty(0,T;X)}$$

for some $\varepsilon < 1$.

Proof. From (4.14) and (2.4) we have

$$\begin{aligned}\|\mathcal{H}(u)(t) - \mathcal{H}(v)(t)\| &\leq \frac{1}{\omega} \int_0^t (-r_{-\omega}(t-s)) \|F(s, u(s)) - F(s, v(s))\| \, ds \\ &\leq \frac{1}{\omega} \int_0^t (-r_{-\omega}(t-s)) \, \eta(s) \|u(s) - v(s)\| \, ds.\end{aligned}$$

Iterating this procedure we have

$$\begin{aligned}&\|\mathcal{H}^k(u)(t) - \mathcal{H}^k(v)(t)\| \\ &\quad \leq \frac{1}{\omega^k} \int_0^t (-r_{-\omega}(t - x_1)) \, \eta(x_1) \int_0^{x_1} (-r_{-\omega}(x_1 - x_2)) \, \eta(x_2) \cdots \\ &\qquad \cdots \int_0^{x_{k-1}} (-r_{-\omega}(x_{k-1} - x_k)) \, \eta(x_k) \|u(x_k) - v(x_k)\| \, dx_k \cdots dx_2 \, dx_1.\end{aligned}$$

Then

$$\begin{aligned}&\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \\ &\quad \leq \|u - v\|_{L^\infty(0,T;X)} \omega^{-k} \, \|\eta\|^k_{L^\infty(0,T)} \\ &\qquad \int_0^T \int_0^{x_1} \cdots \int_0^{x_{k-1}} (-r_{-\omega}(T - x_1)) \cdots (-r_{-\omega}(x_{k-1} - x_k)) \, dx_k \cdots dx_1,\end{aligned}$$

but, by a repeated use of Fubini's theorem, we have

$$\|\mathcal{H}^k(u) - \mathcal{H}^k(v)\|_{L^\infty(0,T;X)} \\ \le \|u - v\|_{L^\infty(0,T;X)} \omega^{-k} \|\eta\|^k_{L^\infty(0,T)} \|(-r_{-\omega})^{*k}\|_{L^1(0,T)}.$$

Finally, by Lemma 3.8, we have that the right-hand side converges to zero, so for sufficiently large k we have the lemma. □

As stated before, this lemma provides the proof of Theorem 2.2. We insist on the following interpretation.

Remark 5.1. Let $u = U(x, F(\cdot, u))$ be a generalized solution to (1.1): then, by definition, this means that there exists a sequence u_μ such that

$$L_\mu(u_\mu - x)(t) + G(u_\mu(t)) = F(t, u(t))$$

and $u_\mu \to u$ in $L^\infty_{\text{loc}}(\mathbf{R}_+; X)$.

6. Dissipative nonlinearity. This section is devoted to prove the results stated in Theorem 1.1, along the lines of the proof of Theorem 7.13 of [**6**]. Let us introduce, for any $\alpha > 0$, the approximating equation

$$L(u_\alpha - x)(t) + G(u_\alpha(t)) = F_\alpha(t, u_\alpha(t)), \tag{6.1}$$

where $F_\alpha(t, \cdot)$ are the Yosida approximations of $F(t, \cdot)$. We denote with $J^{F,t}_\alpha(\cdot)$ the resolvent operators associated to $F(t, \cdot)$.

Let us recall that F_α is Lipschitz continuous; moreover, for any $x, y \in X$ and $x^* \in \partial\|x\|$,

$$\langle F_\alpha(t, x + y), x^*\rangle = \langle F_\alpha(t, x + y) - F_\alpha(t, y), x^*\rangle + \langle F_\alpha(t, y), x^*\rangle \\ \le \langle F_\alpha(t, y), x^*\rangle \le \|F(t, y)\|.$$

From Theorem 2.2 we know that there exists a generalized solution u_α to equation (6.1). Then, there exist sequences $u_{\alpha,\mu}$ and $\delta_{\alpha,\mu}$

$$\delta_{\alpha,\mu} = L_\mu(u_{\alpha,\mu} - x)(t) + G(u_{\alpha,\mu}(t)) - F_\alpha(t, u_{\alpha,\mu}(t))$$

such that

$$\left.\begin{matrix} u_{\alpha,\mu} \to u_\alpha \\ \delta_{\alpha,\mu} \to 0 \end{matrix}\right\} \quad \text{in} \quad L^\infty_{\text{loc}}(\mathbf{R}_+; X).$$

Now, let $y \in D(G)$, then for some $y^* \in \partial\|u_{\alpha,\mu}(t) - y\|$ we get, from

$$\langle L_\mu(u_{\alpha,\mu}(t) - y), y^*\rangle - \langle L_\mu(x - y), y^*\rangle + \langle G(u_{\alpha,\mu}(t)) - G(y), y^*\rangle \\ + \langle G(y), y^*\rangle - \langle F_\alpha(t, u_{\alpha,\mu}(t)), y^*\rangle = \langle \delta_{\alpha,\mu}, y^*\rangle$$

the estimate

$$\mu\Big(\|u_{\alpha,\mu}(t) - y\| - \big(\|u_{\alpha,\mu}(\cdot) - y\| * r_\mu\big)(t)\Big) \\ \leq \omega\|u_{\alpha,\mu}(t) - y\| + s_\mu(t)\|x - y\| + \|G(y)\| + \|F(t, y)\| + \|\delta_{\alpha,\mu}\|.$$

Lemma 3.5 now implies

$$\|u_{\alpha,\mu}(t) - y\| \leq \frac{\omega_\mu}{\omega}\,\frac{d}{dt}\Bigg(\|x - y\| + \Bigg(\frac{1}{\mu}\,[\|G(y)\| + \|F(\cdot, y)\| + \|\delta_{\alpha,\mu}\|] \\ + \big(a * [\|G(y)\| + \|F(\cdot, y)\| + \|\delta_{\alpha,\mu}\|]\big)\Bigg) * s_{-\omega_\mu}\Bigg)(t),$$

and passing to the limit as $\mu \to \infty$, we get

$$\|u_\alpha(t) - y\| \leq \frac{d}{dt}\Big(\big(\|x - y\| + a * [\|G(y)\| + \|F(\cdot, y)\|]\big) * s_{-\omega}\Big)(t).$$

We can simplify this expression. If we consider separately the case $\omega = 0$, then the estimate (6.2) has the simpler form

$$\|u_\alpha(t) - y\| \leq \|x - y\| + (a * [\|G(y)\| + \|F(\cdot, y)\|])\,(t).$$

In the general case $\omega \neq 0$, we get

$$\|u_\alpha(t) - y\| \leq s_{-\omega}(t)\,\|x - y\| - \frac{1}{\omega}\,(r_{-\omega} * [\|G(y)\| + \|F(\cdot, y)\|])\,(t).$$

This tells us that the sequence $\{u_\alpha(\cdot)\}$ is bounded uniformly on bounded sets.

To show the convergence of the sequence, we set, for any $\alpha, \beta > 0$,

$$g^{\alpha,\beta}(t) = u_\alpha(t) - u_\beta(t).$$

Let us consider the functions $g_\mu^{\alpha,\beta} = u_{\alpha,\mu} - u_{\beta,\mu}$, where $u_{\alpha,\mu}$ and $u_{\beta,\mu}$ are the approximating functions solving

$$\begin{aligned} L_\mu(u_{\alpha,\mu} - x)(t) + G(u_{\alpha,\mu}(t)) &= F_\alpha(t, u_\alpha(t)) \\ L_\mu(u_{\beta,\mu} - x)(t) + G(u_{\beta,\mu}(t)) &= F_\beta(t, u_\beta(t)), \end{aligned}$$

respectively; moreover, we have that

$$u_{\alpha,\mu} \longrightarrow u_\alpha \quad \text{and} \quad u_{\beta,\mu} \longrightarrow u_\beta$$

in $L^\infty_{\text{loc}}(\mathbf{R}_+; X)$. Then $g^{\alpha,\beta}$ shall be a generalized solution to the problem

$$Lg^{\alpha,\beta}(t) + G(u_\alpha(t)) - G(u_\beta(t)) = F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)).$$

Now we have, for $y^* \in \partial\|g_\mu^{\alpha,\beta}(t)\|$,

$$\begin{aligned} \left\langle L_\mu g^{\alpha,\beta}(t), y^*\right\rangle + \langle G(u_{\alpha,\mu}(t)) - G(u_{\beta,\mu}(t)), y^*\rangle \\ = \langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle, \end{aligned}$$

which becomes, thanks to (3.14):

$$\begin{aligned} \mu\left(\|g_\mu^{\alpha,\beta}(t)\| - (\|g_\mu^{\alpha,\beta}\| * r_\mu)(t)\right) - \omega\|g_\mu^{\alpha,\beta}(t)\| \\ \le \langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle\,. \end{aligned} \tag{6.3}$$

Let us notice that

$$\begin{aligned} \langle F_\alpha(t, u_\alpha(t)) - F_\beta(t, u_\beta(t)), y^*\rangle &\le \langle F(t, u_{\alpha,\mu}(t)) - F(t, u_{\beta,\mu}(t)), y^*\rangle \\ &\quad + \langle F(t, J_\alpha^{F,t}(u_\alpha(t))) - F(t, u_{\alpha,\mu}(t)) \\ &\quad + F(t, u_{\beta,\mu}(t)) - F(t, J_\beta^{F,t}(u_\beta(t))), y^*\rangle \\ &\le \|F(t, J_\alpha^{F,t}(u_\alpha(t))) - F(t, u_{\alpha,\mu}(t))\| \\ &\quad + \|F(t, J_\beta^{F,t}(u_\beta(t))) - F(t, u_{\beta,\mu}(t))\| \end{aligned}$$

Now, by (6.2) and recalling that F is uniformly bounded on bounded subsets of $\mathbf{R}_+ \times X$, for a fixed $T > 0$, there exists $R > 0$ such that

$$\|u_\alpha(t)\| \le R \quad \text{and} \quad \|F(t, u_\alpha(t))\| \le 2R \quad \text{for all} \quad t \in [0, T],$$

for all α. Then we have

$$\|u_\alpha(t) - J_\alpha^{F,t}(u_\alpha(t))\| \le \frac{1}{\alpha}\, \|F_\alpha(t, u_\alpha(t))\| \le \frac{2}{\alpha}\, R;$$

and, for μ sufficiently large,

$$\|u_{\alpha,\mu}(t)\| \le R \quad \text{for all} \quad t \in [0.T];$$

so we have

$$\|u_\alpha(t) - u_{\alpha,\mu}(t)\| \le 2R, \quad \text{for all} \quad t \in [0.T].$$

Therefore, it follows

$$\begin{aligned}
&\|F(t, J_\alpha^{F,t}(u_\alpha(t))) - F(t, u_{\alpha,\mu}(t))\| \\
&\le \|F(t, J_\alpha^{F,t}(u_\alpha(t))) - F(t, u_\alpha(t))\| + \|F(t, u_\alpha(t)) - F(t, u_{\alpha,\mu}(t))\| \\
&\le \rho_F\left(\frac{2}{\alpha}\, R\right) + \rho_F(\|u_\alpha(t) - u_{\alpha,\mu}(t)\|),
\end{aligned}$$

where ρ_F is the modulus of continuity of $F(t,\cdot)$ restricted to $[0,T] \times B(0,2R)$, i.e., a function such that $\rho_F(s) = \sup\{\|F(t,x_1) - F(t,x_2)\| : t \in [0,T],\ x_1, x_2 \in B(0,2R),\ \|x_1 - x_2\| \le s\}$.

The above construction, starting from (6.3), leads to

$$\begin{aligned}
\mu\Big(\|g_\mu^{\alpha,\beta}(t)\| - (\|g_\mu^{\alpha,\beta}\| * r_\mu)(t)\Big) - \omega\|g_\mu^{\alpha,\beta}(t)\| \\
\le \rho_F\left(\frac{2}{\alpha}\, R\right) + \rho_F\left(\frac{2}{\beta}\, R\right) + \rho_F(\varepsilon_{\alpha,\mu}) + \rho_F(\varepsilon_{\beta,\mu}),
\end{aligned}$$

where

$$\begin{aligned}
\varepsilon_{\alpha,\mu} &= \sup_{t\in[0,T]} \|u_\alpha(t) - u_{\alpha,\mu}(t)\| \le 2R, \\
\varepsilon_{\beta,\mu} &= \sup_{t\in[0,T]} \|u_\beta(t) - u_{\beta,\mu}(t)\| \le 2R.
\end{aligned}$$

Lemma 3.5 now implies

$$\begin{aligned}
\|g_\mu^{\alpha,\beta}(t)\| \le \frac{\omega_\mu}{\omega}\left[\rho_F\left(\frac{2}{\alpha}\, R\right) + \rho_F\left(\frac{2}{\beta}\, R\right) + \rho_F(\varepsilon_{\alpha,\mu}) + \rho_F(\varepsilon_{\beta,\mu})\right] \\
\left(\frac{1}{\mu}\, s_{-\omega_\mu}(t) + \big(a * s_{-\omega_\mu}\big)(t)\right).
\end{aligned}$$

From the above inequality, as we pass to the limit for $\mu \to \infty$, we have

$$\|g^{\alpha,\beta}(t)\| \le \left[\rho_F\left(\frac{2}{\alpha}\,R\right) + \rho_F\left(\frac{2}{\beta}\,R\right)\right]\big(a * s_{-\omega}\big)(t).$$

This yields the convergence of the sequence u_α in $L^\infty_{\mathrm{loc}}(\mathbf{R}_+;X)$ to a function $u \in C(\mathbf{R}_+;X)$, which is easily seen to be a generalized solution to (1.1). The remainder of the proof now follows as in Da Prato and Zabczyk [**6**].

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