JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 18, Number 2, Summer 2006

ON INTEGRAL EQUATION FORMULATIONS OF A CLASS OF EVOLUTIONARY EQUATIONS WITH TIME-LAG

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Dedicated to Kendall Atkinson

ABSTRACT. In discussions of certain neutral delay differential equations in Hale's form, the relationship of the original problem with an integrated form (an integral equation) proves to be helpful in considering existence and uniqueness of a solution and sensitivity to initial data. Although the theory is generally based on the assumption that a solution is continuous, natural solutions of neutral delay differential equations of the type considered may be discontinuous. This difficulty is resolved by relating the discontinuous solution to its restrictions on appropriate (half-open) subintervals where they are continuous and can be regarded as solutions of related integral equations. Existence and unicity theories then follow. Furthermore, it is seen that the discontinuous solutions can be regarded as solutions in the sense of Carathéodory (where this concept is adapted from the theory of ordinary differential equations, recast as integral equations).

1. The forms of integral equation considered. The *integral equations* discussed in this paper are in the form

$$y(t) = g\big(t, y(t), y(t-\tau(t))\big) + \int_{t_0}^{t} f\big(s, y(s), y(s-\tau(s))\big)\mathrm{d}s + z^0 \tag{1.1}$$

or the form

$$y(t) = \gamma\Big(t, y(t-\tau(t)), \int_{t_0}^{t} f\big(s, y(s), y(s-\tau(s))\big)\mathrm{d}s + z^0\Big). \tag{1.2}$$

In either case, the equation holds for $t \in \mathfrak{I}_0$ where $\mathfrak{I}_0$ is $[t_0, T]$ or $[t_0, T)$ (for $t_0 < T \in \mathbf{R} \cup \infty$), and $y(t)$ is prescribed on a suitable initial interval $[t_{-1}, t_0] \subset (-\infty, t_0]$. The situations considered can

Received by the editors on June 1, 2005, and in revised form on September 13, 2005.

give rise to discontinuous solutions. We summarize the main results in subsection 3.1.

2. Related evolutionary equations. We are motivated by the use of equations with time-lag (neutral delay differential equations, "NDDEs") in certain mathematical models. The equations that lead us to (1.1) and (1.2) are of the form

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\big\{y(t)-g\big(t,y(t),y(t-\tau(t))\big)\big\}=f\big(t,y(t),y(t-\tau(t))\big), \tag{2.1a}$$

for $t \in \mathfrak{I}_0$. The special case $y'(t) = f\big(t,y(t),y(t-\tau(t))\big)$ is a delay differential equation, or DDE.

With appropriate assumptions, see subsection 3.2, a particular solution $y(t) \equiv y(\varphi,\tau;t)$ is defined by (2.1a) together with

$$y(t)=\varphi(t)\ (t\in[t_{-1},t_0],\ t_{-1}:=\inf_{t\in\mathfrak{I}_0} t-\tau(t)\in(-\infty,t_0)). \tag{2.1b}$$

We write $[t_{-1},t_0]\cup\mathfrak{I}_0$ as $\mathfrak{I}_{-1}$; we regard $y(\varphi,\tau;t)$ as defined for $t\in\mathfrak{I}_{-1}$.

If $g(t,u,v)$ is not independent of v, e.g., if $\{\partial/\partial v\}g(t,u,v)$ exists and does not vanish identically, this is an example of a form of NDDE often called "Hale's form," see [**10**, Chapter 12], [**14**, pp. 9, 118–120].

Throughout, we regard t as representing 'time,' and, for convenience, consider problems (2.1) with a single time-dependent "lag" $\tau(t)$.

We seek a solution $y(t)\in\mathbf{R}^n$, given $\varphi(t)\in\mathbf{R}^n$ for $t\in[t_{-1},t_0]$ and appropriate functions $f,g:\mathfrak{I}_0\times\mathbf{R}^n\times\mathbf{R}^n\to\mathbf{R}^n$.

Example 2.1. An example of (2.1) taken in [**2**] to illustrate the class of problems reads $(d/dt)\{y(t)-y(t-1)\}=y(t-1)$ (for $t\in\mathfrak{I}_0$, taking a right-hand derivative); $\varphi(t)=t$ $(t\in[-1,0))$, $\varphi(0)=1$. Expressions for a solution on $[0,1]$, $[1,2]$, $[2,3]$ and $[3,4]$ are given in [**2**].

Regarding (2.1), we note that (i) the existence of a one- or two-sided derivative

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\big\{y(t)-g\big(t,y(t),y(t-\tau(t))\big)\big\}$$

does not imply the continuity of $y(t)$ and that (ii) $y(t)$ inherits discontinuities, for certain $t \geq t_0$, from discontinuities at earlier times, through dependency on $y(t-\tau(t))$. Related discussions in the literature refer almost exclusively to continuous, or even differentiable, solutions of (2.1). For some pathologies see [**5, 9** and their citations].

By way of illustration, suppose that $y(\varphi, \tau; t)$ is actually continuous in t for $t \in \mathfrak{I}_{-1}$(with $y(\varphi, \tau; t) = \varphi(t)$ for $t \in [t_{-1}, t_0]$), and that $f(t, u, v)$ and $g(t, u, v)$ are continuous for $t \in \mathfrak{I}_0$, $|u|, |v| < \infty$. Then if $y(\varphi, \tau; t)$ satisfies the integral equation (1.1) where the integral is interpreted in the sense of Riemann, it follows that

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right)\left\{y(t) - g\big(t, y(t), y(t-\tau(t))\big)\right\} = f\big(t, y(t), y(t-\tau(t))\big). \tag{2.2}$$

However, $y'(t)$ will not exist for arbitrary $g(t, u, v)$.

Our remarks have been concentrated on the "implicit" form of NDDE, (2.2). We also refer to the problem

$$y'(t) = f_*\big(t, y(t), y(t-\tau), y'(t-\tau)\big), \quad t \geq t_0, \tag{2.3a}$$

$$y(t) = \varphi(t), \quad t \in [t_{-1}, t_0] \tag{2.3b}$$

which is an *explicit* form of NDDE. Bellen and Zennaro [**4**, p. 5] note that, unless the initial function φ satisfies an appropriate condition[1] at t_0, the solution of (2.3) "remains solely of class C^0 and the solution of (2.3) must be understood in the 'almost everywhere' generalized sense."

Example 2.2. Consider the problem of determining a function y that has a derivative for almost all $t \geq 0$ and satisfies (cf. Example 2.1)

$$y'(t) = y'(t-1) + y(t-1) \quad \text{for almost all} \quad t \in \mathfrak{I}_0 \tag{2.4a}$$

$$y(t) = \varphi(t) \quad \text{for all} \quad t \in [-1, 0]. \tag{2.4b}$$

Suppose that $y_\sharp$ satisfies (2.4) for $T = N$ (where $N \in \mathbf{N}$), and suppose that $c \in \mathbf{R}$ is arbitrary. Then the function y_c with $y_c(t) = y_\sharp(t)$ for $t \in [0, N-1)$ and $y_c(t) = y_\sharp(t) + c$ for $t \in [N-1, N]$, also satisfies (2.4) for $T = N$. Observe that a function that has a derivative for almost all t need not be continuous for all t.

The preceding and similar remarks suggest that one may ask what interpretation of the problem (2.1) is appropriate, and, allied to this, in what sense we seek a solution of (2.1), and whether a 'solution' is then unique. Integral equation formulations, of the type (1.1), play a rôle when we answer such questions, below.

The results presented here require only standard analysis. Some of the insight on which we rely comes from the special case $y(t) = y(\zeta_\ell) + \int_{\zeta_\ell}^{t} f(s, y(s))\mathrm{d}s$, $t \in [\zeta_\ell, \zeta_{\ell+1})$, in which $y(t)$ has a derivative on $[\zeta_\ell, \zeta_{\ell+1})$. Additional insight comes from the "method of steps" used in the discussion of certain DDEs [**13, 14**]; however, the transition from DDEs to NDDEs in Hale's form involves some complications.

2.1 Supplementary remarks. We conclude this section with two additional remarks.

Remark 2.3. Liu [**15**] considers a simplified form of (2.2):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\left\{y(t) - g_*\big(t, y(t-\tau(t))\big)\right\} = f\big(t, y(t), y(t-\tau(t))\big), \quad t \in \Im_0.$$

If $y'(t)$ exists, this equation may be converted to explicit form $y'(t) = f_*\big(t, y(t), y(t-\tau(t)), y'(t-\tau)\big)$ by differentiating $g_*\big(t, y(t-\tau(t))\big)$, assuming the derivatives of τ and g_* are available. However, $y'(t)$ exists for $t \geq t_0$ only for a restricted class of initial functions φ.

Remark 2.4. As an illuminating diversion, we note that Hale's form (2.1) of the NDDE problem can be expressed as a semi-explicit constrained delay differential equation (CDDE), see [**3** and references therein]. Consider, for $t \in \Im_0$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z(t) = f(t, y(t), y(t-\tau(t))), \tag{2.5a}$$

$$z(t) = y(t) - g(t, y(t), y(t-\tau(t))) \tag{2.5b}$$

with $y(t) = \varphi(t)$ for $t \in [t_{-1}, t_0]$. This formulation is completely equivalent to (2.1); however, it will be recognized as having the form of a CDDE—or "delay differential algebraic equation" (DDAE) [**1**]. (The terminology DDAE has some appeal if the function $g(t,u,v)$ is an algebraic function of its arguments u and v.) The equation (2.5b) can be regarded as a constraint.

If $f(t,u,v) = \mathrm{f}(t,u)$ and $g(t,u,v) = \mathrm{g}(t,u)$, both independent of v, the problem (2.5) becomes a "differential algebraic equation" (DAE) [**1, 18**]. It has been observed that "DAEs are not ODEs" [**17**].

Extending the observation in Liu [**15**], there is a variant of (2.5) in which the problem is written in the completely equivalent form

$$\left(\frac{\mathrm{d}}{\mathrm{dt}}\right) z(t) = f\Big(t, z(t) + g\big(t, y(t), y(t-\tau(t))\big),\, y(t-\tau(t))\Big), \tag{2.6a}$$

$$z(t) = y(t) - g\big(t, y(t), y(t-\tau(t))\big), \tag{2.6b}$$

for $t \in \mathfrak{I}_0$, with $y(t) = \varphi(t)$ for $t \in [t_{-1}, t_0]$.

Our discussion will establish, in part, the extent to which one can claim that DAEs, DDEs, and constrained DDEs and NDDEs, can be construed as *integral equations*.

3. Concerning the theory for NDDEs. This section is comprised of the following: subsection 3.1, the principal results; subsection 3.2, formal assumptions and subsection 3.3, initial observations. The remainder of the paper is then divided as follows: Section 4, the integral equation formulation; subsection 4.1, similar problems in ODEs; subsection 4.2, a method of steps; subsection 4.3, extension of Peano's theorem to the NDDE; subsections 4.3.1–4.3.2, three lemmas and a theorem; subsection 4.3.3, equicontinuity and uniform boundedness; subsection 4.4, the extension of Picard iterations to the NDDE; subsection 4.5, sensitivity to initial data; subsection 4.6, solutions in the sense of Carathéodory; Section 5, conclusions; and, finally, references to the literature.

3.1 The principal results. Any solution $y(t) \equiv y(\varphi, \tau; t)$ of (2.1) derives its properties from τ and φ (and f, g). Two definitions of the term 'solution' are given: in Definition 3.14 (a 'natural solution', based on an interpretation of derivatives as right-hand derivatives) and in Definition 4.9 (a solution in the sense of Carathéodory). We use assumptions, collected in subsection 3.2, concerning our class of problems. The reader is asked to provide details of the proofs of lemmata, except Lemma 4.4 for which a reference is cited. The approach adopted differs from that which is conventional in existing monographs, e.g., [**11, 13, 14**]. The principal results are: (a) Theorems 4.5 and 4.6 on the existence of a natural solution; (b) Theorem 4.6 on the uniqueness of a natural solution; (c) Theorem 4.8 relating to the dependence of a natural solution on the original data; (d) Theorem 4.11 relating natural solutions to solutions in the sense of Carathéodory.

3.2 Formal assumptions.

Assumption 3.1. *Throughout,* $\tau \in C(\mathfrak{I}_0)$*,* $0 < \tau_* = \inf_{t\in\mathfrak{I}_0} \tau(t)$*,* $-\infty < t_{-1} = \inf_{t\in\mathfrak{I}_0}\{t - \tau(t)\} < t_0$*, and* $\sup_{t\in\mathfrak{I}_0} \tau(t) \le \tau^* < \infty$*.*

Assumption 3.2. *The initial function* φ *is continuous at points in* $[t_{-1}, t_0]$ *with the exception of a finite set of* $(R+1)$ *ordered points*

$$\zeta_{-R} < \zeta_{-R+1} < \cdots < \zeta_{-1} < \zeta_0 \tag{3.1}$$

in $(t_{-1}, t_0]$ *where it is continuous from the right but suffers bounded jump discontinuities* $\varphi(\zeta_\ell) - \varphi(\zeta_\ell-)$*,* $\ell \in \{-R, 1-R, \dots, -1, 0\}$*. For convenience, we assume* $\zeta_0 = t_0$*.*

The next assumption imposes a further restriction on $\tau(t)$.

Assumption 3.3. *There exists a monotone strictly increasing sequence* $\{\zeta_1, \zeta_2, \zeta_3 \dots\}$ *that has no finite point of accumulation in* $\overline{\mathfrak{I}_0}$*, and a map* $\ell \to k_\ell$ *for* $\ell \in \{0, 1, 2, \dots\}$*, with* $k_\ell < \ell - 1$*, such that*

$$t - \tau(t) \in [\zeta_{k_\ell}, \zeta_{k_\ell+1}) \quad \textit{when} \quad t \in [\zeta_\ell, \zeta_{\ell+1}). \tag{3.2}$$

If $T \notin \{\zeta_\ell\}$, then it should be added to $\{\zeta_\ell\}$ (as $\zeta_{\ell_{\max}+1}$). For $\ell \in \{0, 1, 2, \cdots\}$, we use the *notation*

$$z^\ell := y(\zeta_\ell) - g\big(\zeta_\ell, y(\zeta_\ell), y(\zeta_\ell - \tau(\zeta_\ell))\big). \tag{3.3}$$

Certain of our conclusions extend to problems with multiple time-lags. Note that we can replace Assumption 3.3 by the following stronger hypothesis, which (with Assumption 3.1) guarantees its validity.

Assumption 3.4. *We assume that $t-\tau(t)$ is monotonic increasing.*

We request that the functions f, g satisfy the following.

Assumption 3.5. *We suppose the functions f, g assuming values $f(t,u,v)$ and $g(t,u,v)$ are continuous for $t \in \Im_0$ and $\|u\|, \|v\| < \infty$.*

As an additional (weak) condition on g, we assume:

Assumption 3.6. *There is a function $\gamma(t,v,w)$ that is continuous in v and w for all $t \in \Im_0$, such that if $u = \gamma(t,v,w)$ then u satisfies the equation $u = g(t,u,v) + w$.*

Assumption 3.6 can be viewed as an *index* 1 *condition*, to evoke the terminology for DAEs. Assumption 3.6 does not imply uniqueness of γ and to show uniqueness of $y(t)$ we strengthen the assumption:

Assumption 3.7. *There exists a unique function $\gamma(t,v,w)$ such that if $u = g(t,u,v) + w$ then $u = \gamma(t,v,w)$ (for arbitrary $t \in \Im_0$ and $u, v, w \in \mathbf{R}^n$), where $\gamma(t,v,w)$ satisfies*

$$\|\gamma(t,v',w') - \gamma(t,v'',w'')\| \leq K\{\|v'-v''\| + \|w'-w''\|\} \tag{3.4}$$

uniformly for all $t \in \Im_0$ and all v', v'', w', $w'' \in \mathbf{R}^n$, where $K > 0$.

Example. For $g(t,u,v) = v$, as in Example 2.1, $\gamma(t,v,w) = v + w$.

Additional assumptions, on f and g, follow.

Assumption 3.8. *There exist constants $\Lambda_1(f), \Lambda_2(f) > 0$, such that, uniformly for all $t \in \Im_0$ and all $u_1, u_2, v_1, v_2 \in \mathbf{R}^n$,*

$$\|f(t,u_1,v_1) - f(t,u_2,v_2)\| \leq \Lambda_1(f)\|u_1-u_2\| + \Lambda_2(f)\|v_1-v_2\|. \tag{3.5}$$

Assumption 3.9. *There exist constants* $\Lambda_1(g), \Lambda_2(g) > 0$*, such that, uniformly for all* $t \in \mathfrak{I}_0$ *and all* $u_1, u_2, v_1, v_2 \in \mathbf{R}^n$*,*

$$\|g(t,u_1,v_1) - g(t,u_2,v_2)\| \le \Lambda_1(g)\|u_1-u_2\| + \Lambda_2(g)\|v_1-v_2\|. \tag{3.6}$$

We remark on a possible further strengthening of the conditions on g.

Condition 3.10. *For* $u_1, u_2, v \in \mathbf{R}^n$*,*

$$\begin{gathered}\|g(t,u_1,v) - g(t,u_2,v)\| \le \lambda_1(g)\|u_1 - u_2\| \\ \textit{with} \quad \lambda_1(g) \in [0,1).\end{gathered} \tag{3.7}$$

If (3.6) is strengthened so that (3.7) also holds, then Assumption 3.7 can be omitted, as it follows from (3.6) and (3.7). However, equation (3.7) is not a necessary condition for Assumption 3.7 (for example, $g(t,u,v) = v$ as in Example 2.1) and will not be a part of our general assumptions.

In subsection 4.3, we assume the following condition.

Condition 3.11. *There exist constants* $\kappa_{0,1} > 0$ *such that* $\|f(t,u,v)\| \le \kappa_0\|u\| + \kappa_1$ *for* $t \in \mathfrak{I}_0$ *and* $u, v \in \mathbf{R}^n$*.*

If $f(t,0,v)$ is uniformly bounded (for $t \in \mathfrak{I}_0$ and $\|v\| < \infty$) and Assumption 3.8 is valid, then Condition 3.11 holds.

We adopt *all* the Assumptions 3.1–3.9 unless we state that we are dropping stronger hypotheses but retaining weaker ones.

Remark 3.12. Liu [**15**] considers numerics for the case (see Remark 2.3) where $g\big(t, y(t), y(t-\tau(t))\big)$ is replaced by the simpler form $g_*\big(t, y(t-\tau(t))\big)$. This case lacks an essential feature of our discussion, because the existence of the function γ in Assumption 3.7 becomes a trivial issue (in the simplified case, $\gamma(t,v,w) = g_*(t,v) + w$).

3.3 Initial observations. For a function $\psi(t)$ that is continuous from the right on $[t', t'')$ (so $\psi(t) = \lim_{\delta\searrow 0}\psi(t+\delta)$ for $t \in [t', t'')$), consider, if it exists, the right-hand derivative [**12**] defined as $(d/dt)_+\psi(t) = \lim_{\delta\searrow 0}\{\psi(t+\delta) - \psi(t)\}/\delta$. If ψ is right-continuous on the interval $[t', t'')$, $(d/dt)_+ \int_{t'}^{t} \psi(s)\mathrm{d}s = \psi(t)$ for $t \in [t', t'')$.

Lemma 3.13. (i) *Suppose* $y(t) \equiv y(\varphi, \tau; t)$ *satisfies the integral equation* (1.1) *where the integral is interpreted in the sense of Riemann, and suppose Assumption* 3.5 *is valid. If* $y(t)$ *is right continuous and has a right-hand derivative* $(d/dt)_+y(t)$*, then*

$$\left(\frac{d}{dt}\right)_+ \{y(t) - g\big(t, y(t), y(t-\tau(t))\big)\} = f\big(t, y(t), y(t-\tau(t))\big); \tag{3.8}$$

further,

$$\left(\frac{d}{dt}\right)_+ y(t) = \left(\frac{d}{dt}\right)_+ g\big(t, y(t), y(t-\tau(t))\big) + f\big(t, y(t), y(t-\tau(t))\big).$$

(ii) *Alternatively, suppose* $y(t)$ *possesses the assumed right-continuity and satisfies* (3.8)*, and suppose that* $y(t) - g\big(t, y(t), y(t-\tau(t))\big)$ *is continuous. Then* $y(t)$ *also satisfies* (1.1).

Note that a continuous function with a bounded right-hand derivative is absolutely continuous on compact intervals.

Definition 3.14. A natural solution of (2.1) on $\mathfrak{I}_0$ is a right-continuous function satisfying (3.8) (the derivatives being taken as right-hand derivatives) for $t \in \mathfrak{I}_0$, and such that $y(t) - g\big(t, y(t), y(t-\tau(t))\big)$ is continuous for $t \in \mathfrak{I}_0$.

4. The integral equation formulation. Our concern is to relate the problem (2.1a) to its integrated form

$$y(t) = g\big(t, y(t), y(t-\tau(t))\big) + \int_{t_0}^{t} f\big(s, y(s), y(s-\tau(s))\big)\mathrm{d}s + z^0 \tag{4.1a}$$

for $t \in \Im_0$, with z^0 as in (3.3), with the condition (2.1b), viz.

$$\text{(4.1b)} \qquad y(t) = \varphi(t) \quad (t \in [t_{-1}, t_0]) \quad \text{where} \quad t_{-1} := \inf_{t \in \Im_0} t - \tau(t),$$

and to exploit (4.1). To develop (4.1a) further, Assumption 3.7 is convenient, and we deduce, from (4.1a), the new formulation

$$\text{(4.1c)} \qquad y(t) = \gamma\Big(t, y(t-\tau(t)), \int_{t_0}^{t} f\big(s, y(s), y(s-\tau(s))\big)\mathrm{d}s + z^0\Big).$$

Remark 4.1. There are similarities in our approach and the treatment of Driver [**8**] of the problem of a system of explicit NDDEs

$$\text{(4.2)} \qquad y'(t) = f_\star\big(t, y(t), y(t-\tau_1(t)), y'(t-\tau_2(t))\big),$$

where $\tau_{1,2}(t) > 0$, where Driver assumes that the initial function φ is absolutely continuous on the closed initial interval ($[t_{-1}, t_0]$ in our notation) with $t - \tau_{1,2}(t) \geq t_{-1}$.

4.1 Similar problems for ordinary differential equations. We pause to place our approach in perspective. Where g vanishes identically and $f(t,u,v) = f(t,u)$ the problem of relating (2.1a) to the integrated form reduces to a well-studied problem of the relation between ordinary differential equations (ODEs) $y'(t) = f(t, y(t))$ and the integrated form $y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s$. Related are (i) use of the integral equation to establish existence of a solution (ii) investigation of Picard iteration (or other iterative methods) for the integral equation and (iii) consideration of the solution of the ODE in the sense of Carathéodory.

We observe that: (i) the *Peano* or *Cauchy-Peano* theory relies on the Arzelà-Ascoli theorem (stated below as Lemma 4.1); (ii) the *Picard* or *Picard-Lendelöf* iteration for solution of the integral equation form of the ODE $y'(t) = f(t, y(t))$ has the form $y_{k+1}(t) = y(t_0) + \int_{t_0}^{t} f(s, y_k(s))\,\mathrm{d}s$; and (iii) a solution of $y'(t) = f(t, y(t))$ on $\Im_0$ *in the sense of Carathéodory* is a function that is absolutely continuous (see [**12**]) on compact sub-intervals of $\Im_0$, and satisfies the integral equation $y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s))\,\mathrm{d}s$ on $\mathfrak{I}_0$, with the integral taken in the sense of Lebesgue. See, e.g., [**6, 19, 20**].

4.2 A method of steps. The method of steps for the solution of a DDE extends[2] to potentially discontinuous 'natural' solutions of (2.1) (that is, solutions of (3.8) in the sense of Definition 3.14). We examine the solution on $[\zeta_\ell, \zeta_{\ell+1})$, taking as hypothesis the existence of a (not necessarily unique) solution $y(t)$ satisfying (3.8) on the interval $[t_{-1}, \zeta_\ell)$. The value of this solution at ζ_ℓ is a solution of the equation $y(\zeta_\ell) - g\big(\zeta_\ell, y(\zeta_\ell), y(\zeta_{k_\ell})\big) = \lim_{t\to\zeta_\ell} y(t) - g\big(\zeta_\ell, \lim_{t\to\zeta_\ell} y(t), \lim_{t\to\zeta_\ell} y(t-\tau(t))\big)$, and it follows from a knowledge of $y(t)$ on $[t_{-1}, \zeta_\ell)$ provided that equations of the form $u - g(t, u, v) = w$ have some solution u when given t, v, w.

We shall be seeking proofs by induction, so let us assume that a solution $y(t)$ exists on $[t_{-1}, \zeta_\ell]$. For $t \in [\zeta_\ell, \zeta_{\ell+1})$, we have $t - \tau(t) \in [\zeta_{k_\ell}, \zeta_{k_\ell+1}]$ (by Assumption 3.3, equation (3.2)). We write

$$\varphi_\ell(t) = y(t) \quad \text{for} \quad t \in [\zeta_{k_\ell}, \zeta_{k_\ell+1}) \tag{4.3}$$

and

$$\varphi_\ell(\zeta_{k_\ell+1}) = \lim_{t \nearrow \zeta_{k_\ell+1}} y(t).$$

Thus, $\varphi_\ell(t) = y(t)$ on $[\zeta_{k_\ell}, \zeta_{k_\ell+1})$ but $\varphi_\ell(t)$ lacks a jump that $y(t)$ may be assumed to possess, at $\zeta_{k_\ell+1}$.

If an extension of the solution $y(t)$ to $[\zeta_\ell, \zeta_{\ell+1}]$ exists, it agrees on the half-open interval $[\zeta_\ell, \zeta_{\ell+1})$ with the solution $y_\ell(t)$ of

$$\left(\frac{d}{dt}\right)\Big\{y_\ell(t) - g_\ell\big(t, y_\ell(t)\big)\Big\} = f_\ell\big(t, y_\ell(t)\big) \quad (t \in [\zeta_\ell, \zeta_{\ell+1}]), \tag{4.4a}$$

$$y_\ell(\zeta_\ell) = y(\zeta_\ell), \tag{4.4b}$$

with

$$\left.\begin{aligned} f_\ell\big(t, y_\ell(t)\big) &:= f\big(t, y_\ell(t), \varphi_\ell(t-\tau(t))\big) \\ g_\ell\big(t, y_\ell(t)\big) &:= g\big(t, y_\ell(t), \varphi_\ell(t-\tau(t))\big) \end{aligned}\right\}, \quad t \in [\zeta_\ell, \zeta_{\ell+1}). \tag{4.4c}$$

If we then require that $y_\ell(\zeta_{\ell+1})$ satisfies

$$y_\ell(\zeta_{\ell+1}) - g_\ell\big(\zeta_{\ell+1}, y_\ell(\zeta_{\ell+1})\big) = \lim_{t \nearrow \zeta_{\ell+1}} \Big\{ y(t) - g_\ell\big(t, y(t)\big) \Big\}, \tag{4.5}$$

we define $y(\zeta_{\ell+1}) = y_\ell(\zeta_{\ell+1})$. While $y_\ell(t)$ has a jump at $\zeta_{\ell+1}$, (4.4c) implies that, for $t \in [\zeta_\ell, \zeta_{\ell+1})$, $f_\ell\big(t, y_\ell(t)\big) \equiv f\big(t, y_\ell(t), y_{k_\ell}(t-\tau(t))\big)$ and $g_\ell\big(t, y_\ell(t)\big) \equiv g\big(t, y_\ell(t), y_{k_\ell}(t-\tau(t))\big)$. We write $z_\ell(t) := y_\ell(t) - g_\ell\big(t, y_\ell(t)\big)$, $z(t) = y(t) - g\big(t, y(t), y(t-\tau(t))\big)$; $z_\ell(\zeta_\ell) = z(\zeta_\ell)$ is z^ℓ.

The literature on ODEs suggests two ways to proceed: the first is to follow Peano and establish the existence of a solution (not necessarily unique), using Assumption 3.5. The second is to follow Picard and study suitable iterations using conditions of Lipschitz continuity. Actually, the nonvanishing of g in (4.4) suggests that we should look to the theory of DAEs rather than that of ODEs, but we are unaware of literature that gives extensions of Peano's or Picard's theories to DAEs.

4.3 Extension of Peano's theorem to the NDDE. Condition 3.11 implies that $\|f_\ell(t,u)\| \le \kappa_0 \|u\| + \kappa_1$, and we now require this condition to be satisfied along with Assumption 3.6. Our theory echoes that for ODEs detailed in Reid [**19**, Chapter 1, Section 3]; an alternative is to assume a uniform bound on $\|f_\ell(t,u)\|$ for all possible arguments.

4.3.1 *Three lemmas ....* We state three lemmas that we use to establish what follows.

Lemma 4.2. *If* $|\nu_r| \le \Delta C_0 \sum_{q=0}^{r-1} |\nu_q| + C_1$ *for* $r = 1, 2, \dots, N$ *where* $C_0 > 0$, $C_1 > 0$, *are constants, then*

$$\max_{r \in \{0,1,\dots,N\}} |\nu_r| \le (C_1 + \Delta\, C_0 |\nu_0|) \exp\{C_0 N \Delta\}.$$

Now write $\triangledown\zeta_\ell = \zeta_\ell - \zeta_{\ell-1}$ and pick $\Delta = \triangledown\zeta_\ell / N$ for some positive integer N. Suppose $\mu = r + \theta \le N$ with $r \in \{0, 1, \dots, N-1\}$, $\theta \in [0,1]$, so that $\int_{\zeta_\ell}^{\zeta_\ell + \mu\Delta} \psi(s)\, \mathrm{d}s = \sum_{s=0}^{r-1} \int_{\zeta_\ell + s\Delta}^{\zeta_\ell + (s+1)\Delta} \psi(\sigma)\, \mathrm{d}\sigma + \int_{\zeta_\ell + r\Delta}^{\zeta_\ell + \mu\Delta} \psi(\sigma)\, \mathrm{d}\sigma$. Euler's rule may be applied to each integral term, and Lemma 4.3, below, provides a bound on the error in the resulting approximation, in terms of the modulus of continuity $\omega(\psi;\Delta)$ of the integrand.

Lemma 4.3. *For* $\psi \in C[\zeta_\ell, \zeta_{\ell+1}]$, $\mu = r+\theta$, *as above, we have*

$$\left\| \int_{\zeta_\ell}^{\zeta_\ell+\mu\Delta} \psi(s)\,\mathrm{d}s - \left\{ \Delta \sum_{s=0}^{r-1} \psi(\zeta_\ell + s\Delta) + \theta\Delta\psi(\zeta_\ell + r\Delta) \right\} \right\| \le |\zeta_{\ell+1} - \zeta_\ell|\,\omega(\psi;\Delta).$$

Lemma 4.4 (The Arzelà-Ascoli theorem, see [19, p. 527]). *Let* $\mathcal{F}$ *be a set of uniformly bounded and equicontinuous functions defined on a compact metric space* $\mathcal{X}$*; then any sequence* $\{f_n\} \subseteq \mathcal{F}$ *has a subsequence that is uniformly convergent on* $\mathcal{X}$ *to a continuous function.*

4.3.2 … *and a theorem.*

Theorem 4.5. *Suppose that Assumptions* 3.1–3.6 *and Condition* 3.11 *hold*; *then there exists a natural solution of* (2.1).

Proof. Consider the solution of the problem (4.4) on $[\zeta_{\ell-1}, \zeta_\ell)$; we take $\ell = 1$, then consider $\ell = 2, 3$, etc., in turn.

We construct, for each $\Delta = \triangledown\zeta_\ell/N$ (as above), approximations $y_\Delta(t)$ to $y_\ell(t)$ and $z_\Delta(t)$ to $z_\ell(t)$ where dependencies on ℓ are suppressed in the notation y_Δ, z_Δ. For $\theta \in [0,1]$, we consider the Euler-type equations

(4.6a)
$$z_\Delta(\zeta_\ell + (r+\theta)\Delta) = z_\Delta(\zeta_\ell + r\Delta) + \theta\Delta f_\ell\big(\zeta_\ell + r\Delta, y_\Delta(\zeta_\ell + r\Delta)\big),$$
(4.6b)
$$z_\Delta(t) = y_\Delta(t) - g_\ell\big(t, y_\Delta(t)\big) \equiv y_\Delta(t) - g\big(t, y_\Delta(t), \varphi_{k_\ell}(t-\tau(t))\big).$$

Thus, writing $\mu = r+\theta$ (for $\theta \in [0,1]$, $r \in \{0,1,\cdots,N-1\}$),

(4.7a)
$$z_\Delta(\zeta_\ell + \mu\Delta) = z_\Delta(\zeta_\ell) + \Delta \sum_{s=0}^{r-1} f_\ell(\zeta_\ell + s\Delta, y_\Delta(\zeta_\ell + s\Delta)) + \theta\Delta f_\ell(\zeta_\ell + r\Delta, y_\Delta(\zeta_\ell + r\Delta))$$

$$z_\Delta(\zeta_\ell + \mu\Delta) = y_\Delta(\zeta_\ell + \mu\Delta) - g_\ell(\zeta_\ell + \mu\Delta, y_\Delta(\zeta_\ell + \mu\Delta)). \tag{4.7b}$$

This yields

$$\begin{aligned} y_\Delta(\zeta_\ell + \mu\Delta) = z(\zeta_\ell) &+ g_\ell(\zeta_\ell + \mu\Delta, y_\Delta(\zeta_\ell + \mu\Delta)) \\ &+ \Delta \sum_{s=0}^{r-1} f_\ell(\zeta_\ell + s\Delta, y_\Delta(\zeta_\ell + s\Delta)) \\ &+ \theta\Delta f_\ell(\zeta_\ell + r\Delta, y_\Delta(\zeta_\ell + r\Delta)). \end{aligned} \tag{4.8}$$

From the above, we deduce that the functions $\{y_\Delta(t) \big| \Delta = \triangledown\zeta_\ell/N;\ N = 1, 2, \ldots,\}$ are uniformly bounded and equicontinuous on $[\zeta_\ell, \zeta_{\ell+1}]$.

The details of how this conclusion is reached (using Lemma 4.3 and our stated assumptions) appear in subsection 4.3.3 below. By the Arzelà-Ascoli theorem (Lemma 4.4), the set of uniformly bounded and equicontinuous functions $\{y_\Delta(t)\}$ on $[\zeta_\ell, \zeta_{\ell+1}]$ contains a subsequence $\{y_{\Delta_j}(t)\}$ with a continuous limit $\lim_{\Delta_j \to 0} y_{\Delta_j}(t)$, on $[\zeta_\ell, \zeta_{\ell+1}]$, in the sense of uniform convergence. We can now appeal to Lemma 4.3 and the equicontinuity of $\{y_{\Delta_k}(t)\}$ and hence of $\{f_\ell(t, y_{\Delta_k}(t))\}$. Since (4.8) gives

$$y_\Delta(t) = z^\ell + g_\ell(t, y_\Delta(t)) + \int_{\zeta_\ell}^t f_\ell(s, y_\Delta(s))\mathrm{d}s + \varepsilon(\Delta)$$

and $\lim_{\Delta_k \to 0} \varepsilon(\Delta_k) = 0$, it follows that $\lim_{\Delta_k \to 0} y_{\Delta_k}(t)$ exists on the closed interval $[\zeta_\ell, \zeta_{\ell+1}]$ and on the open interval $[\zeta_\ell, \zeta_{\ell+1})$ it satisfies the integral equation

$$y(t) = z^\ell + g\big(t, y(t), y(t - \tau(t))\big) + \int_{\zeta_\ell}^t f\big(s, y(s), y(s - \tau(s))\big)\,\mathrm{d}s, \tag{4.9}$$

and hence (4.4). Equation (4.5) defines a value $y(\zeta_{\ell+1})$. By induction on ℓ, there follows the existence of a natural solution $y(t)$ on $\mathfrak{I}_0$.

4.3.3 *Equicontinuity and uniform boundedness.* To establish the equicontinuity and uniform boundedness above, we rely on the discrete Gronwall inequality in Lemma 4.2.

• To show uniform boundedness, we could employ (4.8) directly or recast it in terms of γ as we do here. As a convenient shorthand, write

$$\sigma_\mu(\Delta) := \Delta \sum_{s=0}^{r-1} f_\ell(\zeta_\ell + s\Delta, y_\Delta(\zeta_\ell + s\Delta)) + \theta\Delta f_\ell(\zeta_\ell + r\Delta, y_\Delta(\zeta_\ell + r\Delta)).$$

From (4.8),

$$y_\Delta(\zeta_\ell + \mu\Delta) = \gamma(\zeta_\ell + \mu\Delta, \varphi_{k_\ell}(\zeta_\ell + \mu\Delta), z(\zeta_\ell) + \sigma_\mu(\Delta)). \tag{4.10}$$

This is valid for $\mu \in \{r + \theta \,|\, r \in \{0, 1, \dots, N\};\ \theta \in [0, 1];\ \mu \leq N\}$ and in particular for $\mu = 0$ (we have $y_\Delta(\zeta_\ell) = \gamma_\ell(\zeta_\ell, z(\zeta_\ell))$). The expression that we deduce for $y_\Delta(\zeta_\ell + \mu\Delta) - y_\Delta(\zeta_\ell)$ allows us to establish a bound on $y_\Delta(\zeta_\ell + \mu\Delta)$ (using the assumptions of our theorem, a triangle inequality, and the fact that $y_\Delta(\zeta_\ell) = y(\zeta_\ell)$). With

$$\Gamma_\mu(\Delta) := \|\gamma(\zeta_\ell + \mu\Delta, \varphi_{k_\ell}(\zeta_\ell + \mu\Delta), z^\ell) - \gamma(\zeta_\ell, \varphi_{k_\ell}(\zeta_\ell), z^\ell)\| \tag{4.11}$$

(for which a uniform bound exists) we have the result

(4.12)

$$\begin{aligned}
&\|y_\Delta(\zeta_\ell + \mu\Delta)\| \\
&\quad \leq \Gamma_\mu(\Delta) + \|\gamma(\zeta_\ell + \mu\Delta, \varphi_{k_\ell}(\zeta_\ell + \mu\Delta), z^\ell + \sigma_\mu(\Delta)) \\
&\qquad - \gamma(\zeta_\ell + \mu\Delta, \varphi_{k_\ell}(\zeta_\ell + \mu\Delta), z^\ell)\|, \\
&\quad \leq \Gamma_\mu(\Delta) + \Lambda_3(\gamma)\Bigg\{\kappa_0\Bigg(\Delta\sum_{s=0}^{r-1}\|y_\Delta(\zeta_\ell + s\Delta)\| \\
&\qquad\qquad + \theta\Delta\|y_\Delta(\zeta_\ell + r\Delta)\|\Bigg) + \kappa_1\mu\Delta\Bigg\}
\end{aligned}$$

using Condition 3.11 for f. If we set $\theta = 0$ in (4.12), the discrete Gronwall inequality provides a uniform bound for $\{\|y_\Delta(\zeta_\ell + r\Delta)\|\}_{r=0}^N$. Now consider $\theta \in (0, 1)$ and we deduce from (4.12) a uniform bound on the quantities $\{\|y_\Delta(\zeta_\ell + \mu\Delta)\|\}$.

• To show equicontinuity, employ (4.8) with $\mu = \mu'$ and $\mu = \mu''$. Difference the resulting equations[3], and use the uniform boundedness of $\{y_\Delta(\zeta_\ell + s\Delta)\}_{s=0}^N$ and the continuity of f and g, which is uniform on compact intervals.

4.4 Extension of Picard iterations to the NDDE. The iterations obtained with either $r_k = k$ or $r_k = k+1$ in

$$y_{k+1}(t) = g\big(t, y_{r_k}(t), y_k(t-\tau(t))\big) + \int_{t_0}^{t} f\big(s, y_k(s), y_k(s-\tau(s))\big)\,\mathrm{d}s + z^0,$$

are two tentative candidates for the discussion of (4.1a). We amend these iterations, to reflect ideas underpinning the method of steps. Our replacements for the above iterations read, respectively,

(4.13a)

$$y_{k+1}(t) = g\big(t, y_k(t), y(t-\tau(t))\big) + \int_{t_0}^{\zeta_\ell} f\big(s, y(s), y(s-\tau(s))\big)\,\mathrm{d}s + \int_{\zeta_\ell}^{t} f\big(s, y_k(s), y(s-\tau(s))\big)\,\mathrm{d}s + z^\ell;$$

(4.13b)

$$y_{k+1}(t) - g\big(t, y_{k+1}(t), y(t-\tau(t))\big) = \int_{t_0}^{\zeta_\ell} f\big(s, y(s), y(s-\tau(s))\big)\,\mathrm{d}s + \int_{\zeta_\ell}^{t} f\big(s, y_k(s), y(s-\tau(s))\big)\,\mathrm{d}s + z^\ell$$

for $t \in [\zeta_\ell, \zeta_{\ell+1})$. These are based on (4.1a), and the last of these corresponds to an iteration for (4.1d) of the form

$$y_{k+1}(t) = \gamma\Big(t, y(t-\tau(t)), \int_{t_0}^{\zeta_\ell} f\big(s, y(s), y(s-\tau(s))\big)\,\mathrm{d}s + \int_{\zeta_\ell}^{t} f\big(s, y_k(s), y(s-\tau(s))\big)\,\mathrm{d}s + z^\ell\Big) \tag{4.13c}$$

for $t \in [\zeta_\ell, \zeta_{\ell+1})$. Recall that $t-\tau(t) < \zeta_\ell$ when $t \in [\zeta_\ell, \zeta_{\ell+1})$ and $z^{\ell+1} = \lim_{t \nearrow \zeta_{\ell+1}} \{y(t) - g\big(t, y(t), y(t-\tau(t))\big)\}$ for $\ell \in \{0, 1, 2, \dots\}$. The iteration (4.13a) is less than ideal. When f vanishes identically, the iteration reads $y_{k+1}(t) = g\big(t, y_k(t), y(t-\tau(t))\big)$. Convergence of this iteration is normally discussed under a condition of the form (3.7), and we therefore expect[4] to have to strengthen this condition for the general case. On the other hand, (3.7) is not a necessary condition for the existence of γ and it therefore seems desirable to consider (4.13b) or the form (4.13c).

Theorem 4.6. *Suppose that Assumptions* 3.1–3.8 *are valid. Then there exists a unique natural solution of* (2.1)*, which is obtainable by Picard iteration based on* (4.13b) *or* (4.13c)*.*

Proof. On writing $\Delta y_k(t) := y_{k+1}(t) - y_k(t)$, (3.4) and (4.13c) yield

$$\|\Delta y_k(t)\| \le K\Lambda_1(f)\int_{\zeta_\ell}^{t}\|\Delta y_{k-1}(s)\|\,\mathrm{d}s. \tag{4.14}$$

Now $y_{k+1}(t) = y_0(t) + \sum_{r=0}^{k}\Delta y_r(t)$ and by comparison with the exponential series and (4.14) it follows that $\sum_{r=0}^{\infty}\Delta y_r(t)$ converges absolutely (for $t \in [\zeta_\ell, \zeta_{\ell+1})$). From the continuity of γ and f it follows that the limit satisfies the integral equation formulation (4.1c), and hence (4.1a), on $[\zeta_\ell, \zeta_{\ell+1})$. The value $y(\zeta_{\ell+1})$ follows. Using similar inequalities to those above, the assumption that there are two such solutions yields a contradiction.

4.5 Sensitivity to initial data. An approach that can be useful both for the theory and the practical treatment of (2.1) involves the replacement of a discontinuous initial function $\varphi(t)$ by a continuous approximation $\varphi^\delta(t)$ that has the jumps "smoothed out" but retains the original support $[t_{-1}, t_0]$. One may then analyze the change $y(\varphi,\tau;t) - y(\varphi^\delta,\tau;t)$. Let us consider the effect of changing the initial function $\varphi(t)$ to an initial function $\widetilde{\varphi}(t) \in \{\widetilde{\varphi}^\delta(t)\}_{0<\delta<\widehat{\delta}}$ where $\widehat{\delta} > 0$. The functions $\widetilde{\varphi}^\delta(t)$ are to be either continuous on $[t_{-1}, t_0]$ or to have possible jumps at the same points (3.1) as does $\varphi(t)$. For $t \in \mathfrak{I}_{-1}$, we write

$$y(t) \equiv y(\varphi,\tau;t), \quad \widetilde{y}(t) \equiv y(\widetilde{\varphi},\tau;t), \quad \delta y(t) := \widetilde{y}(t) - y(t),$$
$$z^\ell = \lim_{t\to\zeta_\ell} z(t), \quad \widetilde{z}^\ell = \lim_{t\to\zeta_\ell}\widetilde{z}(t), \quad \delta z^\ell = \widetilde{z}^\ell - z^\ell.$$

Here $\widetilde{z}(t) := \bigl\{\widetilde{y}(t) - g\bigl(t, \widetilde{y}(t), \widetilde{y}(t-\tau(t))\bigr)\bigr\}$ but, more conveniently,

$$\widetilde{z}(t) = \widetilde{z}^\ell + \int_{\zeta_\ell}^{t} f\bigl(s, \widetilde{y}(s), \widetilde{y}(s-\tau(s))\bigr)\,\mathrm{d}s, \tag{4.15}$$

for $t \in [\zeta_\ell, \zeta_{\ell+1})$, with an analogous form for $z(t)$. Using (3.5),

$$\|\delta z(t)\| \le \|\delta z^\ell\| + \int_{\zeta_\ell}^{t} \{\Lambda_1(f)\|\delta y(s)\| + \Lambda_2(f)\|\delta y(s-\tau(s))\|\}\,\mathrm{d}s. \tag{4.16}$$

Taking limits in equations of the form (4.16),

$$\|\delta z^{\ell+1}\| \le \|\delta z^\ell\| + \int_{\zeta_\ell}^{\zeta_{\ell+1}} \{\Lambda_1(f)\|\delta y(s)\| + \Lambda_2(f)\|\delta y(s-\tau(s))\|\}\,\mathrm{d}s, \tag{4.17}$$

$\ell = 0, 1, 2, \ldots$. For $t \in [\zeta_\ell, \zeta_{\ell+1})$,

$$\widetilde{y}(t) = \gamma\Big(t, \widetilde{y}(t-\tau(t)), \int_{\zeta_\ell}^{t} f\big(s, \widetilde{y}(s), \widetilde{y}(s-\tau(s))\big)\,\mathrm{d}s + \widetilde{z}^\ell\Big), \tag{4.18}$$

with a corresponding equation for $y(t)$.

Lemma 4.7. *Let Assumptions* 3.1–3.9 *apply. If* $t \in [\zeta_\ell, \zeta_{\ell+1}) \subset \mathfrak{I}_0$,

$$\|\delta y(t)\| \le K \int_{\zeta_\ell}^{t} \Lambda_1(f)\|\delta y(s)\|\,\mathrm{d}s + v_\ell(t) \tag{4.19}$$

with $v_\ell(t) := K\big\{\|\delta z^\ell\| + \|\delta y(t-\tau(t))\| + \int_{\zeta_\ell}^{t} \Lambda_2(f)\|\delta y(s-\tau(s))\|\,\mathrm{d}s\big\}$.

The term $v_\ell(t)$ depends on $\delta y(t)$ for $t \le \zeta_\ell$, and (4.19) yields $\|\delta y(t)\| \le v_\ell(t) + K_1 \int_{\zeta_\ell}^{t} \exp\{K_1(t-s)\} v_\ell(s)\,\mathrm{d}s$ for $t \in [\zeta_\ell, \zeta_{\ell+1})$, with $K_1 = K\Lambda_1(f)$. Thus, there exist positive values $c_{1,2}(\ell)$ such that $\|\delta y(t)\| \le v_\ell(t) + c_1(\ell) \int_{\zeta_\ell}^{\zeta_{\ell+1}} v_\ell(s)\,\mathrm{d}s$, and $\int_{\zeta_\ell}^{t} \|\delta y(s)\|\,\mathrm{d}s \le c_2(\ell) \int_{\zeta_\ell}^{\zeta_{\ell+1}} \|v(s)\|\,\mathrm{d}s$, for $t \in [\zeta_\ell, \zeta_{\ell+1}) \subset \mathfrak{I}_0$. We can now prove the following result by considering successive intervals $[\zeta_\ell, \zeta_{\ell+1})$ and using induction.

Theorem 4.8. *Let Assumptions* 3.1–3.9 *apply. If* $\|\varphi(t) - \widetilde{\varphi}^\delta(t)\| \overset{pw}{\to} 0$ *(pointwise) as* $\delta \to 0$ *for each* $t \in [t_{-1}, t_0]$, *and* $\int_{t_{-1}}^{t_0} \|\varphi(s) - \widetilde{\varphi}^\delta(s)\|\,\mathrm{d}s \to 0$ *as* $\delta \to 0$, *then* $\|y(\varphi, \tau; t) - y(\widetilde{\varphi}^\delta, \tau; t)\| \overset{pw}{\to} 0$, *for each* $t \in \mathfrak{I}_0$.

4.6 Solutions in the sense of Carathéodory.

Definition 4.9. A function $y(t) = y(\varphi, \tau; y)$ that satisfies (4.1a) for $t \in \mathfrak{I}_0$, where $y(t) - g(t, y(t), y(t - \tau(t)))$ is absolutely continuous on compact sub-intervals of $\mathfrak{I}_0$, where the integral is interpreted in the sense of Lebesgue, and also satisfies (4.1b) for all $t \in [t_{-1}, t_0]$, will be called a solution of (2.1) on $\mathfrak{I}_0$ *in the sense of Carathéodory.*

For the extension of the fundamental theorems of calculus in terms of Riemann integrals to Lebesgue integrals, we recall the following:

Lemma 4.10. *If* $\psi \in \mathcal{L}[t', t'']$ *and* $\Psi(t) = \int_{t_0}^{t} \psi(s)\,\mathrm{d}s$, $t \in [t', t'']$, *then* Ψ *is absolutely continuous on* $[t', t'']$. *If* Ψ *is absolutely continuous on* $[t', t'']$, *then* Ψ *is differentiable almost everywhere on* $[t', t'']$, $\Psi' \in \mathcal{L}[t', t'']$ *and* $\int_{t'}^{t} \Psi'(s)\,\mathrm{d}s = \Psi(t) - \Psi(t')$ *for all* $t \in [t', t'']$.

Given $\gamma(t, u, v)$, a solution of (2.1) on $\mathfrak{I}_0$ in the sense of Carathéodory also satisfies (4.1c) for almost all $t \in \mathfrak{I}_0$, and vice-versa in the case that $y(t) - g(t, y(t), y(t - \tau(t)))$ is absolutely continuous.

It has long been appreciated that solutions of DDEs and NDDEs may have discontinuous *derivatives*, [**16**]. The discontinuous natural solutions considered by Baker and Paul [**2**] (who give some illustrative examples) have right-hand derivatives everywhere on $\mathfrak{I}_0$, and conventional derivatives almost everywhere on $\mathfrak{I}_0$ (the conventional derivative fails to exist only at the points ζ_ℓ where the solution has jump discontinuities).

Theorem 4.11. *With the given assumptions, natural solutions of* (2.1)*, in the sense of Definition* 3.14 (*with derivatives taken as right-hand derivatives*)*, are solutions in the sense of Carathéodory as stated in Definition* 4.9.

5. Conclusions. The principal results established here comprise: (a) a theorem on the existence of a natural solution; (b) a theorem on the uniqueness of a natural solution; (c) a theorem relating to the dependence of a natural solution on the original data; (d) the relationship between natural solutions and solutions in the sense of Carathéodory. The use of integral equations underpins the discussion. The extension of our results to NDDEs of a more general type, or not satisfying our hypotheses, presents opportunities for further investigation.

ENDNOTES

1. See also [**13, 14**].

2. In the usual method of steps [**13, 14**] a solution on $[t_{-1}, t_k]$ is extended to an interval $[t_k, t_{k+1}]$ such that $t - \tau(t) < t_k$ for $t \in [t_k, t_{k+1}]$, $k \in \{0, 1, 2, \dots, \}$.

3. It is convenient to consider first the case $\mu' = r + \theta'$, $\mu'' = r + \theta''$.

4. If f vanishes, and if $g(t, u, v)$ is continuously differentiable with respect to its second argument and the spectral radius of the matrix $(\partial/\partial u)g\big(t, y(t), y(t - \tau(t))\big)$ is greater than unity, convergence does not take place for arbitrary starting values.

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