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ON THE REGULARITY OF SOLUTIONS TO VOLTERRA FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH WEAKLY SINGULAR KERNELS

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Dedicated to Professor Ken Atkinson

ABSTRACT. We study the regularity properties of solutions for various classes of Volterra functional integro-differential equations with nonvanishing delays and weakly singular kernels. In particular, we characterize equations in which "supersmoothing," smoothing, or no smoothing occurs at the primary discontinuity points induced by the nonvanishing delay. These results will play a crucial role in the design and analysis of methods for the numerical solution of such functional equations with nonsmooth solutions.

1. Introduction. In this paper we analyze the regularity properties of solutions to Volterra functional integro-differential equations (VFIDEs) with weakly singular kernels and containing a delay function $\theta = \theta(t) := t - \tau(t)$ satisfying the following conditions (D1)–(D3) on a given (compact) interval $J := [t_0, T]$:

(D1) $\tau \in C^d(J)$ for some $d \geq 0$;

(D2) $\tau(t) \geq \tau_0 > 0$ for all $t \in J$;

(D3) θ is strictly increasing on J.

The points $\{\xi_\mu\}$ induced by (D2) and defined implicitly by the recursion

$$(1.1) \qquad \theta(\xi_\mu) = \xi_\mu - \tau(\xi_\mu) = \xi_{\mu-1} \quad (\mu = 1, \dots), \quad \text{with} \quad \xi_0 := t_0,$$

are called the primary discontinuity points corresponding to θ; they play a crucial role in the subsequent regularity analysis. Note that

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condition (D2) implies the "separation property" $\xi_{\mu+1} - \xi_\mu \geq \tau_0 > 0$ for all $\mu \geq 0$. We will assume, without loss of generality, that T in J is such that $T = \xi_{M+1}$ for some $M \geq 1$ and define the subintervals $J^{[\mu]} := (\xi_\mu, \xi_{\mu+1}]$, $\mu = 0, 1, \ldots, M$, with $\bar{J}^{[\mu]} := [\xi_\mu, \xi_{\mu+1}]$.

As an illustration we present two typical examples of nonvanishing delays that frequently arise in applications, see also [**4**, Section 1].

Example 1.1. Constant delay $\tau > 0$:

$$\theta(t) = t - \tau \Longrightarrow \xi_\mu = t_0 + \mu\tau, \quad \mu = 0, 1, \ldots, M+1.$$

Example 1.2. Nonvanishing proportional delay $\tau(t) = (1-q)t$, $t_0 > 0$:

$$\theta(t) = qt = t - (1-q)t, \quad 0 < q < 1$$
$$\Longrightarrow \xi_\mu = \frac{1}{q^\mu}\, t_0, \quad \mu = 0, 1, \ldots, M+1.$$

It is well known that delay (or retarded) differential equations with nonvanishing delays typically possess solutions exhibiting a significant reduction in regularity locally, at the points $\{\xi_\mu\}$: in $\bar{J}^{[\mu]}$ the solution y lies in C^{d+1} (provided the given functions are in $C^d(J)$), but $y^{(\mu+1)}$ is not continuous at $t = \xi_\mu$ when $\mu < d$; in the case of neutral DEs we have $y \in C^{d+1}(\bar{J}^{[\mu]})$, with y' discontinuous at $t = \xi_\mu$, $\mu = 0, 1, \ldots, M$. Details may be found in, e.g., Bellman and Cooke [**2**], El'sgol'ts and Norkin [**15**], Neves and Feldstein [**22**], Hale [**16**], de Gee [**14**], Willé and Baker [**25**], Hale and Verduyn Lunel [**17**], Bellen and Zennaro [**1**, Chapter 2] and, for regular delay Volterra integro-differential equations, in Brunner and Zhang [**10**].

In sharp contrast to the above, the presence of a Volterra integral operator containing a nonvanishing delay and a weakly singular kernel induces an additional local low-regularity phenomenon: solutions now have unbounded derivatives at the points $t = \xi_\mu^+$. To be more precise, consider the Volterra (-Hammerstein) integral operators defined by

$$(\mathcal{V}_{\theta,\alpha}f)(t) := \int_{t_0}^{t} k_\alpha(t-s)G(s, f(\theta(s)))\,ds, \tag{1.2}$$

$$(\mathcal{U}_{\theta,\alpha}f)(t) := \int_{t_0}^{t} k_\alpha(t-s)U(s, f'(\theta(s)))\,ds, \tag{1.3}$$

and

$$(\mathcal{W}_{\theta,\alpha}f)(t) := \int_{\theta(t)}^{t} k_\alpha(t-s)G(s, f(s))\,ds. \tag{1.4}$$

(VFIDEs corresponding to more general (non-Hammerstein) nonlinear Volterra integral operators can be shown to possess solutions with analogous regularity properties.) Here, the convolution kernel has the form

$$k_\alpha(t-s) := \begin{cases} k_0(t-s) & \text{if } \alpha = 0, \\ (t-s)^{-\alpha} & \text{if } 0 < \alpha < 1, \end{cases} \tag{1.5}$$

and the functions k_0, G and U are assumed to be smooth (more precise conditions will be stated later). In this paper we will focus on the following initial-value problems associated with the above integral operators:

$$\begin{aligned} &y'(t) = F(t, y(t), y(\theta(t)) + (\mathcal{V}_{\theta,\alpha}y)(t), \quad t \in J, \\ &y(t) = \phi(t), \quad t \le t_0, \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} &y'(t) = F(t, y(t), y(\theta(t)) + (\mathcal{U}_{\theta,\alpha}y)(t)), \quad t \in J, \\ &y(t) = \phi(t), \quad t \le t_0 \end{aligned} \tag{1.7}$$

(Sections 2 and 3), and

$$\begin{aligned} \frac{d}{dt}\left[y(t) - (\mathcal{V}_{\theta,\alpha}y)(t)\right] = F(t, y(t), y(\theta(t))), \quad t \in J, \\ y(t) = \phi(t), \quad t \le t_0. \end{aligned} \tag{1.8}$$

(Section 4). Motivated by a class of VFIDEs arising in applications, see [**3, 4, 11, 18, 19** and their references] we then turn, in Section 5, to the "implicit" neutral-type initial-value problem,

$$\text{(1.9)} \qquad \frac{d}{dt}\left[y(t) - (\mathcal{W}_{\theta,\alpha} y)(t)\right] = F(t, y(t), y(\theta(t))), \quad t \in I,$$
$$y(t) = \phi(t), \quad t \le t_0.$$

Since, by (D2) and (D3), we have $\theta(t) \ge \theta(t_0) = t_0 - \tau(t_0) =: \vartheta_0$, $t \in J$, the initial function ϕ is defined on the interval $[\vartheta_0, t_0]$. For brevity we will usually not specify this interval explicitly but simply write (as in the description of the above initial-value problems) $t \le t_0$. It will always be assumed that ϕ is at least continuous on the interval $[\vartheta_0, t_0]$ (more precise regularity assumptions will be stated in the following sections); hence, there will be no "secondary" discontinuity points induced by ϕ.

VFIDEs that are related to (1.6)–(1.9) and whose solutions have analogous regularity properties (for example, VFIDEs of the form (1.8) and (1.9) in which F has the additional argument

$$(\mathcal{V}f)(t) := \int_{t_0}^{t} b(t,s) K(s, y(s))\, ds,$$

with smooth b and K) will be briefly described in remarks following the proofs of the various regularity theorems.

A summary of the regularity results, in the form of a table, is presented in Section 6. Section 7 contains comments on related VFIDEs and on open problems.

We conclude this section with some remarks. The first one concerns notation. Typically, solutions to VFIDEs with (integrable) algebraic kernel singularities, cf. (1.5), and nonvanishing delays have unbounded (higher-order) derivatives at the points $x = \xi_\mu^+$ but are smooth on the subintervals $J^{(\mu)} = (\xi_\mu, \xi_{\mu+1}]$. Thus, in analogy to Brunner, Pedas and Vainikko [**7, 8**], the setting for our subsequent regularity analysis of (1.6)–(1.9) is given by the particular Hölder spaces defined by

(1.10)

$$C^{d,\,1+\nu}(J^{[\mu]}) \\ := \left\{ f \in C^d(J^{[\mu]}) : \ |f^{(j)}(t)| \le C_j \left\{ \begin{matrix} 1 \text{ if } \ j < 1+\nu \\ (t-\xi_\mu)^{1+\nu-j} \text{ if } \ j > 1+\nu \end{matrix} \right\}, \ t \in J^{[\mu]} \right\},$$

with $d \in \mathbf{N}$, $-1 < \nu < \infty$, $\nu \notin \mathbf{N}$, and constants $C_j < \infty$. As we shall see, the value of d may depend on μ in the case where the solution y experiences smoothing along the (global) interval J. If $d = 0$, we will write $C^{0,\,1+\nu}(J^{[\mu]}) =: C^{1+\nu}(J^{[\mu]})$.

The second remark is to remind the reader of an important regularity result for solutions of the "classical" VIDE with weakly singular kernel,

(1.11)

$$y'(t) = a(t)y(t) + g(t) + \lambda \int_{t_0}^{t} k_\alpha(t-s)y(s)\,ds, \quad t \in J; \quad y(t_0) = y_0,$$

with $0 < \alpha < 1$, corresponding to $\nu = -\alpha$. It was shown in Brunner, Pedas and Vainikko [**7**], see also Brunner [**5**, Chapter 7], that if $a, g \in C^{d,\,1-\alpha}(t_0, T]$, then the unique solution y of this initial-value problem lies in the space $C^{d+1,\,2-\alpha}(t_0, T]$. The same regularity result is true when the data are in $C^d(J)$. Thus, the second derivative of the solution of (1.13) is unbounded at $t = t_0^+$; it behaves like

$$|y''(t)| \le C_0(t-t_0)^{-\alpha}, \quad t \in (t_0, T]. \tag{1.12}$$

These regularity results have been crucial in the design and analysis of high-order numerical methods for weakly singular Volterra integral and integro-differential equations; see, for example, Tang [**23, 24**], Brunner, Pedas and Vainikko [**7, 8**], Cao, Herdman and Xu [**12**] and Brunner [**5**, Chapter 7].

It is obvious that the above regularity property is inherited, on the interval $J^{[0]}$, by the solutions of (1.6) and (1.7) and, with y'' and $2-\alpha$ replaced respectively by y' and $1-\alpha$, by the solutions of (1.8) and (1.9). However, due to the presence of the primary discontinuity points $\{\xi_\mu\}$ induced by the nonvanishing delay, analogous regularity results for the subintervals $J^{[\mu]}$, $\mu \geq 1$, play an equally crucial role when extending high-order numerical methods, e.g., hybrid collocation methods [**12**], or discontinuous Galerkin methods [**9**], to Volterra functional integro-differential equations with weakly singular kernels.

5. VFIDEs with supersmoothing. We start our regularity analysis by first considering a particular case of the Volterra functional integro-differential equation (1.6), namely,

$$\begin{aligned} &y'(t) = F(t, y(t)) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in I, \quad 0 < \alpha < 1, \\ &y(t) = \phi(t), \quad t \leq t_0. \end{aligned} \tag{2.1}$$

On the first subinterval $\bar{J}^{[0]} = [t_0, \xi_1]$ the above problem reduces to an initial-value problem for an ODE with nonsmooth forcing term,

$$y'(t) = F(t, y(t)) + \Phi_0(t), \quad t \in \bar{J}^{[0]}; \quad y(t_0) = \phi(t_0), \tag{2.2}$$

where

$$\Phi_0(t) := (\mathcal{V}_{\theta,\alpha}\phi)(t) = \int_{t_0}^{t} k_\alpha(t-s) G(s, \phi(\theta(s)))\, ds.$$

For C^d-data F, G, ϕ and θ, this function Φ_0 lies in $C^{d,\,1-\alpha}(J^{[0]})$; hence, it follows that the solution y of (2.1) satisfies

$$y \in C^{d+1,\,2-\alpha}(J^{[0]}). \tag{2.3}$$

Note that if $F(t,y) = a(t)y + g(t)$ and $G(s,z) = \lambda(s)z$, with $a, g, \lambda \in C^d(J)$, then the variation-of-constants formula corresponding to the linear version of (2.2),

$$y(t) = r(t, t_0)\phi(t_0) + \int_{t_0}^{t} r(t,s)\{g(s) + \Phi_0(s)\}\, ds, \quad t \in (\bar{J}^{[0]}),$$

with $r(t,s) := \exp\left(\int_s^t a(v)\, dv\right)$, explicitly reveals the $C^{d+1,\,2-\alpha}$-regularity of y on $J^{[0]}$.

The local regularity properties of y on the subsequent subintervals $J^{[\mu]}$ are described in the following theorem.

Theorem 2.1. *Assume*:

(i) θ *is subject to the hypothesis* (D1)–(D3) *of Section* 1;

(ii) $\phi \in C^d[\theta(t_0), t_0]$;

(iii) $F = F(t, y)$ *and* $G = G(s, z)$ *possess continuous partial derivatives of order* d *on* $I \times \mathbf{R}$ *and are such that the initial-value problem* (2.1) *possesses a unique solution* y *on* J.

Then the (*local*) *regularity of* y *is given by*

$$y \in C^{d+1,\, 2-\alpha}(J^{[0]})$$

and, for $\mu = 1, \ldots, M$, *by*

$$y \in C^{d+1,\, 2\mu+1-\alpha}(J^{[\mu]}).$$

Remark 2.1. We note in passing that the (global) analogue of Theorem 2.1 for $\alpha = 0$ was established in Brunner and Zhang [**10**, Theorem 3.1]: it essentially states that on $(\xi_{\mu-1}, \xi_{\mu+1})$, $\mu = 1, \ldots, M$, the solution corresponding to arbitrarily smooth data is in $C^{2\mu}$ but not in $C^{2\mu+1}$.

Proof. Set $\overline{G}(s) := G(s, y(\theta(s)))$. The following lemma will prove useful in the analysis of the degree of regularity of solutions to (2.1) and to more general VFIDEs with weakly singular kernels. Its straightforward proof is based on repeated integration by parts.

Lemma 2.2. *For sufficiently smooth* G, θ *and* ϕ,

(2.4)

$$\begin{aligned} H_\mu(t) &:= \int_{\xi_\mu}^t (t-s)^{-\alpha}\overline{G}(s)\,ds \\ &= \frac{1}{1-\alpha}\,\overline{G}(\xi_\mu)(t-\xi_\mu)^{1-\alpha} + \frac{1}{(1-\alpha)(2-\alpha)}\,\overline{G}'(\xi_\mu)(t-\xi_\mu)^{2-\alpha} \\ &\quad + \cdots + \frac{1}{(1-\alpha)_{m+1}}\,\overline{G}^{(m)}(\xi_\mu)(t-\xi_\mu)^{m+1-\alpha} \\ &\quad + \frac{1}{(1-\alpha)_{m+1}}\int_{\xi_\mu}^t \overline{G}^{(m+1)}(s)(t-s)^{m+1-\alpha}\,ds, \quad t \in J^{[\mu]}, \end{aligned}$$

with $(1-\alpha)_m := (1-\alpha)(2-\alpha)\cdots(m-\alpha)$.

(1) As we have already observed, in (2.2) and (2.3), the regularity of the solution to (2.1) on the interval $J^{[0]}$ is given by $y \in C^{d+1,\,2-\alpha}(J^{[0]})$ (recall also the remark following (1.11)). Furthermore, we always have $y(t_0) = \phi(t_0)$, while in general the equality $y'(t_0^+) = \phi'(t_0^-)$ is not true: the continuity of the derivative of the solution at the initial point $t = t_0$ is guaranteed only if $\phi(t)$ satisfies the condition $\phi'(t_0^-) = F(t_0, \phi(t_0))$. This is analogous to the situation in delay differential equations.

(2) We now turn to the interval $J^{[1]}$. Rewriting (2.1) on this interval as

$$y'(t) = F(t, y(t)) + H_0(t), \quad t \in J^{[1]}, \tag{2.5}$$

with given initial value $y(\xi_1)$, and recalling (2.4) we have

$$H_0(t) = \frac{1}{1-\alpha}\,\overline{G}(0)(t-t_0)^{1-\alpha} + \frac{1}{1-\alpha}\int_{t_0}^{t} \overline{G}'(s)(t-s)^{1-\alpha}\,ds.$$

Thus, we obtain

$$H_0'(t) = \overline{G}(0)(t-t_0)^{-\alpha} + \int_{t_0}^{t} \overline{G}'(s)(t-s)^{-\alpha}\,ds,$$

implying that y'' is continuous at the points ξ_μ, $\mu \geq 1$. But $y^{(3)}$ is discontinuous at ξ_1 and

$$\int_{t_0}^{t} \overline{G}'(s)(t-s)^{-\alpha}\,ds \in C^{1-\alpha}(J^{[1]}). \tag{2.6}$$

This is true since, for all $t \in J^{[1]}$,

$$\begin{aligned}
\left| \int_{t_0}^{t} \overline{G}'(s)(t-s)^{-\alpha}\,ds - \int_{t_0}^{\xi_1} \overline{G}'(s)(\xi_1-s)^{-\alpha}\,ds \right| \\
&\leq \left| \int_{\xi_1}^{t} \overline{G}'(s)(t-s)^{-\alpha}\,ds \right| \\
&\quad + \left| \int_{t_0}^{\xi_1} \left(\overline{G}'(s)(t-s)^{-\alpha} - \overline{G}'(s)(\xi_1-s)^{-\alpha} \right) ds \right| \\
&\leq L_{\overline{G}'} \left\{ \frac{2(t-\xi_1)^{1-\alpha}}{1-\alpha} + \frac{{\xi_1}^{1-\alpha}}{1-\alpha} - \frac{t^{1-\alpha}}{1-\alpha} \right\} \\
&\leq \frac{3L_{\overline{G}'}}{1-\alpha}\,(t-\xi_1)^{1-\alpha},
\end{aligned}$$

where $L_{\overline{G}'}$ denotes an upper bound of $|\overline{G}'(s)|$ in $\bar{J}^{[1]}$. Hence, $y \in C^{d+1,\,3-\alpha}(J^{[1]})$.

(3) Suppose now that $y \in C^{d+1,\,2\mu+1-\alpha}(J^{[\mu]})$, $\mu \geq 1$. To analyze the regularity of y on the interval $J^{[\mu+1]}$, we write (2.1) in the form

$$y'(t) = F(t, y(t)) + \int_{t_0}^{\xi_\mu} (t-s)^{-\alpha}\overline{G}(s)\,ds + H_\mu(t), \quad t \in \bar{J}^{[\mu+1]}, \tag{2.7}$$

where

$$H_\mu(t) := \int_{\xi_\mu}^{t} (t-s)^{-\alpha}\overline{G}(s)\,ds.$$

By (2.4) we can express $H_\mu(t)$ in the form

$$\begin{aligned} H_\mu(t) &= \frac{1}{1-\alpha}\,\overline{G}(\xi_\mu)(t-\xi_\mu)^{1-\alpha} \\ &\quad + \frac{1}{(1-\alpha)(2-\alpha)}\,\overline{G}'(\xi_\mu)(t-\xi_\mu)^{2-\alpha} \\ &\quad + \cdots + \frac{1}{(1-\alpha)_{2\mu+1}}\,\overline{G}^{(2\mu)}(\xi_\mu)(t-\xi_\mu)^{2\mu+1-\alpha} \\ &\quad + \frac{1}{(1-\alpha)_{2\mu+1}} \int_{\xi_\mu}^{t} \overline{G}^{(2\mu+1)}(s)(t-s)^{2\mu+1-\alpha}\,ds. \end{aligned} \tag{2.8}$$

Thus we can calculate the derivatives of order up to $2\mu+1$ of $H_\mu(t)$:

$$\begin{aligned} H_\mu'(t) &= \overline{G}(\xi_\mu)(t-\xi_\mu)^{-\alpha} + \frac{1}{1-\alpha}\,\overline{G}'(\xi_\mu)(t-\xi_\mu)^{1-\alpha} \\ &\quad + \cdots + \frac{1}{(1-\alpha)_{2\mu}}\,\overline{G}^{(2\mu)}(\xi_\mu)(t-\xi_\mu)^{2\mu-\alpha} \\ &\quad + \frac{1}{(1-\alpha)_{2\mu}} \int_{\xi_\mu}^{t} \overline{G}^{(2\mu+1)}(s)(t-s)^{2\mu-\alpha}\,ds. \end{aligned} \tag{2.9}$$

$$\vdots$$

$$\begin{aligned} H_\mu^{(2\mu+1)}(t) &= (\alpha)_{2\mu}\overline{G}(\xi_\mu)(t-\xi_\mu)^{-\alpha-2\mu} + \cdots + \overline{G}^{(2\mu)}(\xi_\mu)(t-\xi_\mu)^{-\alpha} \\ &\quad + \frac{1}{(1-\alpha)_{2\mu}} \int_{\xi_\mu}^{t} \overline{G}^{(2\mu+1)}(s)(t-s)^{-\alpha}\,ds. \end{aligned} \tag{2.10}$$

This reveals that $y \in C^{d+1,\,2(\mu+1)+1-\alpha}(J^{[\mu+1]})$, as asserted.

Remark 2.2. The $(2\mu + 1)$st derivative of the solution of (2.1) is unbounded at $t = \xi_\mu^+$: it behaves like

$$|y^{(2\mu+1)}(t)| \leq C\,(t - \xi_\mu)^{-\alpha}, \quad t \in J^{[\mu]}. \tag{2.11}$$

Compare also Remark 2.1 for the case $\alpha = 0$: here, $y^{(2\mu+1)}$is not continuous at $t = \xi_\mu$ but is of course bounded at $t = \xi_\mu^\pm$.

Remark 2.3. If the term $y(s)$ is also present in the kernel function G of the Volterra integral operator $\mathcal{V}_{\theta,\alpha}$, that is, if in (1.2) $G = G(s, y(\theta(s))$ is replaced by $G := G(s, y(s), y(\theta(s))$, then the regularity of the corresponding solution does not change: it is the one given in Theorem 2.1.

The following example shows that for special choices of F in (2.1), the local regularity of y in $J^{[\mu]}$ may even be higher, at least for certain values of $\alpha \in (0, 1)$.

Example 2.1. Consider the initial-value problem

$$\begin{aligned} &y'(t) = g(t) + \lambda \int_{t_0}^{t} (t - s)^{-\alpha} y(\theta(s))\, ds, \quad t \in J, \quad 0 < \alpha < 1, \\ &y(t) = \phi(t) = \phi_0 = \text{const.}, \quad t \leq t_0. \end{aligned} \tag{2.12}$$

On $J^{[0]}$ the above VFIDE reduces to

$$y'(t) = g(t) + \lambda\phi_0 \int_{t_0}^{t} (t - s)^{-\alpha}\, ds,$$

and hence we find

$$y(t) = \phi_0 + \int_{t_0}^{t} g(s)\, ds + \frac{\lambda\phi_0}{1 - \alpha}\,(t - t_0)^{2-\alpha}, \quad t \in J^{[0]}.$$

On $J^{[1]}$, equation (2.12) may be written as

$$y'(t) = g(t) + \lambda \int_{t_0}^{\xi_1} (t - s)^{-\alpha} \phi(\theta(s))\, ds + \lambda \int_{\xi_1}^{t} (t - s)^{-\alpha} y(\theta(s))\, ds.$$

A simple calculation shows that

(2.13)

$$y'(t) = g(t) + \frac{\lambda\phi_0}{1-\alpha}(t-t_0)^{1-\alpha} + \frac{\lambda}{1-\alpha}\int_{t_0}^{\theta(t)}(t-\theta^{-1}(s))^{1-\alpha}g(s)\,ds$$
$$+ \frac{\lambda^2\phi_0}{(1-\alpha)(2-\alpha)}\int_{\xi_1}^{t}(t-s)^{-\alpha}(\theta(s)-t_0)^{2-\alpha}\,ds, \quad t \in J^{[1]}.$$

This implies that, for any $\phi_0 \neq 0$,

$$y \in \begin{cases} C^{d+1,\,3-\alpha}(J^{[1]}) & \text{if } g \not\equiv 0, \\ C^{d+1,\,2(2-\alpha)}(J^{[1]}) & \text{if } g \equiv 0. \end{cases}$$

In particular, for the linear delay functions $\theta(t) = t - \tau$ (Example 1.1) and $\theta(t) = qt$, $0 < q < 1$, (Example 1.2) we have, respectively, $\theta^{-1}(t) = t+\tau$ and $\theta^{-1}(t) = t/q$; thus, the last integral in (2.13) becomes

$$\int_{\xi_1}^{t}(t-s)^{-\alpha}(s-\xi_1)^{2-\alpha}\,ds = B(1-\alpha, 3-\alpha)(t-\xi_1)^{3-2\alpha},$$

with $B(\cdot,\cdot)$ denoting the Euler beta function. While the regularity corresponding to $g \not\equiv 0$ confirms the result in Theorem 2.1, we see that $g \equiv 0$ yields a higher regularity at $t = \xi_1^+$, since $3 - \alpha < 4 - 2\alpha$, $0 < \alpha < 1$; in the latter case we have $y \in C^3(\bar{J}^{[1]})$ for $0 < \alpha \le 1/2$. For general μ, we find that

$$y \in C^{d+1,(\mu+1)(2-\alpha)}(J^{[\mu]}),$$

whenever $g(t) \equiv 0$ and $\phi_0 \neq 0$. This implies that in this case y exhibits the higher degree of regularity $(\mu+1)(2-\alpha)$ at $t = \xi_\mu^+$, compared to $2\mu + 1 - \alpha$ in Theorem 2.1, for all values of α satisfying $\alpha \in (0, 1/\mu)$.

3. VFIDEs with regular smoothing.

We now direct our attention to the general initial-value problem (1.6) (which we recall for the convenience of the reader),

$$(3.1) \qquad \begin{aligned} &y'(t) = F(t, y(t), y(\theta(t))) + (\mathcal{V}_{\theta,\alpha}y)(t), \quad t \in J, \\ &y(t) = \phi(t), \quad t \le t_0, \end{aligned}$$

with

$$(3.2)\qquad (\mathcal{V}_{\theta,\alpha}f)(t) := \int_{t_0}^{t} k_\alpha(t-s)G(s,f(\theta(s)))\,ds, \quad 0<\alpha<1.$$

Since, in contrast to (2.1), the first term on the right-hand side of (3.1) now also depends on $y(\theta(t))$, we no longer have the supersmoothing property of Theorem 2.1, as Theorem 3.1 reveals.

Theorem 3.1. *Assume that θ and ϕ in* (3.1) *satisfy the assumptions* (i)–(ii) *of Theorem* 2.1*, and let $F = F(t,y,z)$, $G = G(t,z)$ possess continuous partial derivatives of order d on $J \times \mathbf{R} \times \mathbf{R}$ and $J \times \mathbf{R}$, respectively, and be such that the initial-value problem* (3.1) *has a unique solution on J. Then, for $0<\alpha<1$ and F with $\partial F/\partial z \not\equiv 0$, the local regularity of the solution of the initial-value problem* (3.1) *is given by*

$$y \in C^{d+1,\,\mu+2-\alpha}(J^{[\mu]}), \quad \mu = 0,1,\dots,M.$$

Proof. Due to the presence of the term $y(\theta(t))$ in F, the degree of smoothing at $t=\xi_\mu$ of the solution to equation (3.1) is now governed by the "DDE part" of the VFIDE (3.1): supersmoothing is no longer possible, and we have only "regular" smoothing given by $y \in C^{d+1,\,\mu+1-\alpha}(J^{[\mu]})$, $\mu = 0,1,\dots,M$. This can be readily verified by combining the analysis of the regularity of solutions to delay differential equations,

$$z'(t) = F(t,z(t),z(\theta(t))),$$

see, e.g., Bellen and Zennaro [**1**], with the arguments employed in the proof of Theorem 2.1.

Remark 3.1. The regularity result of Theorem 3.1 remains valid if the function F in (4.1) also depends on the memory term $(\mathcal{V}y)(t)$ introduced in Section 1.

Consider now the initial-value problem (1.7): here, the Volterra operator $\mathcal{U}_{\theta,\alpha}$ contains the kernel function $U(s,y'(\theta(s)))$, cf. (1.3), in contrast to $G(s,y(\theta(s)))$ in $\mathcal{V}_{\theta,\alpha}$ in (1.6). It turns out that in this case there is a slight modification in the smoothing of y at the points $\{\xi_\mu\}$, as Theorem 3.2 shows.

Theorem 3.2. *Under the assumptions of Theorem*3.1*, with* U *replacing* G*, the solution of the initial-value problem* (1.7) *possesses the local regularity properties described by*

$$y \in C^{d+1,\,2-\alpha}(J^{[0]})$$

and, for $\mu = 1, \dots, M$*,*

$$y \in C^{d+1,\,\mu+1-\alpha}(J^{[\mu]}).$$

Proof. (1) We first consider the regularity of the solution of (1.7) at the point $\xi_0 := t_0$. The initial condition implies that, for any ϕ, we have $y(t_0) = \phi(t_0)$. In general, however, $\phi'(t_0^-) \neq y'(t_0^+)$. Hence, y is continuous at ξ_μ ($\mu \geq 0$), but the derivative y' is in general not continuous at ξ_0. Moreover, for C^d-data F, G and ϕ, we have

$$y' \in C^{d,1-\alpha}(J^{[0]}) \Longrightarrow y \in C^{d+1,\,2-\alpha}(J^{[0]}).$$

(2) Turning to the interval $J^{[1]}$, it is clear that y' is continuous at ξ_1; hence y' is continuous at ξ_μ for $\mu \geq 2$. Moreover, we can show, by adapting the argument used to verify (2.6), that

$$H_1(t) := \int_{\xi_1}^{t} (t-s)^{-\alpha}\overline{U}(s)\,ds \in C^{1-\alpha}(J^{[1]}),$$

where $\overline{U}(s) := U(s, y'(\theta(s)))$. Hence the solution of (1.7) satisfies $y \in C^{d+1,\,2-\alpha}(J^{[1]})$.

(3) Suppose now that $y \in C^{d+1,\,\mu+1-\alpha}(J^{[\mu]})$, $\mu \geq 1$. In order to verify that $y \in C^{d+1,\,\mu+2-\alpha}(J^{[\mu+1]})$, we rewrite the equation (1.7) in the form

$$y'(t) = F(t, y(t), y(\theta(t))) + \int_{t_0}^{\xi_\mu} (t-s)^{-\alpha}\overline{U}(s)\,ds + H_\mu(t), \quad t \geq \xi_\mu.$$

In analogy to the proof of Theorem 2.1, we express the term $H_\mu(t) := \int_{\xi_\mu}^t (t-s)^{-\alpha}\overline{U}(s)\,ds$ as

$$\begin{aligned} H_\mu(t) &= \frac{1}{1-\alpha}\,\overline{U}(\xi_\mu)(t-\xi_\mu)^{1-\alpha} \\ &\quad+ \frac{1}{(1-\alpha)(2-\alpha)}\,\overline{U}'(\xi_\mu)(t-\xi_\mu)^{2-\alpha} \\ &\quad+ \cdots + \frac{1}{(1-\alpha)_{\mu+1}}\,\overline{U}^{(\mu)}(\xi_\mu)(t-\xi_\mu)^{\mu+1-\alpha} \\ &\quad+ \frac{1}{(1-\alpha)_{\mu+1}}\int_{\xi_\mu}^t \overline{U}^{(\mu)}(s)(t-s)^{\mu+1-\alpha}\,ds. \end{aligned} \tag{3.3}$$

Thus, we can calculate the derivatives of order up to $\mu+1$ of $H_\mu(t)$: we obtain

$$\begin{aligned} H_\mu'(t) = \overline{U}(\xi_\mu)(t-\xi_\mu)^{-\alpha} &+ \frac{1}{1-\alpha}\,\overline{U}'(\xi_\mu)(t-\xi_\mu)^{1-\alpha} \\ &+ \cdots + \frac{1}{(1-\alpha)_\mu}\,\overline{U}^{(\mu)}(\xi_\mu)(t-\xi_\mu)^{\mu-\alpha} \\ &+ \frac{1}{(1-\alpha)_\mu}\int_{\xi_\mu}^t \overline{U}^{(\mu+1)}(s)(t-s)^{\mu-\alpha}\,ds, \end{aligned} \tag{3.4}$$

and hence,

$$\begin{aligned} H_\mu^{(\mu+1)}(t) &= (\alpha)_\mu\overline{U}(\xi_\mu)(t-\xi_\mu)^{-\alpha-\mu} + \cdots + \overline{U}^{(\mu)}(\xi_\mu)(t-\xi_\mu)^{-\alpha} \\ &\quad+ \frac{1}{(1-\alpha)_\mu}\int_{\xi_\mu}^t \overline{U}^{(\mu)}(s)(t-s)^{-\alpha}\,ds. \end{aligned} \tag{3.5}$$

Hence, $y^{(\mu+2)}(t)$ is continuous at $\xi_{\mu+1}$. Furthermore, we see that

$$\int_{\xi_\mu}^t \overline{U}^{(\mu+1)}(s)(t-s)^{-\alpha}\,ds \in C^{1-\alpha}(J^{[\mu+1]}),$$

and this implies $y \in C^{d+1,\,\mu+2-\alpha}(J^{[\mu+1]})$. This completes the proof of Theorem 3.2. □

Example 3.1. As an illustration we use the analogue of the VFIDE of Example 2.1,

$$\begin{aligned} y'(t) &= g(t) + \lambda \int_{t_0}^{t} (t-s)^{-\alpha} y'(\theta(s))\, ds, \quad t \in J, \\ y(t) &= \phi_0 + \phi_1 \cdot (t - t_0), \quad t \leq t_0. \end{aligned}$$

On $J^{[0]}$ we have

$$y'(t) = g(t) + \frac{\lambda \phi_1}{1-\alpha}\,(t - t_0)^{1-\alpha},$$

and this shows that $y \in C^{d+1,\,2-\alpha}(J^{[0]})$.

A straightforward calculation then gives, for $t \in J^{[1]}$,

$$\begin{aligned} y'(t) = g(t) &+ \frac{\lambda \phi_1}{1-\alpha}\,(t-t_0)^{1-\alpha} - \frac{\lambda \phi_1}{1-\alpha}\,(t-\xi_1)^{1-\alpha} \\ &+ \int_{\xi_1}^{t} (t-s)^{-\alpha} g(\theta(s))\, ds \\ &+ \frac{\lambda^2 \phi_1}{1-\alpha} \int_{\xi_1}^{t} (t-s)^{-\alpha} (\theta(s) - t_0)^{1-\alpha}\, ds. \end{aligned}$$

This yields $y' \in C^{d,\,1-\alpha}(J^{[1]})$, and thus $y \in C^{d+1,\,2-\alpha}(J^{[1]})$. Smoothing will take effect on $J^{[2]}$, as can readily be verified.

4. VFIDEs with no smoothing.

We first consider the neutral initial-value problem

$$(4.1) \qquad \begin{aligned} y'(t) &= F(t, y(t), y'(\theta(t))) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in J, \\ y(t) &= \phi(t), \quad t \leq t_0. \end{aligned}$$

Theorem 4.1. *Let θ be subject to the hypotheses* (D1)–(D3) *of Section* 1; *$\phi \in C^d[\theta(t_0), t_0]$, withd ≥ 1; and assume that $F = F(t, y, w)$ and $G = G(t, z)$ satisfies the assumptions of Theorem* 3.1. *Then the solution y of* (4.1) *does not exhibit any smoothing at the points $\{\xi_\mu\}$: we have*

$$y \in C^{d,\,2-\alpha}(J^{[\mu]}) \quad \textit{for all} \quad \mu = 0, 1, \ldots, M.$$

Proof. Since there is no smoothing in the solution of a neutral delay differential equation at $t = \xi_\mu$, as μ increases [**1, 15, 16**], the presence of the Volterra integral operator $\mathcal{V}_{\theta,\alpha}$ in (4.1) will in general not change this. In $J^{[0]}$, the initial-value problem (4.1) reduces to

$$y'(t) = F(t, y(t), \phi'(t)) + (\mathcal{V}_{\theta,\alpha}\phi)(t), \quad y(t_0) = \phi(t_0),$$

and, for $\phi \in C^d(J)$, $d \geq 1$, it follows, as in the proof of Theorem 2.1, that

$$y' \in C^{d-1,\,1-\alpha}(J^{[0]}) \Longrightarrow y \in C^{d,\,2-\alpha}(J^{[0]}).$$

For $\mu \geq 1$ we readily see, by proceeding along the lines of the proof of Theorem 1 and by recalling the remark at the start of the present proof, that smoothing at $t = \xi_\mu$ is in general not possible if $\partial F(t, y, w)/\partial w \not\equiv 0$. In other words, the solution of (4.1) has the property that $y \in C^{d,\,2-\alpha}(J^{[\mu]})$ for $\mu \geq 1$, too.

Remark 4.1. The regularity result of Theorem 4.1 remains true if in (4.1) the operator $\mathcal{V}_{\theta,\alpha}$ is replaced by $\mathcal{U}_{\theta,\alpha}$, cf. (1.3).

We now turn to the neutral VFIDE (1.8),

$$\begin{aligned} \frac{d}{dt}\left[y(t) - (\mathcal{V}_{\theta,\alpha}y)(t)\right] &= F(t, y(t), y(\theta(t))), \quad t \in J, \quad 0 < \alpha < 1, \\ y(t) &= \phi(t), \quad t \leq t_0, \end{aligned} \tag{4.2}$$

where we encounter nonsmoothing similar to that in the solution to (4.1).

Theorem 4.2. *Under the assumptions of Theorem* 3.1 *the solution* y *of the initial-value problem* (4.2) *satisfies*

$$y \in C^{d,\,1-\alpha}(J^{[\mu]}) \quad \textit{for} \quad \mu = 0, 1, \ldots, M.$$

Proof. (I) Setting

$$z(t) := y(t) - (\mathcal{V}_{\theta,\alpha}y)(t),$$

the initial-value problem (4.2) can be rewritten as

$$\begin{aligned} z'(t) &= Q(t, z(t), y(\theta(t))), \quad t \in J, \\ z(t_0) &= \phi(t_0); \\ y(t) &= z(t) + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in J, \end{aligned}$$

with $y(t) = \phi(t)$ if $t \leq t_0$. Here, the function Q is defined by

$$Q(t, z, w) := F(t, z(t) + (\mathcal{V}_{\theta,\alpha} y)(t), w)$$

(compare also Brunner [**6**, Section 1]). Locally, on $\bar{J}^{[\mu]}$, we have

$$\begin{aligned} z'(t) &= Q(t, z(t), y(\theta(t))), \quad t \in J^{[\mu]}), \\ z(\xi_\mu) &= y(\xi_\mu) - (\mathcal{V}_{\theta,\alpha} y)(\xi_\mu). \end{aligned} \tag{4.3}$$

This shows that on each $J^{[\mu]}$ we are faced with an initial-value problem for an ODE with nonsmooth ($C^{1-\alpha}$) forcing function induced by the term $(\mathcal{V}_{\theta,\alpha} y)(t)$. Hence, we can again resort to Lemma 2.2, as, e.g., in the proof of Theorem 2.1; here, however, these forcing terms do not become more regular at $t = \xi_\mu^+$ as μ increases.

(II) Alternatively, the result of Theorem 4.2 can also be established by considering the *integrated form* of (4.2),

$$\begin{aligned} &y(t) = \phi(t_0) + \int_{t_0}^{t} F(s, y(s), y(\theta(s)))\, ds + (\mathcal{V}_{\theta,\alpha} y)(t), \quad t \in J, \\ &y(t) = \phi(t), \quad t \leq t_0. \end{aligned} \tag{4.4}$$

This is reminiscent of (1.6), with the difference that the role of $y'(t)$ in (1.6) is now assumed by $y(t)$. Thus, the regularity of the solution to (4.4) can again be established by resorting to Lemma 2.1: as a consequence of the observation in the previous sentence, the degree of regularity now decreases by one, and we obtain $y \in C^{d,\,1-\alpha}(J^{[\mu]})$ for $\mu = 0, 1, \ldots, M$.

In order to make these arguments more transparent, the reader may also wish to consider the linear version of (4.2),

$$\frac{d}{dt}\,[y(t) - (\mathcal{V}_{\theta,\alpha} y)(t)] = a(t)y(t) + b(t)y(\theta(t)) + g(t),$$

with $\mathcal{V}_{\theta,\alpha}$ as in Example 2.1, cf. (2.12). Equation (4.3) then becomes

$$z'(t) = a(t)z(t) + a(t)(\mathcal{V}_{\theta,\alpha}y)(t) + b(t)y(\theta(t)) + g(t).$$

Using the variation-of-constants formula, with $r(t,s) := \exp(\int_s^t a(v)\,dv)$, the solution on $\bar{J}^{[\mu]}$ can be written as

$$z(t) = r(t,\xi_\mu)z(\xi_\mu) + \int_{\xi_\mu}^t r(t,s)\{(\mathcal{V}_{\theta,\alpha}y)(s) + b(s)y(\theta(s)) + g(s)\}\,ds.$$

This shows that $z \in C^{d+1,\,2-\alpha}(J^{[0]})$. Hence, by definition of y,

$$y(t) = z(t) + (\mathcal{V}_{\theta,\alpha}y)(t),$$

with $y(t) = \phi(t)$ when $t \le t_0$, we obtain $y \in C^{d,\,1-\alpha}(J^{[0]})$. An analogous argument can be used on $J^{[\mu]}$, $\mu = 1,\dots,M$, to establish the regularity result of Theorem 4.2.

Remark 4.2. The regularity result of Theorem 4.2 remains valid, under appropriately modified regularity assumptions on $F = F(t,y,z,w)$, for solutions of the neutral VFIDEs

$$\frac{d}{dt}\,[y(t) - (\mathcal{V}_{\theta,\alpha}y)(t)] = F(t, y(t), y(\theta(t)), (\mathcal{V}y)(t)). \tag{4.5}$$

This follows readily from either of the above proofs: the presence of the Volterra integral operator $\mathcal{V}$,

$$(\mathcal{V}y)(t) := \int_{t_0}^t b(t,s)K(s,y(s))\,ds,$$

with smooth kernel $b(t,s)$ and smooth G, clearly does not affect the regularity of the solution of this more general VFIDE.

5. "Implicit" neutral VFIDEs.

This paper was motivated in part by a class of VFIDEs arising in the mathematical modeling of certain aeroelastic systems (see, e.g., [**3, 11, 12, 18, 19**] for details and additional references). The generic form of the resulting initial-value problem is

$$\begin{aligned} &\frac{d}{dt}\,[y(t) - (\mathcal{W}_{\theta,\alpha}y)(t)] = F(t, y(t), y(\theta(t))), \quad t \in J := [t_0, T], \\ &y(t) = \phi(t), \quad t \le t_0, \end{aligned} \tag{5.1}$$

with $\mathcal{W}_{\theta,\alpha}$ defined in (1.4); it may be viewed as the "implicit" counterpart of equation (1.8) (or (4.5) if F also depends on $(\mathcal{V}y)(t)$). The regularity properties of its solution are, not too surprisingly, very similar to those for (1.8) (Theorem 4.2).

Theorem 5.1. *Assume that the given functions θ, G, F, ϕ in* (5.1) *and* (1.4) *are subject to the assumptions in Theorem* 4.2. *Then there is no smoothing in the solution y of* (5.1): *locally, we have*

$$y \in C^{d,\,1-\alpha}(J^{[\mu]}) \quad \textit{for all} \quad \mu = 0, 1, \dots, M.$$

Proof. Consider first the regularity of the solution for (5.1) at the point $\xi_0 := t_0$. It is possible to choose $y(t_0) = \phi(t_0)$. The continuity of the derivative of the solution can be guaranteed at the initial point ξ_0 only for $\phi(t)$ satisfying the condition

$$\phi'(t_0^-) = \frac{d}{dt}\left(\int_{\theta(t)}^{t} (t-s)^{-\alpha} G(s, \phi(s))\, ds \right) + F(t_0, \phi(t_0), \phi(\theta(t_0))).$$

On the interval $J^{[0]}$ the initial-value problem (5.1) reduces to

$$\frac{d}{dt}\left[y(t) - (\mathcal{W}_{\theta,\alpha} y)(t)\right] = F(t, y(t)\phi(t),$$

with

$$(\mathcal{W}_{\theta,\alpha} y)(t) = \int_{\theta(t)}^{t_0} k_\alpha(t-s) G(s, \phi(s))\, ds + \int_{t_0}^{t} k_\alpha(t-s) G(s, y(s))\, ds.$$

Equivalently, we may use the integrated form of (5.1),

$$y(t) = \Phi_0 + (\mathcal{W}_{\theta,\alpha} y)(t) + \int_{t_0}^{t} F(s, y(s), y(\theta(s)))\, ds,$$

with

$$\Phi_0 := \phi(t_0) - (\mathcal{W}_{\theta,\alpha} y)(t_0),$$

as the basis for our regularity analysis. In either case, it follows that $y \in C^{d,\,1-\alpha}(J^{[0]})$, see also Lubich [**21**] or Brunner [**5**, Section 6.1].

Hence, at $t = \xi_0^+$ the derivative of its solution y is unbounded: it behaves like $C(t-\xi_0)^{-\alpha}$ for some $C \neq 0$.

Consider now the regularity at the point $t = \xi_1^+$. We write (5.1) in the form

$$\begin{aligned} y'(t) &= F(t, y(t), y(\theta(t))) + \frac{d}{dt}\,(H_0(t))\,, \quad t \in J, \\ y(t) &= \phi(t), \quad t \leq t_0, \end{aligned}$$

with

$$H_0(t) := \int_{\theta(t)}^{t} (t-s)^{-\alpha}\overline{G}(s)\,ds,$$

and $\overline{G}(s) := G(s, y(s))$. It follows from (2.4) (Lemma 2.2) that

$$H_0(t) = \frac{1}{1-\alpha}\,\overline{G}(\theta(t))(t-\theta(t))^{1-\alpha} + \frac{1}{1-\alpha}\int_{\theta(t)}^{t} \overline{G}'(s)(t-s)^{1-\alpha}\,ds.$$

Thus,

$$\begin{aligned} \frac{d}{dt}\,(H_0(t)) &= \overline{G}(\theta(t))(1-\theta'(t))(t-\theta(t))^{-\alpha} \\ &\quad + \frac{1}{1-\alpha}\,\overline{G}'(\theta(t))\theta'(t)(t-\theta(t))^{1-\alpha} \\ &\quad + \int_{\theta(t)}^{t} \overline{G}'(s)(t-s)^{-\alpha}\,ds. \end{aligned}$$

Hence, y' is discontinuous at the point ξ_1 and thus there is no smoothing to the solution of (5.1).

To prove that $y \in C^{d,\,1-\alpha}(J^{[\mu]})$, it is sufficient to verify that

$$\left| \int_{\theta(t)}^{t} (t-s)^{-\alpha} G'(y(s))\,ds \right| \leq C(t-\xi_1)^{1-\alpha}. \tag{5.2}$$

Since $\overline{G}(t) = G(t, y(t))$ is continuous for $t \in J$, we obtain

$$\begin{aligned} \left| \int_{\theta(t)}^{t} (t-s)^{-\alpha} G(s, y(s))\,ds \right| &\leq L_G \int_{\theta(t)}^{t} (t-s)^{-\alpha}\,ds \\ &\leq C(t-\theta(t))^{1-\alpha} \leq C(t-\xi_1)^{1-\alpha}. \end{aligned} \tag{5.3}$$

In the last step of (5.3), we have used the fact that θ is strictly increasing in I, that is,

$$\xi_1 = \theta(\xi_2) > \theta(t) > \theta(\xi_1) = \xi_0, \quad \forall\, t \in (\xi_1, \xi_2).$$

This allows us to establish the estimate

$$|y(t)| \le C(t - \xi_\mu)^{1-\alpha}, \quad t \in J^{[\mu]}$$

which is valid for $\mu = 0, 1, \dots, M$.

Remark 5.1. The regularity result of Theorem 5.1 remains true if the right-hand side F in (5.1) also depends on $(\mathcal{V}y)(t)$.

6. Summary of regularity results. In Table 1 we provide an overview of the regularity results derived in Sections 2–5.

TABLE 1. Regularity and smoothing of solutions to weakly singular VFIDEs.

VFIDE (C^d-data, $d \ge 0$; $0 < \alpha < 1$)	Eq. #	Regularity in $J^{[\mu]} := (\xi_\mu, \xi_{\mu+1}]$, $\mu \ge 0$	Theorem / Remark
$y'(t) = F(t, y(t)) + (\mathcal{V}_{\theta,\alpha}y)(t)$	(2.1)	$C^{d+1,\, 2\mu+1-\alpha}$ (supersmoothing)	Thm. 2.1 Rem. 2.3
$y'(t) = F(t, y(t), y(\theta(t))) + (\mathcal{V}_{\theta,\alpha}y)(t)$	(3.1)	$C^{d+1,\, \mu+2-\alpha}$	Thm. 3.1 Rem. 3.1
$y'(t) = F(t, y(t), y(\theta(t))) + (\mathcal{U}_{\theta,\alpha}y)(t)$	(1.7)	$\{ \begin{smallmatrix} C^{d+1,\, 2-\alpha} & \text{if } \mu = 0 \\ C^{d+1,\, \mu+1-\alpha} & \text{if } \mu \ge 1 \end{smallmatrix}.$ (regular smoothing)	Thm. 3.2
$y'(t) = F(t, y(t), y(\theta(t)), y'(\theta(t))) + (\mathcal{V}_{\theta,\alpha}y)(t)$	(4.1)	$C^{d,\, 2-\alpha}$	Thm. 4.1 Rem. 4.1
$d[y(t) - (\mathcal{V}_{\theta,\alpha}y)(t)]/dt = F(t, y(t), y(\theta(t)), (\mathcal{V}y)(t))$	(1.8)	$C^{d,\, 1-\alpha}$	Thm. 4.2 Rem. 4.2
$d[y(t) - (\mathcal{W}_{\theta,\alpha}y)(t)]/dt = F(t, y(t), y(\theta(t)), (\mathcal{V}y)(t))$	(5.1)	$C^{d,\, 1-\alpha}$ (no smoothing)	Thm. 5.1 Rem. 5.1

7. Concluding remarks. Similar regularity and smoothing properties can be derived for solutions to initial-value problems for VFIDEs with logarithmic kernels, $k_1(t-s) = \log(t-s)$, cf. (1.5) with $\alpha = 1$. On $J^{[0]}$ these results follow from the ones in Brunner, Pedas and Vainikko [**7, 8**]: if the given functions in the "classical" (linear) VIDE,

$$y'(t) = a(t)y(t) + g(t) + \int_{t_0}^{t} \log(t-s)K(t,s)y(s)\,ds, \quad t \in J \tag{7.1}$$

(with $K(t,t) \neq 0,\ t \in J$) are in C^d, then the solutions of (7.1) satisfy

$$\begin{aligned} y &\in C^{d+1,1}(t_0,T] \\ &:= \{y \in C^{d+1}(t_0,T] : |y^{(j)}(t)| \leq C_j(t-t_0)^{1-j} \text{ if } j \geq 2,\ t > t_0\}. \end{aligned}$$

Details on the extension of the results in Sections 2–6 to VFIDEs with Volterra operators $\mathcal{V}_{\theta,1}$, $\mathcal{U}_{\theta,1}$ and $\mathcal{W}_{\theta,1}$ are left to the reader.

Due to limitations of space we defer the analysis of the regularity and smoothing properties of solutions to the considerably more complex VFIDE

$$\begin{aligned} \frac{d}{dt}\left[(\mathcal{W}_{\theta,\alpha}y)(t)\right] &= F(t,y(t),y(\theta(t)),(\mathcal{V}y)(t)), \quad t \in (t_0,T], \\ y(t) &= \phi(t), \quad t \leq t_0, \end{aligned} \tag{7.2}$$

with $\mathcal{W}_{\theta,\alpha}$ given by (1.4). An important special case of (7.2) corresponds to $F(t,y,z) = g(t)$: the resulting VFIDE

$$\frac{d}{dt}\left[(\mathcal{W}_{\theta,\alpha}y)(t)\right] = g(t), \quad t \in (t_0,T], \tag{7.3}$$

can be rewritten as

$$\int_{\theta(t)}^{t} k_\alpha(t-s)G(s,y(s))\,ds = \Phi_0 + \int_{t_0}^{t} g(s)\,ds, \quad t \in (t_0,T],$$

with

$$\Phi_0 := \int_{\theta(t_0)}^{t_0} k_\alpha(t_0-s)G(s,\phi(s))\,ds.$$

This is equivalent to

$$\int_{t_0}^{t} k_\alpha(t-s)G(s,y(s))\,ds = h(t), \quad t \in (t_0,T], \tag{7.4}$$

where

$$h(t) := \Phi_0 + \int_{t_0}^{t} g(s)\,ds - \int_{\theta(t)}^{t_0} k_\alpha(t-s)G(s,\phi(s))\,ds.$$

Equation (7.4) represents a nonlinear first-kind Volterra integral equation with weakly singular kernel; since we have $h(t_0) = 0$, it has a bounded solution in $J = [t_0, T]$, with $y(t_0^+) \neq \phi(t_0)$ in general.

For results on the existence of solutions of (7.2), with linear $G(s,y) = b(s)y$ and $\theta(t) = t - \tau$, the reader is referred to Kappel and Zhang [**20**, equation (7.3)] and Ito and Turi [**19**]; see also Brunner [**3**] for additional references. The existence and uniqueness of solutions for nonlinear Volterra equations of the form (7.4) is studied in Deimling [**13**].

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