

THE p -VERSION OF THE BOUNDARY ELEMENT METHOD FOR A THREE-DIMENSIONAL CRACK PROBLEM

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ABSTRACT. We study the p -version of the boundary element method for a crack problem in linear elasticity with Dirichlet boundary conditions. The unknown jump of the traction has strong edge singularities and is approximated by solving an integral equation of the first kind with weakly singular operator. We prove a quasi-optimal a priori error estimate in the energy norm. For sufficiently smooth given data, this gives a convergence like $cp^{-1+\varepsilon}$ with $\varepsilon > 0$. Here, p denotes the polynomial degree of the piecewise polynomial functions used to approximate the unknown.

1. Introduction and formulation of the problem. We analyze the convergence of the p -version of the boundary element method (BEM) with weakly singular integral operator for problems in \mathbf{R}^3 . That is, we study approximation properties of piecewise polynomial functions on surfaces in a negative order Sobolev space (order $-1/2$).

The p -version uses piecewise polynomial spaces on fixed meshes and increases the polynomial degrees, whereas the more conventional h -version improves approximations by mesh refinement and by using piecewise polynomials of lower, fixed degrees. It is well known that an appropriate combination of mesh refinement and polynomial degree distribution (the hp -version with geometrically graded meshes) may lead to an exponential rate of convergence, even in the presence of singularities (for the FEM, see [9, 10] and for the BEM we refer to [13–15, 17]). The approximation strategy of such hp -methods is to use polynomial degrees of lowest order where solutions behave singularly and to use high order polynomials where solutions are smooth. With

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respect to theory, this strategy has the advantage that it completely avoids the approximation analysis of singular functions by high order polynomials. This is different for the p -version which makes its analysis more involved. Actually, only relatively little is known for problems in three dimensions. In this paper we fill one of the gaps in theory by studying for the first time the p -version of the BEM with weakly singular operators on surfaces. For numerical results studying the h -, p -, and hp -versions, we refer to [14].

We also note that high order polynomials have much better approximations properties for wave problems in the sense that they reduce very efficiently the pollution effect of the oscillatory behavior of solutions, see [1, 16]. Therefore, the p -version (combined with mesh refinements but using high degrees everywhere) becomes attractive for wave problems.

The p -version (and hp -version with quasi-uniform meshes) of the BEM on curves have been widely studied, see [11, 12, 20, 21]. As mentioned before, there are very few results in three dimensions, i.e., on surfaces. The case of hypersingular operators on polyhedral surfaces (the energy space is $H^{1/2}$) is analyzed in [18]. There, the optimal convergence of the p -version has been shown by making use of the H^1 -regularity of the solution. In [5] we consider hypersingular operators on open surfaces, where no H^1 -regularity can be assumed, and prove optimal a priori error estimates. The case of weakly singular integral operators on surfaces has been an open problem so far. Here we study this situation for the model problem of linear elasticity with a crack Γ that has a smooth boundary. The solution exhibits in general strong edge singularities not being L_2 -regular.

Throughout the paper Γ denotes an open smooth surface in \mathbf{R}^3 with smooth boundary curve γ (γ is locally C^∞). Let $H^t(\Gamma)$ and $\tilde{H}^t(\Gamma)$ be the usual Sobolev spaces, see Section 3 for a definition. We will use these notations for scalar functions as well as for vector functions. The latter will be denoted by boldface symbols, the norms and inner products for them are defined componentwise.

Let us formulate the model problem. We consider the Dirichlet boundary value problem for the displacement field $\mathbf{u} = (u_1, u_2, u_3)$ of a homogeneous, isotropic, elastic material covering the domain

$\Omega_\Gamma := \mathbf{R}^3 \setminus \bar{\Gamma}$: For given $\mathbf{u}_1, \mathbf{u}_2 \in H^{1/2}(\Gamma)$ with $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{1/2}(\Gamma)$ find \mathbf{u} satisfying

$$(1.1) \quad \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega_\Gamma,$$

$$(1.2) \quad \mathbf{u}|_{\Gamma_1} = \mathbf{u}_1, \quad \mathbf{u}|_{\Gamma_2} = \mathbf{u}_2,$$

$$(1.3) \quad \mathbf{u}(x) = o(1), \quad \frac{\partial}{\partial x_j} \mathbf{u}(x) = o(|x|^{-1}), \quad j = 1, 2, 3, \quad |x| \rightarrow \infty.$$

Here, $\Gamma_i, i = 1, 2$, are the two sides of Γ and $\mu > 0, \lambda > -2/3\mu$ are the given Lamé constants. The corresponding Neumann data of the linear elasticity problem are the tractions

$$\mathbf{T}(\mathbf{u}) := \lambda(\operatorname{div} \mathbf{u})\mathbf{n} + 2\mu \frac{\partial \mathbf{u}}{\partial \mathbf{n}} + \mu \mathbf{n} \times \operatorname{curl} \mathbf{u} \quad \text{on } \Gamma_i, \quad i = 1, 2,$$

where \mathbf{n} is the normal vector on Γ pointing into a specified direction.

The problem (1.1)–(1.3) can be formulated as an integral equation of the first kind, see, e.g., [7, 19]: $\mathbf{u} \in H_{\text{loc}}^1(\mathbf{R}^3 \setminus \bar{\Gamma})$ is the solution of the Dirichlet problem (1.1)–(1.3) if and only if the jump of the traction $\mathbf{t} := \mathbf{T}(\mathbf{u})|_{\Gamma_1} - \mathbf{T}(\mathbf{u})|_{\Gamma_2} \in \tilde{H}^{-1/2}(\Gamma)$ solves the weakly singular integral equation

$$(1.4) \quad \mathbf{Vt}(x) := \int_\Gamma \mathbf{E}(y, x)\mathbf{t}(y) \, ds_y = \mathbf{g}(x), \quad x \in \Gamma$$

where

$$\mathbf{g}(x) = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2)(x) + \int_\Gamma \mathbf{T}_y \mathbf{E}(y, x)(\mathbf{u}_1 - \mathbf{u}_2)(y) \, ds_y.$$

Here,

$$\mathbf{E}(y, x) = \frac{\lambda + 3\mu}{8\pi\mu(\lambda + 2\mu)} \left(\frac{1}{|x - y|} I + \frac{\lambda + \mu}{\lambda + 3\mu} \frac{(x - y)(x - y)^T}{|x - y|^3} \right)$$

denotes the fundamental solution of (1.1) with the identity matrix I . The solution \mathbf{t} of (1.4) yields the solution to problem (1.1)–(1.3) via the representation or Betti’s formula

$$\mathbf{u}(x) = \int_\Gamma \left(\mathbf{E}(y, x)\mathbf{t}(y) - (\mathbf{T}_y \mathbf{E}(y, x))^T (\mathbf{u}_1(y) - \mathbf{u}_2(y)) \right) ds_y, \quad x \notin \Gamma.$$

In what follows, together with usual space coordinates $(x_1, x_2, x_3) = x \in \Gamma$, we will use surface coordinates (s, ρ) in a small neighborhood of γ on Γ such that s , respectively ρ , varies in tangential, respectively normal, direction to γ . Thus, the boundary curve γ is described by the equation $\rho = 0$, and in a sufficiently small neighborhood of γ one has $s = s(x)$ and $\rho = \rho(x)$. Throughout the paper, we will specify this small neighborhood of γ as the boundary strip Γ_δ of Γ such that, for small $\delta > 0$,

$$\Gamma_\delta = \{x \in \Gamma; 0 < \rho(x) < \delta\}.$$

Let us cite the following regularity result from [7].

Proposition 1.1. *Let $|\sigma| < 1/2$ and $\mathbf{u}_j \in H^{3/2+\sigma}(\Gamma)$, $j = 1, 2$, with $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{3/2+\sigma}(\Gamma)$. Then the solution $\mathbf{t} \in \tilde{H}^{-1/2}(\Gamma)$ of the integral equation (1.4) has the form*

$$(1.5) \quad \mathbf{t} = \boldsymbol{\beta}(s)\rho^{-1/2}\chi(\rho) + \mathbf{t}_0$$

with vector functions $\boldsymbol{\beta} \in H^{1/2+\sigma}(\gamma)$ and $\mathbf{t}_0 \in \tilde{H}^{1/2+\sigma'}(\Gamma)$ for any $\sigma' < \sigma$. Furthermore, $\chi \in C_0^\infty(\mathbf{R})$ denotes a cutoff function with $0 \leq \chi \leq 1$ and $\chi = 1$ near zero.

In the next section we formulate the p -version of the BEM for the approximate solution of (1.4) and state the main result which proves an almost optimal convergence rate, Theorem 2.1. Technical details and the proof of Theorem 2.1 are given in Section 3.

2. The p -version of the BEM. Below p will always denote a polynomial degree, and C is a generic positive constant independent of p .

In order to define finite-dimensional subspaces of $\tilde{H}^{-1/2}(\Gamma)$, we use a regular parameter representation $x = X(u)$, $u \in U$, U being a compact region in \mathbf{R}^2 whose boundary is mapped onto γ . On U we use a fixed regular mesh $\mathcal{T} = \{U_j; j = 1, \dots, J\}$ of quadrilaterals and triangles which are in general curvilinear such that U is completely discretized. We assume that, for each $j = 1, \dots, J$, there exists a smooth one-to-one mapping M_j such that $\bar{U}_j = M_j(\bar{K})$ with $K = Q$ or T (here, $Q = (-1, 1)^2$ and $T = \{\xi = (\xi_1, \xi_2); 0 < \xi_1 < 1, 0 < \xi_2 < \xi_1\}$) denote

the reference square and triangle, respectively). The Jacobians of M_j are assumed to be bounded from above and below by positive constants independent of j .

Using the parameter representation X we have a fixed regular mesh $\Delta = \{\Gamma_j = X(U_j); j = 1, \dots, J\}$ on Γ . The union of the elements of Δ touching the boundary curve γ will be denoted by A_γ , i.e., $\bar{A}_\gamma = \cup\{\bar{\Gamma}_j; \bar{\Gamma}_j \cap \gamma \neq \emptyset\}$. We assume that, close to the boundary γ , the mesh is fine enough such that $\bar{A}_\gamma \subset (\Gamma_{\delta/2} \cup \gamma)$. We also assume that the cutoff function χ in (1.5) is chosen such that $\text{supp}(\beta(s)\rho^{-1/2}\chi(\rho)) \subset \bar{A}_\gamma$.

Now, for given integer p , we define the space $S_p(\Gamma)$ of piecewise polynomials on Γ . For $K = Q$ or $K = T$ let $\mathcal{Q}_p(K)$ be the set of polynomials of degree $\leq p$ (in each variable for $K = Q$ and of total degree $\leq p$ on T). Furthermore, for $K = I$ an interval, $\mathcal{Q}_p(I)$ denotes the set of polynomials of degree $\leq p$ on I . We will also use the set $\mathcal{R}_p(\Gamma_j)$ of polynomials of degree $\leq p$ in each variable s and ρ on the elements $\Gamma_j \subset A_\gamma \subset \Gamma_{\delta/2}$. Then, using the notation $\mathbf{v}_j = \mathbf{v}|_{\Gamma_j}$, we define

$$S_p(\Gamma) := \{ \mathbf{v}; \mathbf{v}_j \in [\mathcal{R}_p(\Gamma_j)]^3 \text{ if } \Gamma_j \subset A_\gamma, \text{ and } (\mathbf{v}_j \circ X \circ M_j) \in [\mathcal{Q}_p(K)]^3, \\ K = Q \text{ or } T, \text{ if } \Gamma_j \subset (\Gamma \setminus A_\gamma) \}$$

(here, we denote by $[\cdot]^3$ the sets of vector functions with corresponding polynomial components).

One has $S_p(\Gamma) \subset \tilde{H}^{-1/2}(\Gamma)$, and the p -version of the boundary element Galerkin method is as follows: *For given p , find $\mathbf{t}_p \in S_p(\Gamma)$ such that*

$$(2.1) \quad \langle \mathbf{Vt}_p, \mathbf{v} \rangle = \langle \mathbf{g}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in S_p(\Gamma),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\tilde{H}^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

As it is well known, this method converges quasi-optimally, see [6], i.e., there exists a constant $C > 0$ such that for all polynomial degrees p the following holds

$$(2.2) \quad \|\mathbf{t} - \mathbf{t}_p\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C \inf\{\|\mathbf{t} - \mathbf{v}\|_{\tilde{H}^{-1/2}(\Gamma)}; \mathbf{v} \in S_p(\Gamma)\}.$$

We now present the main result giving an a priori error estimate.

Theorem 2.1. *Let $|\sigma| < 1/2$ and $\mathbf{u}_j \in H^{3/2+\sigma}(\Gamma)$, $j = 1, 2$, with $\mathbf{u}_1 - \mathbf{u}_2 \in \tilde{H}^{3/2+\sigma}(\Gamma)$. Then the following a priori error estimate holds*

$$(2.3) \quad \|\mathbf{t} - \mathbf{t}_p\|_{\tilde{H}^{-1/2}(\Gamma)} \leq Cp^{-\alpha}, \quad \alpha = 1/2 + \sigma - \varepsilon, \quad \varepsilon > 0,$$

where $C > 0$ depends on ε but not on p . Here, \mathbf{t} is the solution of (1.4) and \mathbf{t}_p is the boundary element approximation to \mathbf{t} given by (2.1).

This error estimate is quasi-optimal for sufficiently smooth given data. More precisely, if σ is large enough, then there exists for any $\varepsilon > 0$ a constant $c > 0$ such that the p -version converges like $cp^{-1+\varepsilon}$. A convergence like cp^{-1} would be optimal, cf. the results in [5, 18]. The sub-optimality of (2.3) is due to Proposition 1.1 which states the regularity of the term β in the representation of the exact solution only in standard Sobolev spaces, which are not appropriate to obtain optimal results. For numerical results (dealing with the scalar version of the Laplace operator) which underline the a priori error estimate we refer to [14].

The proof of Theorem 2.1 is given in the next section.

3. Technical details. Before proving Theorem 2.1, we define Sobolev spaces and collect several auxiliary results.

Let $L_2(\mathbf{R}^n)$ be the usual Lebesgue space of square integrable functions on \mathbf{R}^n , $n \geq 1$, equipped with the standard norm $\|\cdot\|_{L_2(\mathbf{R}^n)}$. For $t \in \mathbf{R}$, we define the Sobolev space $H^t(\mathbf{R}^n)$ with norm

$$\|u\|_{H^t(\mathbf{R}^n)} = \left\| (1 + |\xi|^2)^{t/2} \hat{u} \right\|_{L_2(\mathbf{R}^n)}.$$

Here $\hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} u(x) e^{-i x \cdot \xi} dx$ denotes the Fourier transform of the function $u \in L_2(\mathbf{R}^n)$, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, and $x \cdot \xi = x_1 \xi_1 + \dots + x_n \xi_n$.

Then for a Lipschitz domain $\Omega \subset \mathbf{R}^n$ and $t > 0$ we set

$$H^t(\Omega) = \{u = \varphi|_{\Omega}; \varphi \in H^t(\mathbf{R}^n)\}$$

with norm

$$\|u\|_{H^t(\Omega)} = \inf_{\substack{\varphi \in H^t(\mathbf{R}^n) \\ u = \varphi|_{\Omega}}} \|\varphi\|_{H^t(\mathbf{R}^n)}$$

and

$$\tilde{H}^t(\Omega) = \{u \in H^t(\mathbf{R}^n); \text{supp } u \subset \bar{\Omega}\}$$

with norm

$$\|u\|_{\tilde{H}^t(\Omega)} = \|u\|_{H^t(\mathbf{R}^n)}.$$

For $t < 0$ the spaces $H^t(\Omega)$ and $\tilde{H}^t(\Omega)$ are the dual spaces of $\tilde{H}^{-t}(\Omega)$ and $H^{-t}(\Omega)$, respectively, with $L_2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space. When Ω is bounded and $t > 0$ we will also use the space $H_0^t(\Omega)$ being the closure of $C_0^\infty(\Omega)$ with respect to the norm in $H^t(\Omega)$.

Note that $H^t(\Omega) = \tilde{H}^t(\Omega) = H_0^t(\Omega)$ if $0 \leq t < 1/2$, and $\tilde{H}^t(\Omega) = H_0^t(\Omega)$ if $t - 1/2$ is not an integer, see [8]. Moreover, in the latter case, the norms $\|\cdot\|_{\tilde{H}^t(\Omega)}$ and $\|\cdot\|_{H^t(\Omega)}$ are equivalent.

For an open surface Γ in \mathbf{R}^3 , we define $H^s(\Gamma)$ via a regular parameter representation $x = X(y)$, $x \in \Gamma$, $y \in \Omega \subset \mathbf{R}^2$, and by using the definition for Ω as a subset of \mathbf{R}^2 . On an interval or a smooth curve γ we define the Sobolev space $H^s(\gamma)$ similarly (using periodic functions in the case of closed curves).

The Sobolev spaces satisfy the interpolation property, see [4]: let $t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$, and $t = (1 - \theta)t_1 + \theta t_2$ for $0 < \theta < 1$, then

$$H^t(\Omega) = (H^{t_1}(\Omega), H^{t_2}(\Omega))_\theta \quad \text{and} \quad \tilde{H}^t(\Omega) = \left(\tilde{H}^{t_1}(\Omega), \tilde{H}^{t_2}(\Omega)\right)_\theta.$$

Here we use the real K-method of interpolation where, for two normed spaces A_0 and A_1 , the interpolation space $(A_0, A_1)_s$, $0 < s < 1$, is equipped with the norm

$$\|a\|_{(A_0, A_1)_s} := \left(\int_0^\infty t^{-2s} \inf_{a=a_0+a_1} (\|a_0\|_{A_0}^2 + t^2 \|a_1\|_{A_1}^2) \frac{dt}{t} \right)^{1/2}.$$

Lemma 3.1. *Let $\Omega \subset \mathbf{R}^2$ be a Lipschitz domain. If $u \in \tilde{H}^t(\Omega)$ with $0 \leq t \leq 1$, then for $i = 1, 2$, $\partial u / \partial x_i \in \tilde{H}^{t-1}(\Omega)$, and*

$$\|\partial u / \partial x_i\|_{\tilde{H}^{t-1}(\Omega)} \leq C \|u\|_{\tilde{H}^t(\Omega)},$$

where $C > 0$ is independent of u .

On an interval, this statement is proved in [21, Lemma 3.5]. In two dimensions the proof is similar and is skipped.

Lemma 3.2. *Let Ω, Ω_1 be two Lipschitz domains in \mathbf{R}^n , $n = 1, 2, 3$, and $\Omega_1 \subset \Omega$. Then, for $0 \leq t < 1/2$, the following holds*

$$(3.1) \quad \|u\|_{\tilde{H}^{-t}(\Omega_1)} \leq C \|u\|_{\tilde{H}^{-t}(\Omega)} \quad \forall u \in \tilde{H}^{-t}(\Omega),$$

where the constant $C > 0$ is independent of u .

Proof. For $0 \leq t < 1/2$, the identity $H_0^t(\Omega_1) = H^t(\Omega_1)$ holds, see, e.g., [8]. Let us consider the function $v \in H^t(\Omega_1) = H_0^t(\Omega_1)$ and denote by \bar{v} the extension of v by zero outside Ω_1 . Then $\bar{v} \in H^t(\Omega) = H_0^t(\Omega)$ and

$$\|\bar{v}\|_{H^t(\Omega)} \leq C \left(\|\bar{v}\|_{H_0^t(\Omega_1)} + \|\bar{v}\|_{H_0^t(\Omega \setminus \Omega_1)} \right) = C \|v\|_{H_0^t(\Omega_1)} = C \|v\|_{H^t(\Omega_1)}.$$

The inequality above is due to [22, Lemma 3.2] when defining the Sobolev spaces by complex interpolation. The proof presented there works the same way for the real interpolation method, see [2, Theorem 4.1]. Then (3.1) follows by using the duality $\tilde{H}^{-t}(\Omega_1) = (H^t(\Omega_1))'$. \square

Lemma 3.3. *Let $f \in H^t(K)$ for real $t > 0$ with $K = I \subset \mathbf{R}$, respectively $K = Q$ or $K = T$ in \mathbf{R}^2 . Then there exists a sequence $f_p \in \mathcal{Q}_p(K)$, $p = 0, 1, 2, \dots$, such that*

$$\|f - f_p\|_{L_2(K)} \leq C p^{-t} \|f\|_{H^t(K)}.$$

For a proof of Lemma 3.3, we refer to [3].

Lemma 3.4 [22, Lemma 3.3]. *Let $f(x) \in \tilde{H}^{-t_1}(I_1)$ and $g(y) \in \tilde{H}^{-t_2}(I_2)$ with $0 \leq t_1, t_2 \leq 1$. Then $f(x)g(y) \in \tilde{H}^{-t_1-t_2}(I_1 \times I_2)$ and*

$$\|f(x)g(y)\|_{\tilde{H}^{-t_1-t_2}(I_1 \times I_2)} \leq C \|f(x)\|_{\tilde{H}^{-t_1}(I_1)} \|g(y)\|_{\tilde{H}^{-t_2}(I_2)}.$$

The constant C is independent of f and g .

To analyze the approximation of the singular part of \mathbf{t} in (1.5) we first study singularities on an interval. Let us consider the singular function

$$(3.2) \quad \psi(x) = (1+x)^{\lambda-1}\chi(x), \quad x \in I = (-1, 1),$$

where $\lambda > 0$ is real, $\chi \in C^\infty(I)$ is a cutoff function with $\chi(x) = 1$ for $x \in (-1, -1+d]$ and $\chi(x) = 0$ for $x \geq -1+2d$, $0 < d \leq 1/4$.

Observe that $\psi \in \tilde{H}^t(I)$ for $-1 \leq t < \min\{0, \lambda - 1/2\}$, in particular, $\psi \in \tilde{H}^{-1/2}(I)$.

Theorem 3.1. *Let $\psi(x)$ be given by (3.2) with $\lambda > 0$. Then there exists a sequence $\psi_p \in \mathcal{Q}_p(I)$, $p = 1, 2, \dots$, such that for $-1 \leq t < \min\{0, \lambda - 1/2\}$,*

$$(3.3) \quad \|\psi - \psi_p\|_{\tilde{H}^t(\tilde{I})} \leq C p^{-2(\lambda-1/2-t)}, \quad \tilde{I} = (-1, 0).$$

Proof. Introducing a C^∞ cutoff function $\tilde{\chi}(x)$ such that

$$(3.4) \quad \tilde{\chi}(x) = 1 \quad \text{for } x \in [-1, 0] \quad \text{and} \quad \tilde{\chi}(x) = 0 \quad \text{for } x \geq 1/2,$$

we define

$$\Psi(x) := \tilde{\chi}(x) \int_{-1}^x \psi(\xi) d\xi, \quad \hat{\Psi}(x) := (1-x)^{-1}\Psi(x), \quad x \in I = (-1, 1).$$

Then $\Psi(-1) = \hat{\Psi}(-1) = 0$, and, due to (3.4), $\Psi(x) = \hat{\Psi}(x) = 0$ for $x \in [1/2, 1]$. Moreover, on the interval $\tilde{I} = (-1, 0)$, one has

$$(3.5) \quad \Psi'(x) = \psi(x), \quad x \in \tilde{I}.$$

Further, using integration by parts we obtain

$$(3.6) \quad \begin{aligned} \hat{\Psi}(x) &= (1-x)^{-1}\tilde{\chi}(x) \int_{-1}^x \psi(\xi) d\xi \\ &= (1-x)^{-1}\tilde{\chi}(x) \int_{-1}^x (1+\xi)^{\lambda-1}\chi(\xi) d\xi \\ &= \frac{(1+x)^\lambda\chi(x)\tilde{\chi}(x)}{\lambda(1-x)} - \frac{\tilde{\chi}(x)}{\lambda(1-x)} \int_{-1}^x (1+\xi)^\lambda\chi'(\xi) d\xi \\ &=: F(x) - G(x). \end{aligned}$$

For the polynomial approximation of the function $(1+x)^\lambda \chi(x)$, we refer to [5, Theorem 3.1] if $0 < \lambda \leq 1/2$ and to [18, Theorem 5.1] if $\lambda > 1/2$ (actually, we apply here the scaled versions of these theorems). We also note that the factor $(\lambda^{-1}(1-x)^{-1}\tilde{\chi}(x)) \in C^\infty(I)$ does not alter the singular behavior of the function $(1+x)^\lambda \chi(x)$, and the mentioned results of [5] and [18] remain valid for the function $F(x)$ in (3.6). Thus there exists a polynomial $F_p \in \mathcal{Q}_p(I)$ such that $F_p(-1) = F(-1) = 0$ and

$$(3.7) \quad \|F - F_p\|_{H^t(I)} \leq C p^{-2(\lambda+1/2-t)}, \quad 0 \leq t < \min\{1, \lambda + 1/2\}.$$

There holds $G \in C_0^\infty(I)$ because $\chi'(\xi) = 0$ for $\xi \in (-1, -1+d)$ and $\tilde{\chi}(x) = 0$ for $x \geq 1/2$. Therefore, for the approximation of G we use the standard result [3, Lemma 3.2]: there exists a polynomial $G_p \in \mathcal{Q}_p(I)$ such that $G_p(\pm 1) = G(\pm 1) = 0$, and for arbitrary $\tau > 0$,

$$(3.8) \quad \|G - G_p\|_{H^t(I)} \leq C p^{-\tau}, \quad 0 \leq t \leq 1.$$

Let us define $\Psi_p(x) := (1-x)(F_p(x) - G_p(x))$. Then $\Psi_p \in \mathcal{Q}_{p+1}(I)$, $\Psi_p(\pm 1) = 0$ and, for $0 \leq t < \min\{1, \lambda + 1/2\}$, we deduce from (3.6)–(3.8),

$$(3.9) \quad \|\Psi - \Psi_p\|_{H^t(I)} \leq C \|\hat{\Psi} - (F_p - G_p)\|_{H^t(I)} \leq C p^{-2(\lambda+1/2-t)}.$$

Hence,

$$(3.10) \quad \begin{aligned} \|\Psi - \Psi_p\|_{\tilde{H}^t(I)} &\leq C \|\Psi - \Psi_p\|_{H^t(I)} \leq C p^{-2(\lambda+1/2-t)}, \\ t &\in [0, \min\{1, \lambda + 1/2\}) \setminus \{1/2\}, \end{aligned}$$

because $(\Psi - \Psi_p) \in H_0^t(I) = \tilde{H}^t(I)$ for these values of t .

Now we define the polynomial ψ_p as

$$(3.11) \quad \psi_p(x) := \Psi_p'(x), \quad x \in I.$$

Then $\psi_p \in \mathcal{Q}_p(I)$, and recalling (3.5) we have $\psi - \psi_p = (\Psi - \Psi_p)'$ on \tilde{I} . Therefore, using sequentially the one-dimensional versions of Lemmas 3.2 and 3.1, and then estimate (3.10), we obtain for any fixed $t' \in (1/2, \min\{1, \lambda + 1/2\})$,

$$(3.12) \quad \begin{aligned} \|\psi - \psi_p\|_{\tilde{H}^{t'-1}(\tilde{I})} &= \|(\Psi - \Psi_p)'\|_{\tilde{H}^{t'-1}(\tilde{I})} \leq C \|(\Psi - \Psi_p)'\|_{\tilde{H}^{t'-1}(I)} \\ &\leq C \|\Psi - \Psi_p\|_{\tilde{H}^{t'}(I)} \leq C p^{-2(\lambda+1/2-t')}. \end{aligned}$$

Thus we have proved (3.3) for $t \in (-1/2, \min\{0, \lambda - 1/2\})$.

On the other hand, applying Lemma 3.1 and inequality (3.9) with $t = 0$, we have

$$\begin{aligned} \|\psi - \psi_p\|_{\tilde{H}^{-1}(\bar{I})} &= \|(\Psi - \Psi_p)'\|_{\tilde{H}^{-1}(\bar{I})} \leq C \|\Psi - \Psi_p\|_{H^0(\bar{I})} \\ &\leq C \|\Psi - \Psi_p\|_{H^0(I)} \leq C p^{-2(\lambda+1/2)}. \end{aligned}$$

Since $-1/2 < t' - 1 < \min\{0, \lambda - 1/2\}$ in (3.12), interpolation between $\tilde{H}^{-1}(\bar{I})$ and $\tilde{H}^{t'-1}(\bar{I})$ gives (3.3) for any $t \in [-1, \min\{0, \lambda - 1/2\})$. \square

Remark 3.1. Since $\Psi_p(-1) = 0$ in the proof of Theorem 3.1, one has by (3.11),

$$\int_{-1}^x \psi_p(\xi) d\xi = \Psi_p(x).$$

Therefore we can rewrite (3.9) with $t = 0$ as follows

$$\begin{aligned} (3.13) \quad \|\Psi - \Psi_p\|_{L_2(I)} &= \left\| \tilde{\chi}(x) \int_{-1}^x \psi(\xi) d\xi - \int_{-1}^x \psi_p(\xi) d\xi \right\|_{L_2(I)} \\ &\leq C p^{-2(\lambda+1/2)}, \end{aligned}$$

where $\psi(x)$ is given by (3.2), and $\psi_p(x)$ is a polynomial approximation to $\psi(x)$.

Moreover, $\Psi(x) \in L_2(I)$ (because $\Psi(x) \sim (1+x)^\lambda$, $\lambda > 0$), and (3.13) yields

$$(3.14) \quad \|\Psi_p\|_{L_2(I)} = \left\| \int_{-1}^x \psi_p(\xi) d\xi \right\|_{L_2(I)} \leq C.$$

Now we prove the main result of the paper.

Proof of Theorem 2.1. Due to the regularity result of Proposition 1.1 and the quasi-optimal convergence (2.2) of the BEM, one only needs to find a piecewise polynomial function that approximates \mathbf{t} in (1.5) with the upper bound stated by (2.3).

For elements at the boundary γ we need covering rectangles in surface coordinates. Let $\Gamma_j \subset A_\gamma$ be an element touching the boundary γ . Since $A_\gamma \subset (\Gamma_{\delta/2} \cup \gamma)$, there exist two points on γ with coordinates $(s_1, 0)$ and $(s_2, 0)$ such that

$$\Gamma_j \subset Q_j = \{(s, \rho) \in \Gamma_{\delta/2}; s_1 < s < s_2, 0 < \rho < \delta/2\}.$$

First, we define an approximation $\mathbf{t}_{0,p}$ to the vector function $\mathbf{t}_0 \in \tilde{H}^\alpha(\Gamma) \subset H^\alpha(\Gamma)$ (hereafter, $\alpha = 1/2 + \sigma - \varepsilon > 0$ with sufficiently small $\varepsilon > 0$). If $\Gamma_j \subset (\Gamma \setminus A_\gamma)$, we apply Lemma 3.3 componentwise on the square (or triangle) K such that $\Gamma_j = X(M_j(K))$. However, if $\Gamma_j \subset A_\gamma$, we apply Lemma 3.3 on $Q_j \supset \Gamma_j$. Since Γ is smooth, the function \mathbf{t}_0 on $\Gamma_\delta \supset A_\gamma$ has the same Sobolev-regularity in terms of coordinates (s, ρ) as in terms of space variables $x = X(u)$. Therefore, recalling the definition of $S_p(\Gamma)$ and applying Lemma 3.3 as indicated above, we find $\mathbf{t}_{0,p} \in S_p(\Gamma)$ such that

$$(3.15) \quad \begin{aligned} \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{\tilde{H}^{-1/2}(\Gamma_j)} &\leq \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(\Gamma_j)} \leq C p^{-\alpha} \|\mathbf{t}_0\|_{H^\alpha(\Gamma_j)} \\ &\leq C p^{-\alpha} \end{aligned}$$

if $\Gamma_j \subset (\Gamma \setminus A_\gamma)$, and

$$(3.16) \quad \begin{aligned} \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{\tilde{H}^{-1/2}(\Gamma_j)} &\leq \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(\Gamma_j)} \leq \|\mathbf{t}_0 - \mathbf{t}_{0,p}\|_{L_2(Q_j)} \\ &\leq C p^{-\alpha} \|\mathbf{t}_0\|_{H^\alpha(Q_j)} \leq C p^{-\alpha} \end{aligned}$$

if $\Gamma_j \subset A_\gamma$.

Now we consider the singular term $\beta(s)\psi(\rho) = \beta(s)\rho^{-1/2}\chi(\rho)$ in (1.5). Let $\Gamma_j \subset A_\gamma$, and $\Gamma_j \subset Q_j$ as above. Then using the one-dimensional version of Lemma 3.3 we approximate the function $\beta(s) \in H^{1/2+\sigma}(\gamma)$: there exists $\beta_p(s) \in [\mathcal{Q}_p(s_1, s_2)]^3$ satisfying

$$(3.17) \quad \begin{aligned} \|\beta - \beta_p\|_{L_2(s_1, s_2)} &\leq C p^{-(1/2+\sigma)} \|\beta\|_{H^{1/2+\sigma}(s_1, s_2)} \\ &\leq C p^{-(1/2+\sigma)} \|\beta\|_{H^{1/2+\sigma}(\gamma)}. \end{aligned}$$

For the singular function $\psi(\rho)$, we apply Theorem 3.1, scaled to the interval $(0, \delta)$, with $\lambda = 1/2$: there exists a polynomial $\psi_p(\rho) \in \mathcal{Q}_p(0, \delta)$ satisfying

$$(3.18) \quad \|\psi - \psi_p\|_{\tilde{H}^{-t}(0, \delta/2)} \leq C p^{-2t}, \quad 0 < t \leq 1.$$

Since $\psi(\rho) \in \tilde{H}^{-t}(0, \delta/2)$ for $t \in (0, 1]$, we estimate by (3.18)

$$(3.19) \quad \|\psi_p\|_{\tilde{H}^{-t}(0, \delta/2)} \leq C, \quad 0 < t \leq 1,$$

with a constant $C > 0$ depending on t . Furthermore, introducing a C^∞ cutoff function $\tilde{\chi}(\rho)$ such that, cf. (3.4),

$$\tilde{\chi}(\rho) = 1 \quad \text{for } \rho \in [0, \delta/2] \quad \text{and} \quad \tilde{\chi}(\rho) = 0 \quad \text{for } \rho \geq 3\delta/4$$

and, arguing as in the proof of Theorem 3.1, we obtain the inequalities similar to (3.13) and (3.14)

$$(3.20) \quad \|\Psi - \Psi_p\|_{L_2(0, \delta)} = \left\| \tilde{\chi}(\rho) \int_0^\rho \psi(r) dr - \int_0^\rho \psi_p(r) dr \right\|_{L_2(0, \delta)} \leq C p^{-2},$$

$$(3.21) \quad \|\Psi_p\|_{L_2(0, \delta)} = \left\| \int_0^\rho \psi_p(r) dr \right\|_{L_2(0, \delta)} \leq C.$$

Then, making use of Lemma 3.2 (that remains valid with $\Omega_1 = \Gamma_j \subset Q_j = \Omega$), Lemma 3.4, the triangle inequality, and estimates (3.17)–(3.19), we derive for some fixed $t' \in (0, 1/2)$

$$(3.22) \quad \begin{aligned} & \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-t'}(\Gamma_j)} \\ & \leq \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-t'}(Q_j)} \\ & \leq C \left(\|\beta(\psi - \psi_p)\|_{\tilde{H}^{-t'}(Q_j)} + \|(\beta - \beta_p)\psi_p\|_{\tilde{H}^{-t'}(Q_j)} \right) \\ & \leq C \left(\|\beta\|_{L_2(s_1, s_2)} \|\psi - \psi_p\|_{\tilde{H}^{-t'}(0, \delta/2)} + \|\beta - \beta_p\|_{L_2(s_1, s_2)} \|\psi_p\|_{\tilde{H}^{-t'}(0, \delta/2)} \right) \\ & \leq C p^{-\min\{1/2+\sigma, 2t'\}} \|\beta\|_{H^{1/2+\sigma}(\gamma)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} \\ & = \left\| \frac{\partial}{\partial \rho} \left(\beta(s)\tilde{\chi}(\rho) \int_0^\rho \psi(r) dr - \beta_p(s) \int_0^\rho \psi_p(r) dr \right) \right\|_{\tilde{H}^{-1}(\Gamma_j)}, \end{aligned}$$

because $\beta(s)\tilde{\chi}(\rho) = \beta(s)$ on Γ_j . Then applying Lemma 3.1 in terms of coordinates $(s, \rho) \in \Gamma_j$, we have

$$\begin{aligned} & \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} \\ & \leq C \left\| \beta(s)\tilde{\chi}(\rho) \int_0^\rho \psi(r) dr - \beta_p(s) \int_0^\rho \psi_p(r) dr \right\|_{H^0(\Gamma_j)} \\ & \leq C \|\beta(s)\Psi(\rho) - \beta_p(s)\Psi_p(\rho)\|_{H^0(Q_j)}, \end{aligned}$$

where $\Psi(\rho) = \tilde{\chi}(\rho) \int_0^\rho \psi(r) dr$ and $\Psi_p(\rho) = \int_0^\rho \psi_p(r) dr$ as in (3.20).

Hence

$$\begin{aligned} \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} & \leq C \left(\|\beta\|_{L_2(s_1, s_2)} \|\Psi - \Psi_p\|_{L_2(0, \delta/2)} \right. \\ & \quad \left. + \|\beta - \beta_p\|_{L_2(s_1, s_2)} \|\Psi_p\|_{L_2(0, \delta/2)} \right), \end{aligned}$$

and we estimate by using (3.17), (3.20) and (3.21),

$$\begin{aligned} \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-1}(\Gamma_j)} & \leq C p^{-\min\{1/2+\sigma, 2\}} \|\beta\|_{H^{1/2+\sigma}(\gamma)} \\ & = C p^{-(1/2+\sigma)} \|\beta\|_{H^{1/2+\sigma}(\gamma)}. \end{aligned}$$

Since $|\sigma| < 1/2$, we may take t' in (3.22) such that $0 < 1/2 + \sigma \leq 2t' < 1$. Then, interpolating between $\tilde{H}^{-1}(\Gamma_j)$ and $\tilde{H}^{-t'}(\Gamma_j)$, we prove for any $\Gamma_j \subset A_\gamma$

$$(3.23) \quad \|\beta\psi - \beta_p\psi_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} \leq C p^{-(1/2+\sigma)} \|\beta\|_{H^{1/2+\sigma}(\gamma)}.$$

Now let us define the approximating function \mathbf{v}_p on Γ as follows:

$$\begin{aligned} \mathbf{v}_p|_{\Gamma_j} & = \beta_p\psi_p + \mathbf{t}_{0,p}|_{\Gamma_j} & \text{if } \Gamma_j \subset A_\gamma, \\ \mathbf{v}_p|_{\Gamma_j} & = \mathbf{t}_{0,p}|_{\Gamma_j} & \text{if } \Gamma_j \subset (\Gamma \setminus A_\gamma). \end{aligned}$$

Then $\mathbf{v}_p \in S_p(\Gamma)$ and, due to (3.15), (3.16) and (3.23), we obtain for any element $\Gamma_j \subset \Gamma$

$$(3.24) \quad \begin{aligned} \|\mathbf{t} - \mathbf{v}_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} & = \|(\beta\psi + \mathbf{t}_0) - \mathbf{v}_p\|_{\tilde{H}^{-1/2}(\Gamma_j)} \leq C p^{-\alpha}, \\ \alpha & = 1/2 + \sigma - \varepsilon > 0 \end{aligned}$$

(here we also used the assumption that $\text{supp}(\beta(s)\psi(\rho)) \subset \bar{A}_\gamma$).

Since

$$\|\mathbf{t} - \mathbf{v}_p\|_{\tilde{H}^{-1/2}(\Gamma)} \leq C \sum_{j=1}^J \|\mathbf{t} - \mathbf{v}_p\|_{\tilde{H}^{-1/2}(\Gamma_j)},$$

see [22], the desired upper bound in (2.3) follows from (3.24). This proves the theorem. \square

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