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STEADY STATE OSCILLATION PROBLEMS IN THE THEORY OF ELASTICITY FOR CHIRAL MATERIALS

DAVID NATROSHVILI, LEVAN GIORGASHVILI
AND SHOTA ZAZASHVILI

Dedicated to the memory of Professor Ilia Vekua

ABSTRACT. Mathematical problems of the theory of steady state oscillations of hemitropic (chiral) elastic materials are considered. In the case of unbounded domains, the generalized Sommerfeld-Kupradze type radiation conditions are introduced and in the space of radiating solutions the uniqueness results are established. Applying the potential method and the theory of pseudodifferential equations, the unique solvability in various function spaces of the Dirichlet, Neumann and mixed boundary value problems for the steady state oscillation equations are proved. Regularity properties and representability of solutions by layer potentials are analyzed in the cases of smooth and Lipschitz domains.

1. Introduction. A solid which is not isotropic with respect to inversion is called *noncentrosymmetric, acentric, hemitropic*, or *chiral*. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules, as well as on a large scale, as in composites with helical or screw-shaped inclusions (for details see, e.g., [**1, 20** and the references therein]).

Mathematical models describing the chiral properties of elastic materials have been proposed by Aero and Kuvshinski [**1,2**] (for the history of the problem see also [**27, 36, 38, 46** and the references therein]).

Particular problems of the elasticity theory of hemitropic continuum related to the present paper have been considered in [**10, 20, 22–24, 36, 37, 39, 46**]. In [**33, 34**] the basic boundary value problems

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(BVP) of *statics* and *pseudo-oscillations* of the elasticity theory for hemitropic bodies with smooth and Lipschitz boundaries are studied by the potential method.

The main goal of the present investigation is to study the three-dimensional basic BVPs of *steady state oscillations*, i.e., the time harmonic dependent dynamic case, of the elasticity theory of hemitropic bodies by the layer potentials method. It should be noted that in comparison with the static and pseudo-oscillation cases here essential difficulties arise in the proof of uniqueness and existence of solutions (for arbitrary values of the oscillation parameter and arbitrary boundary data). To this end, in this paper the *generalized Sommerfeld-Kupradze type radiation conditions* are formulated which play a crucial role to establish the uniqueness results in the case of exterior boundary value problems. Further, the boundary integral (pseudodifferential) operators generated by the single and double layer potentials are studied and their ellipticity and normal solvability properties are established. Based on the results obtained, the uniqueness and existence theorems of solutions to the basic BVPs of steady state oscillations are proved in various Hölder ($C^{k,\alpha}$), Sobolev-Slobodetski ($W^s_{p,\mathrm{loc}}$), Bessel potential ($H^s_{p,\mathrm{loc}}$), and Besov ($B^s_{p,q,\mathrm{loc}}$) function spaces.

The paper is organized as follows.

In Section 2 we give an overview concerning the basic mechanical characteristics of the theory of elasticity of chiral materials. We show that an arbitrary solution to the differential equations of steady state oscillations can be represented as a sum of metaharmonic vector-functions corresponding to *real wave numbers* and formulate the generalized Sommerfeld-Kupradze type radiation conditions $SK(\Omega^-)$. We give the weak and classical formulations of the exterior BVPs and prove the corresponding uniqueness theorems in the class of radiating vectors. The boundary conditions are understood either in the classical sense or in the usual or generalized trace sense.

In Section 3 we derive the general integral representation formula of a radiating solution by means of the radiating layer potentials. We analyze the mapping properties of these potentials and the corresponding boundary integral (pseudodifferential) operators. In particular, we describe in detail Fredholm properties of these pseudodifferential operators on manifolds without boundary and with boundary.

In Section 4, applying special representation formulas, we reduce the basic exterior BVPs of steady state oscillations to the *equivalent strongly elliptic boundary integral* (*pseudodifferential*) *equations* with zero index and prove their unique solvability for an arbitrary value of the oscillation parameter. These results yield unique solvability of the original BVPs for arbitrary values of the oscillation parameter and for arbitrary boundary data. We establish also the almost best regularity results for solutions to the Dirichlet, Neumann and mixed type BVPs for domains with smooth boundaries. In the case of domains with Lipschitz boundaries by the potential method we show existence of radiating *weak solutions* to the BVPs in the Sobolev-Slobodetski function space $W^1_{2,\,loc}(\Omega^-)\cap SK(\Omega^-)$.

2. Auxiliary material and uniqueness results.

2.1. *Constitutive equations.* Let $\Omega^+\subset\mathbf{R}^3$ be a bounded, simply connected domain with a piecewise smooth connected Lipschitz boundary $S:=\partial\Omega^+$ and $\overline{\Omega^+}=\Omega\cup S$. Then it follows that $\Omega^-:=\mathbf{R}^3\backslash\overline{\Omega^+}$ is also simply connected and $\partial\Omega^-=\partial\Omega^+$. Further, let $\overline{\Omega}\in\{\overline{\Omega^+},\,\overline{\Omega^-}\}$ be filled with an elastic material possessing the hemitropic properties, see [**1, 2**].

Denote by $u=(u_1,u_2,u_3)^\top$ and $\omega=(\omega_1,\omega_2,\omega_3)^\top$ the *displacement vector* and the *micro-rotation vector*, respectively; here and in what follows the symbol $(\cdot)^\top$ denotes transposition. Note that the micro-rotation vector in the hemitropic elasticity theory is kinematically distinct from the *macro-rotation vector* $\operatorname{curl} u/2$.

The *force stress tensor* $\{\tau_{pq}\}$ and the *couple stress tensor* $\{\mu_{pq}\}$ in the linear theory are as follows (the constitutive equations)

$$\begin{aligned}\tau_{pq}=\tau_{pq}(U)&:=(\mu+\alpha)\,\frac{\partial u_q}{\partial x_p}+(\mu-\alpha)\,\frac{\partial u_p}{\partial x_q}\\ &+\lambda\delta_{pq}\operatorname{div}u+\delta\,\delta_{pq}\operatorname{div}\omega\\ &+(\kappa+\nu)\,\frac{\partial\omega_q}{\partial x_p}+(\kappa-\nu)\,\frac{\partial\omega_p}{\partial x_q}-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k,\end{aligned}$$

$$\mu_{pq}=\mu_{pq}(U):=\delta\,\delta_{pq}\operatorname{div}u+(\kappa+\nu)\left[\frac{\partial u_q}{\partial x_p}-\sum_{k=1}^{3}\varepsilon_{pqk}\omega_k\right]+\beta\,\delta_{pq}\operatorname{div}\omega$$
$$+(\kappa-\nu)\left[\frac{\partial u_p}{\partial x_q}-\sum_{k=1}^{3}\varepsilon_{qpk}\omega_k\right]+(\gamma+\varepsilon)\,\frac{\partial\omega_q}{\partial x_p}+(\gamma-\varepsilon)\,\frac{\partial\omega_p}{\partial x_q},$$

where $U=(u,\omega)^\top$, δ_{pq} is the Kronecker delta, ε_{pqk} is the permutation (Levi-Civitá) symbol, and α, β, γ, δ, λ, μ, ν, κ and ε are the material constants, see [**1**].

The components of the force stress vector $\tau^{(n)}$ and the couple stress vector $\mu^{(n)}$, acting on a surface element with a normal vector $n=(n_1,n_2,n_3)$, read as

$$\tau_q^{(n)}(U)=\sum_{p=1}^{3}\tau_{pq}(U)\,n_p,\qquad \mu_q^{(n)}(U)=\sum_{p=1}^{3}\mu_{pq}(U)\,n_p,\quad q=1,2,3.$$

Further we introduce the generalized stress operator (6×6 matrix differential operator)

$$\begin{aligned}T(\partial,n)&=\begin{bmatrix}T^{(1)}(\partial,n) & T^{(2)}(\partial,n)\\ T^{(3)}(\partial,n) & T^{(4)}(\partial,n)\end{bmatrix}_{6\times 6},\\ T^{(j)}&=\Big[T_{pq}^{(j)}\Big]_{3\times 3},\quad j=\overline{1,4},\end{aligned}\tag{2.1}$$

where $\partial=(\partial_1,\,\partial_2,\,\partial_3)$ with $\partial_j=\partial/\partial x_j$, $\partial/\partial n$ denotes the directional derivative along the vector n (normal derivative),

$$\begin{aligned}T_{pq}^{(1)}(\partial,n)&=(\mu+\alpha)\,\delta_{pq}\,\frac{\partial}{\partial n}+(\mu-\alpha)n_q\,\frac{\partial}{\partial x_p}+\lambda n_p\,\frac{\partial}{\partial x_q},\\
T_{pq}^{(2)}(\partial,n)&=(\kappa+\nu)\,\delta_{pq}\,\frac{\partial}{\partial n}+(\kappa-\nu)n_q\,\frac{\partial}{\partial x_p}\\
&\qquad+\delta n_p\,\frac{\partial}{\partial x_q}-2\alpha\sum_{k=1}^{3}\varepsilon_{pqk}n_k,\\
T_{pq}^{(3)}(\partial,n)&=(\kappa+\nu)\,\delta_{pq}\,\frac{\partial}{\partial n}+(\kappa-\nu)n_q\,\frac{\partial}{\partial x_p}+\delta n_p\,\frac{\partial}{\partial x_q},\\
T_{pq}^{(4)}(\partial,n)&=(\gamma+\varepsilon)\,\delta_{pq}\,\frac{\partial}{\partial n}+(\gamma-\varepsilon)n_q\,\frac{\partial}{\partial x_p}\\
&\qquad+\beta n_p\,\frac{\partial}{\partial x_q}-2\nu\sum_{k=1}^{3}\varepsilon_{pqk}n_k.\end{aligned}$$

It can be easily checked that

$$\Big(\tau^{(n)}(U),\, \mu^{(n)}(U)\Big)^{\top} = T(\partial, n)\, U.$$

Denote by $T_0(\partial, n)$ the principal homogeneous part (6×6 matrix) of the differential operator $T(\partial, n)$, i.e.,

$$T_0(\partial, n) = \begin{bmatrix} T_0^{(1)}(\partial, n) & T_0^{(2)}(\partial, n) \\ T_0^{(3)}(\partial, n) & T_0^{(4)}(\partial, n) \end{bmatrix}_{6\times 6}, \qquad T_0^{(j)} = \Big[T_{0pq}^{(j)}\Big]_{3\times 3},$$

$$\begin{aligned} T_{0pq}^{(1)}(\partial, n) &= (\mu + \alpha)\, \delta_{pq} \frac{\partial}{\partial n} + (\mu - \alpha) n_q \frac{\partial}{\partial x_p} + \lambda n_p \frac{\partial}{\partial x_q}, \\ T_{0pq}^{(2)}(\partial, n) &= (\kappa + \nu)\, \delta_{pq} \frac{\partial}{\partial n} + (\kappa - \nu) n_q \frac{\partial}{\partial x_p} + \delta n_p \frac{\partial}{\partial x_q}, \\ T_{0pq}^{(3)}(\partial, n) &= (\kappa + \nu)\, \delta_{pq} \frac{\partial}{\partial n} + (\kappa - \nu) n_q \frac{\partial}{\partial x_p} + \delta n_p \frac{\partial}{\partial x_q}, \\ T_{0pq}^{(4)}(\partial, n) &= (\gamma + \varepsilon)\, \delta_{pq} \frac{\partial}{\partial n} + (\gamma - \varepsilon) n_q \frac{\partial}{\partial x_p} + \beta n_p \frac{\partial}{\partial x_q}. \end{aligned} \tag{2.2}$$

We have the evident equality

$$T(\partial_x, n)\, U = T_0(\partial_x, n)\, U + 2[\alpha\, n \times \omega,\, \nu\, n \times \omega]^{\top}, \tag{2.3}$$

where the symbol $\times$ denotes the cross product of two vectors.

2.2. *The basic equations.* The equations of dynamics of the hemitropic theory of elasticity have the form

$$\sum_{p=1}^{3} \partial_p\, \tau_{pq}(x,t) + \varrho\, F_q(x,t) = \varrho\, \frac{\partial^2 u_q(x,t)}{\partial t^2},$$

$$\sum_{p=1}^{3} \partial_p\, \mu_{pq}(x,t) + \sum_{l,r=1}^{3} \varepsilon_{qlr}\, \tau_{lr}(x,t) + \varrho\, G_q(x,t) = \mathcal{I}\, \frac{\partial^2 \omega_q(x,t)}{\partial t^2},$$

$$q = 1, 2, 3,$$

where t is the time variable, $F = (F_1, F_2, F_3)^{\top}$ and $G = (G_1, G_2, G_3)^{\top}$ are the body force and body couple vectors per unit mass, ϱ is the mass density of the elastic material, and $\mathcal{I}$ is a constant characterizing the so called spin torque corresponding to the interior micro-rotations, i.e., the moment of inertia per unit volume.

Using the relations (2.1)–(2.3) we can rewrite the above dynamic equations in terms of the displacement and micro-rotation vectors.

If all the quantities involved in these equations are harmonic time dependent, i.e., $u(x,t)=u(x)\,\exp\{-i\,t\sigma\}$, $\omega(x,t)=\omega(x)\,\exp\{-i\,t\sigma\}$, $F(x,t)=F(x)\,\exp\{-i\,t\sigma\}$ and $G(x,t)=G(x)\,\exp\{-i\,t\sigma\}$, with $\sigma\in\mathbf{R}$ and $i=\sqrt{-1}$, we obtain the *steady state oscillation equations* of the hemitropic theory of elasticity:

(2.4)

$$
\begin{aligned}
&(\mu+\alpha)\Delta u(x)+(\lambda+\mu-\alpha)\mathrm{grad}\,\mathrm{div}\,u(x)+(\kappa+\nu)\Delta\omega(x)\\
&\quad+(\delta+\kappa-\nu)\mathrm{grad}\,\mathrm{div}\,\omega(x)+2\alpha\mathrm{curl}\,\omega(x)+\varrho\,\sigma^2u(x)=-\varrho\,F(x),\\
&(\kappa+\nu)\Delta u(x)+(\delta+\kappa-\nu)\mathrm{grad}\,\mathrm{div}\,u(x)+2\alpha\mathrm{curl}\,u(x)+(\gamma+\varepsilon)\Delta\omega(x)\\
&\quad+(\beta+\gamma-\varepsilon)\mathrm{grad}\,\mathrm{div}\,\omega(x)+4\nu\mathrm{curl}\,\omega(x)+(\mathcal{I}\,\sigma^2-4\alpha)\,\omega(x)=-\varrho\,G(x),
\end{aligned}
$$

where $\Delta=\partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator and u, ω, F and G are complex-valued vector functions and σ is a *frequency parameter*.

If $\sigma=\sigma_1+i\,\sigma_2$ is complex with $\sigma_2\neq 0$, then the above equations are called the *pseudo-oscillation equations*, while for $\sigma=0$ they represent the *equilibrium equations of statics*.

Throughout the paper we deal with the basic BVPs for the steady state oscillation equations and assume that

$$
\sigma>0,\quad \mathcal{I}\,\sigma^2-4\alpha>0. \tag{2.5}
$$

Let us introduce the matrix differential operator corresponding to the system (2.4):

$$
L(\partial,\sigma):=\begin{bmatrix} L^{(1)}(\partial,\sigma), & L^{(2)}(\partial,\sigma)\\ L^{(3)}(\partial,\sigma), & L^{(4)}(\partial,\sigma)\end{bmatrix}_{6\times 6}, \tag{2.6}
$$

where

(2.7)

$$
\begin{aligned}
&L^{(1)}(\partial,\sigma):=[(\mu+\alpha)\,\Delta+\varrho\,\sigma^2]\,I_3+(\lambda+\mu-\alpha)\,Q(\partial),\\
&L^{(2)}(\partial,\sigma)=L^{(3)}(\partial,\sigma):=(\kappa+\nu)\,\Delta\,I_3+(\delta+\kappa-\nu)\,Q(\partial)+2\alpha\,R(\partial),\\
&L^{(4)}(\partial,\sigma):=[(\gamma+\varepsilon)\,\Delta+(\mathcal{I}\sigma^2-4\alpha)]\,I_3+(\beta+\gamma-\varepsilon)\,Q(\partial)+4\nu\,R(\partial).
\end{aligned}
$$

Here and in the sequel I_k stands for the $k \times k$ unit matrix and

$$R(\partial) := \begin{bmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{bmatrix}_{3\times 3}, \qquad Q(\partial) := [\,\partial_k \partial_j\,]_{3\times 3}. \tag{2.8}$$

It is easy to see that

$$R(\partial)u = \operatorname{curl} u, \qquad Q(\partial)\, u = \operatorname{grad}\operatorname{div} u. \tag{2.9}$$

Equations (2.4) can be rewritten in matrix form as

$$\begin{gathered} L(\partial,\sigma)\, U(x) = \Phi(x), \quad U = (u,\omega)^\top, \\ \Phi = (\Phi^{(1)},\Phi^{(2)})^\top := (-\varrho\, F(x), -\varrho\, G(x))^\top. \end{gathered}$$

Further, let us remark that the differential operator

$$L(\partial) := L(\partial, 0) \tag{2.10}$$

corresponds to the static equilibrium case, while the differential operator

$$L_0(\partial) := \begin{bmatrix} L_0^{(1)}(\partial), & L_0^{(2)}(\partial) \\ L_0^{(3)}(\partial), & L_0^{(4)}(\partial) \end{bmatrix}_{6\times 6} \tag{2.11}$$

with

$$\begin{aligned} &L_0^{(1)}(\partial) := (\mu+\alpha)\Delta\, I_3 + (\lambda+\mu-\alpha)\, Q(\partial), \\ &L_0^{(2)}(\partial) = L_0^{(3)}(\partial) := (\kappa+\nu)\Delta\, I_3 + (\delta+\kappa-\nu)\, Q(\partial), \\ &L_0^{(4)}(\partial) := (\gamma+\varepsilon)\Delta\, I_3 + (\beta+\gamma-\varepsilon)\, Q(\partial), \end{aligned} \tag{2.12}$$

represents the principal homogeneous part of the operators (2.6) and (2.10).

It is evident that

$$\begin{gathered} L(\partial,\sigma)\, U - L(\partial)\, U = (\varrho\,\sigma^2\, u,\, \mathcal{I}\,\sigma^2\,\omega)^\top, \\ L(\partial,\sigma) = L_0(\partial) + L_1(\partial) + L_2(\partial), \end{gathered} \tag{2.13}$$

where $L_0(\partial)$ is given by (2.11) and

$$L_1(\partial) := \begin{bmatrix} [0]_{3\times 3}, & 2\alpha\, R(\partial) \\ 2\alpha\, R(\partial), & 4\nu\, R(\partial) \end{bmatrix}_{6\times 6}, \tag{2.14}$$
$$L_2(\partial) := \begin{bmatrix} \varrho\,\sigma^2\, I_3, & [0]_{3\times 3} \\ [0]_{3\times 3}, & (\mathcal{I}\,\sigma^2 - 4\alpha)\, I_3 \end{bmatrix}_{6\times 6}.$$

Let us remark that the operators $L(\partial,\sigma)$ for real σ^2, $L(\partial)$, and $L_0(\partial)$ are formally self-adjoint.

2.3. *Green's formulae.* For real-valued vectors $U := (u,\omega)^\top$, $U' := (u',\omega')^\top \in [C^2(\overline{\Omega^+})]^6$ there holds Green's formula [**33**]

$$\int_{\Omega^+} [L(\partial)U\cdot U' + E(U,U')]\, dx = \int_{\partial\Omega^+} [T(\partial,n)U]^+ \cdot [U']^+\, dS, \tag{2.15}$$

where n is the outward unit normal vector to $\partial\Omega^+$, the symbols $[\cdot]^\pm$ denote the limits on S from $\Omega^\pm$, $E(\cdot\,,\,\cdot)$ is the so called *energy bilinear form*

(2.16)

$$\begin{aligned} E(U,U') = E(U',U) = \sum_{p,q=1}^{3} \big\{ &(\mu+\alpha)u'_{pq}u_{pq} + (\mu-\alpha)u'_{pq}u_{qp} \\ &+ (\kappa+\nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + (\kappa-\nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) \\ &+ (\gamma+\varepsilon)\,\omega'_{pq}\omega_{pq} + (\gamma-\varepsilon)\,\omega'_{pq}\omega_{qp} + \delta(u'_{pp}\omega_{qq} + \omega'_{qq}u_{pp}) \\ &+ \lambda u'_{pp}u_{qq} + \beta\omega'_{pp}\omega_{qq}\big\} \end{aligned}$$

with

$$u_{pq} = \partial_p u_q - \sum_{k=1}^{3} \varepsilon_{pqk}\,\omega_k, \quad \omega_{pq} = \partial_p\,\omega_q, \quad p,q = 1,2,3. \tag{2.17}$$

Here and in what follows $a\cdot b$ denotes the usual scalar product of two (in general) complex vectors $a,b\in\mathbf{C}^m$,

$$a\cdot b = \sum_{j=1}^{m} a_j\,\overline{b_j}, \tag{2.18}$$

where the over bar denotes complex conjugation. The above Green formula immediately follows from the identity

$$L(\partial)U \cdot U' + E(U,U') = \sum_{p,q=1}^{3} \partial_p \left[u'_q \, \tau_{pq}(U) + \omega'_q \, \mu_{pq}(U)\right].$$

From (2.16) and (2.17) we get

$$\begin{aligned} E(U,U) = {} & \frac{3\lambda + 2\mu}{3} \left(\operatorname{div} u + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \operatorname{div} \omega \right)^2 \\ & + \frac{1}{3} \left(3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) (\operatorname{div} \omega)^2 \\ & + \frac{\mu}{2} \sum_{k,j=1,\, k\neq j}^{3} \left[\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]^2 \\ & + \frac{\mu}{3} \sum_{k,j=1}^{3} \left[\frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} + \frac{\kappa}{\mu} \left(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \right]^2 \\ & + \left(\gamma - \frac{\kappa^2}{\mu} \right) \sum_{k,j=1,\, k\neq j}^{3} \left[\frac{1}{2} \left(\frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right)^2 \right. \\ & \left. + \frac{1}{3} \left(\frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right)^2 \right] + \left(\varepsilon - \frac{\nu^2}{\alpha} \right) (\operatorname{curl} \omega)^2 \\ & + \alpha \left(\operatorname{curl} u + \frac{\nu}{\alpha} \operatorname{curl} \omega - 2\omega \right)^2 . \end{aligned} \tag{2.19}$$

From physical considerations (positive definiteness of the potential energy density (2.19) with respect to the variables (2.17)), it follows that the material constants satisfy the inequalities, cf. [**2**]

$$\begin{gathered} \mu > 0, \quad \alpha > 0, \quad 3\lambda + 2\mu > 0, \quad \mu\,\gamma - \kappa^2 > 0, \quad \alpha\,\varepsilon - \nu^2 > 0, \\ (3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2\kappa)^2 > 0, \end{gathered}$$

whence

$$\begin{aligned} & \gamma > 0, \quad \varepsilon > 0, \quad \lambda + \mu > 0, \quad \beta + \gamma > 0, \\ & d_1 := (\mu + \alpha)(\gamma + \varepsilon) - (\kappa + \nu)^2 > 0, \\ & d_2 := (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\kappa)^2 > 0. \end{aligned} \tag{2.20}$$

Throughout the paper L_p, W_p^r, H_p^s, and $B_{p,q}^s$, with $r \geq 0$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, denote the well-known Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov function spaces, respectively, see, e.g., [**25, 44**]. We will use the abbreviations $W_2^r = W^r$, $H_2^s = H^s$, $H^0 = L_2$. We recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \geq 0$, $s \in \mathbf{R}$, for any positive and noninteger t, and for any nonnegative integer k.

Moreover, for an open submanifold $\mathcal{M}$ with boundary, which is a proper part of the surface $S = \partial\Omega^+$, we put

$$\begin{aligned} H_p^s(\mathcal{M}) &:= \left\{ f\,|_{\mathcal{M}} \, : \, f \in H_p^s(S) \right\}, \\ \widetilde{H}_p^s(\mathcal{M}) &:= \{ f \in H_p^s(S) \, : \, \operatorname{supp} f \subset \overline{\mathcal{M}} \}, \\ B_{p,q}^s(\mathcal{M}) &:= \left\{ f\,|_{\mathcal{M}} \, : \, f \in B_{p,q}^s(S) \right\}, \\ \tilde{B}_{p,q}^s(\mathcal{M}) &:= \{ f \in B_{p,q}^s(S) \, : \, \operatorname{supp} f \subset \overline{\mathcal{M}} \}, \end{aligned}$$

where $f|_{\mathcal{M}}$ denotes the restriction of f to $\mathcal{M}$. Recall that $H_{p'}^{-s}(\mathcal{M})$, respectively $B_{p',q'}^{-s}(\mathcal{M})$, is the space adjoint to $\widetilde{H}_p^s(\mathcal{M})$, respectively $\tilde{B}_{p,q}^s(\mathcal{M})$, and vice versa, where $1 < p, q < \infty$ and $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$.

If $U = (u,\, \omega)^\top = (u^{(1)} + i\, u^{(2)},\, \omega^{(1)} + i\, \omega^{(2)})^\top = U^{(1)} + i\, U^{(2)}$ is complex-valued, where $U^{(j)} = (u^{(j)},\, \omega^{(j)})^\top$ $(j = 1, 2)$ are real-valued vectors, then

$$E(U, \overline{U}) = E(U^{(1)}, U^{(1)}) + E(U^{(2)}, U^{(2)}),$$

and, due to the positive definiteness of the energy form for real-valued vector functions with respect to the variables (2.17), we have

$$E(U, \overline{U}) \geq c_1 \sum_{p,q=1}^{3} \left[\,|\, \partial_p u_q \,|^2 + |\, \partial_p \omega_q \,|^2 \,\right] - c_2 \sum_{p=1}^{3} |\, \omega_p \,|^2, \tag{2.21}$$

where c_1 and c_2 are positive numbers depending only on the material constants, and $u_{pq}^{(j)}$ and $\omega_{pq}^{(j)}$ are defined by formulae (2.17) with $u^{(j)}$ and $\omega^{(j)}$ for u and ω.

From (2.21) it follows that, for an arbitrary complex-valued vector $U \in [H^1(\overline{\Omega^+})]^6$ and an arbitrary Lipschitz region Ω^+,

$$\mathcal{B}_{\Omega^+}(U,\overline{U}) := \int_{\Omega^+} E(U,\overline{U})\,dx$$
$$\geq c_1 \int_{\Omega^+} \Big\{ \sum_{p,q=1}^{3} \big[\, |\partial_p u_q|^2 + |\partial_p \omega_q|^2 \big] \Big\}\,dx - c_2 \int_{\Omega^+} \sum_{p=1}^{3} |\omega_p|^2\,dx,$$

i.e., the following Korn's type inequality

$$\mathcal{B}_{\Omega^+}(U,\overline{U}) \,\geq\, c_1^* \,||U||^2_{[H^1(\Omega^+)]^6} \,-\, c_2^* \,||U||^2_{[H^0(\Omega^+)]^6} \tag{2.22}$$

holds, cf. [**12**, Part I, Section 12], [**26**, Chapter 10], where $||\cdot||_{[H^s(\Omega)]^6}$ denotes the norm in the Sobolev-Slobodetski space $[H^s(\Omega)]^6$, and c_1^* and c_2^* are positive numbers depending only on the material constants.

These results imply that the differential operators $L(\partial,\sigma)$, $L(\partial)$, and $L_0(\partial)$ are *strongly elliptic* and the following inequality (*the accretivity condition*)

$$c_1' \,|\xi|^2\,|\eta|^2 \,\leq\, L_0(\xi)\,\eta\cdot\eta = \sum_{k,j=1}^{6} \{L_0(\xi)\}_{kj}\,\eta_j\,\overline{\eta_k} \,\leq\, c_2' \,|\xi|^2\,|\eta|^2$$

holds with some constants $c_k' > 0$, $k = 1, 2$, for arbitrary $\xi \in \mathbf{R}^3$ and arbitrary complex vector $\eta \in \mathbf{C}^6$, cf., e.g., [**12**, Part I, Section 5], [**26**, Chapter 4, Lemma 4.5].

Remark 2.1. For an arbitrary complex parameter σ and complex-valued vectors $U, U' \in [C^2(\overline{\Omega^+})]^6$, we have, cf. (2.13), (2.15),

$$\int_{\Omega^+} \Big[\, L(\partial,\sigma)\,U\cdot\overline{U'} - U\cdot\overline{L(\partial,\sigma)\,U'}\,\Big]\,dx$$
$$= \int_{\partial\Omega^+} \big\{ [\,TU]^+ \cdot [\,\overline{U'}\,]^+ - [\,U]^+\cdot[\,\overline{TU'}\,]^+ \big\}\,dS.$$

Remark 2.2. By standard arguments, Green's formula (2.15) can be extended to general Lipschitz domains and to complex-valued vector functions $U \in [W^1_p(\Omega^+)]^6$ and $U' \in [W^1_{p'}(\Omega^+)]^6$ with $1/p + 1/p' = 1$ and $L(\partial)U \in [L_p(\Omega^+)]^6$, cf. [**4, 25, 26, 35**]

$$\int_{\Omega^+} \big[L(\partial)U \cdot U' + E(U, \overline{U'}\,) \big]\, dx = \big\langle\, [T(\partial, n)U]^+,\, [\,\overline{U'}\,]^+ \big\rangle_{\partial\Omega^+}\,, \tag{2.23}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle_{\partial\Omega^+}$ denotes the duality between the spaces $[B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ and $[B^{1/p}_{p',p'}(\partial\Omega^+)]^6$, which extends the usual (*real*) L_2-scalar product for $f, g \in [L_2(S)]^6$

$$\langle f, g\,\rangle_S = \sum_{k=1}^{6} \int_S f_k\, g_k\, dS =: (f, \overline{g})_{[L_2(S)]^6} \quad \text{for} \quad f, g \in [L_2(S)]^6. \tag{2.24}$$

Note that $[U']^+ \in [B^{1-1/p'}_{p',p'}(\partial\Omega^+)]^6$ is the trace of U' on $\partial\Omega^+$. Clearly, in this case the functional $[T(\partial, n)U]^+ \in [B^{-1/p}_{p,p}(\partial\Omega^+)]^6$ is correctly determined by the relation (2.23). This functional will be referred to as the trace of stress vector on $\partial\Omega^+$.

In the case of unbounded domain Ω^- we can apply the same approach for a vector $U \in [W^1_{p,\,loc}(\Omega^-)]^6$ with $L(\partial)U \in [L_{p,\,loc}(\Omega^-)]^6$ to define the generalized trace functional $[T(\partial, n)U]^- \in [B^{-1/p}_{p,p}(\partial\Omega^-)]^6$ by Green's formula

$$\int_{\Omega^-} \big[L(\partial)U \cdot U' + E(U, \overline{U'}) \big]\, dx = -\, \big\langle\, [T(\partial, n)U]^-, [\,\overline{U'}\,]^- \big\rangle_{\partial\Omega^-}\,, \tag{2.25}$$

where U' is an arbitrary vector from the space $[W^1_{p',\,\mathrm{comp}}(\Omega^-)]^6$ and $[U']^- \in [B^{1-1/p'}_{p',p'}(\partial\Omega^-)]^6$ is the trace of U' on $\partial\Omega^-$.

2.4. *Fundamental solutions. Generalized Sommerfeld-Kupradze type radiation conditions.* Let us consider the pseudodifferential (differential) operator

$$\det\, L(\partial, \sigma) := \mathcal{F}^{-1}\, \det L(-i\,\xi, \sigma)\, \mathcal{F},$$

where $\mathcal{F} = \mathcal{F}_{x\to\xi}$ and $\mathcal{F}^{-1} = \mathcal{F}^{-1}_{\xi\to x}$ denote the direct and inverse generalized Fourier transform in the space of tempered distributions (Schwartz space $\mathcal{S}'(\mathbf{R}^3)$).

As is shown in [**33**] the function $\det L(-i\,\xi,\sigma)$ for $\xi = r\,\hat{\xi}$ with $\hat{\xi} \in \mathbf{R}^3$, $|\,\hat{\xi}\,| = 1$, and $r \in \mathbf{C}$ admits the representation

$$\det L(-i\,\xi,\sigma) = \Phi_1(r)\,\Phi_2(r) = d_1^2\,d_2\,\prod_{j=1}^{6}\big(r^2 - k_j^2\big),$$

where d_1 and d_2 are given by (2.20),

$$\begin{aligned}
\Phi_1(r) := {} & r^4[(\lambda+2\mu)(\beta+2\gamma) - (\delta+2\kappa)^2] - r^2[(\beta+2\gamma)\varrho\,\sigma^2 \\
& + (\lambda+2\mu)(\mathcal{I}\sigma^2 - 4\alpha)] + \varrho\,\sigma^2(\mathcal{I}\sigma^2 - 4\alpha),\\
\Phi_2(r) := {} & r^8[(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2]^2 - r^6\{[\,4\alpha(\kappa+\nu) - 4\nu(\mu+\alpha)]^2 \\
& + 2[\,(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2][(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) \\
& + (\gamma+\varepsilon)\varrho\,\sigma^2 + 4\alpha^2]\} \\
& + r^4\{[(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma+\varepsilon)\varrho\sigma^2 + 4\alpha^2]^2 \\
& + 32\nu\,\varrho\,\sigma^2(\nu\mu - \alpha\kappa) \\
& + 2\varrho\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)[(\mu+\alpha)(\gamma+\varepsilon) - (\kappa+\nu)^2]\} - r^2\{16\nu^2\varrho^2\sigma^4 \\
& + 2\varrho\,\sigma^2(\mathcal{I}\sigma^2 - 4\alpha)[(\mu+\alpha)(\mathcal{I}\sigma^2 - 4\alpha) + (\gamma+\varepsilon)\varrho\,\sigma^2 + 4\alpha^2]\} \\
& + \varrho^2\sigma^4(\mathcal{I}\sigma^2 - 4\alpha)^2.
\end{aligned}$$

Here $\pm k_1$ and $\pm k_2$ are the roots of the equation $\Phi_1(r) = 0$, while $\pm k_3$, $\pm k_4$, $\pm k_5$ and $\pm k_6$ are roots of the equation $\Phi_2(r) = 0$. Note that, due to the evenness of the functions $\Phi_1(r)$ and $\Phi_2(r)$ and since $\sigma \in \mathbf{R}$, it is evident that if $r = k$ is a root of the equation $\det L(-i\,\xi,\sigma) = 0$ then so are $r = -k$ and $r = \overline{k}$. Moreover, (2.5) implies that $k_l \neq 0$ for $l = \overline{1,6}$.

Now we prove

Lemma 2.3. *All the roots of the equation* $\det L(-i\,r\,\hat{\xi},\sigma) = \Phi_1(r)\Phi_2(r) = 0$ *are real provided* (2.5) *is fulfilled,* $\hat{\xi} \in \mathbf{R}^3$ *and* $|\hat{\xi}| = 1$.

Proof. Let us assume that $k = t + i\,\tau \neq 0$, with $t, \tau \in \mathbf{R}$, solves the equation

$$\det L(-i\,k\,\hat{\xi},\sigma) = \Phi_1(k)\,\Phi_2(k) = 0, \quad \text{i.e.,} \quad k \in \{\pm k_1, \cdots, \pm k_6\},$$

where $\hat{\xi} \in \mathbf{R}^3$ and $|\hat{\xi}| = 1$. We will show that $\tau = 0$.

It is evident that the simultaneous system of linear algebraic equations

$$L(-i\,k\,\hat{\xi},\sigma)\,\eta = 0 \tag{2.26}$$

has then a nontrivial solution with respect to the unknown vector $\eta = (\eta_1, \cdots, \eta_6)^\top \in \mathbf{C}^6$.

Denote

$$\eta' := (\eta_1, \eta_2, \eta_3)^\top, \quad \eta'' := (\eta_4, \eta_5, \eta_6)^\top, \quad \text{i.e.,} \quad \eta = (\eta', \eta'')^\top \neq 0.$$

Rewrite the relation (2.26) as

$$L(-i\,k\,\hat{\xi},\sigma)\,\eta = (-i\,k)^2\,L_0(\,\hat{\xi}\,)\,\eta + (-i\,k)\,L_1(\,\hat{\xi}\,)\,\eta + L_2(\,\hat{\xi}\,)\,\eta = 0$$

and multiply it by the nonzero vector η to obtain, see (2.18),

$$(-i\,k)^2\,L_0(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta + (-i\,k)\,L_1(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta + L_2(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta = 0. \tag{2.27}$$

Taking into account that $L_0(\,\hat{\xi}\,)$ is a positive definite matrix and applying the relations (2.8), (2.9), (2.11) and (2.14), we easily derive from (3.27) by dividing it by $i\,k$ and separating the real part

$$-\tau\,L_0(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta - \Re\big\{L_1(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta\big\} - \frac{\tau}{|\,k\,|^2}\,L_2(\,\hat{\xi}\,)\,\eta\,\cdot\,\eta = 0,$$

i.e.,

$$\begin{aligned}\tau\,L_0(\hat{\xi})\,\eta\,\cdot\,\eta + \frac{\tau}{|k|^2}\,\{\varrho\,\sigma^2\,\eta'\cdot\eta' + (\mathcal{I}\,\sigma^2 - 4\alpha)\;\eta''\cdot\eta''\}\\ + \Re\,\{2\alpha\,[\hat{\xi}\times\eta''\,]\cdot\eta' + 2\alpha\,[\hat{\xi}\times\eta'\,]\cdot\eta'' + 4\nu\,[\hat{\xi}\times\eta''\,]\cdot\eta''\} = 0.\end{aligned} \tag{2.28}$$

One can easily check that

$$\begin{aligned}{}[\hat{\xi}\times\eta''\,]\cdot\eta' &= -[\hat{\xi}\times\overline{\eta'}\,]\cdot\overline{\eta''} = -\overline{[\hat{\xi}\times\eta'\,]\cdot\eta''},\\ [\hat{\xi}\times\eta''\,]\cdot\eta'' &= -[\hat{\xi}\times\overline{\eta''}\,]\cdot\overline{\eta''} = -\overline{[\hat{\xi}\times\eta''\,]\cdot\eta''},\end{aligned} \tag{2.29}$$

which yield

$$\begin{gathered}{}[\hat{\xi}\times\eta''\,]\cdot\eta' + [\hat{\xi}\times\eta'\,]\cdot\eta'' = i\,2\,\Im\,\{[\hat{\xi}\times\eta''\,]\cdot\eta'\},\\ \Re\,\{[\hat{\xi}\times\eta''\,]\cdot\eta''\} = 0.\end{gathered} \tag{2.30}$$

By these relations from (2.28) we then get

$$\tau \Big\{ L_0(\hat{\xi})\, \eta \cdot \eta + \frac{1}{|k|^2} \big[\varrho\, \sigma^2 |\eta'|^2 + (\mathcal{I}\, \sigma^2 - 4\alpha)|\eta''|^2 \big] \Big\} = 0,$$

whence $\tau = 0$ due to positive definiteness of the matrix $L_0(\,\hat{\xi}\,)$ and (2.5). □

By Lemma 2.3 we conclude that, since all the roots k_l are real, we can choose $k_l > 0$ for all $l = \overline{1,6}$ and decompose the operator $\det L(\partial, \sigma)$ as follows

$$\det L(\partial, \sigma) = d_1^2\, d_2 \prod_{l=1}^{6} \big(\Delta + k_l^2\big).$$

From now on we assume that

$$k_l > 0, \quad k_l \neq k_j \quad \text{for} \quad l \neq j. \tag{2.31}$$

It is evident that any solution to the equation $L(\partial, \sigma)U(x) = 0$ in Ω^- satisfies also the equation

$$[\det L(\partial, \sigma)]\, U(x) = d_1^2\, d_2 \prod_{l=1}^{6} \big(\Delta + k_l^2\big)\, U(x) = 0 \quad \text{in} \quad \Omega^-.$$

Therefore, due to the results in [**45**] we conclude that U can be represented in the form

$$U(x) = \sum_{l=1}^{6} U^{(l)}(x) \quad \text{in} \quad \Omega^-, \tag{2.32}$$

where $U^{(l)} = (U_1^{(l)}, \cdots, U_6^{(l)})^\top$ solves the following Helmholtz equation

$$\big(\Delta + k_l^2\big)\, U^{(l)} = 0 \quad \text{in} \quad \Omega^-. \tag{2.33}$$

We say that a solution U to the equation $L(\partial, \sigma)\, U(x) = 0$ in Ω^- satisfies the *Sommerfeld-Kupradze radiation condition* at infinity (belongs to the class $SK(\Omega^-)$) if U is represented in the form (2.32) and each component $U_p^{(l)}$ $(p=\overline{1,6})$ of the vector $U^{(l)}$ satisfies the uniform radiation condition at infinity

$$\frac{\partial}{\partial|x|}\,U_p^{(l)}(x)-i\,k_l\,U_p^{(l)}(x)=o\,(\,|x|^{-1})\quad\text{as}\quad|x|\to\infty,\quad l=\overline{1,6}.$$

Such solutions will be referred to as *radiating*.

As is well known, for radiating solutions and for sufficiently large $|x|$, as $|x|\to\infty$, there hold the relations (for details see, e.g., [**45**])

$$U^{(l)}(x)=\frac{\exp\{i\,k_l\,|x|\}}{|x|}\,U_\infty^{(l)}(\hat{x})+\mathcal{O}\,(\,|x|^{-2}),\tag{2.34}$$

$$\frac{\partial}{\partial|x|}\,U_p^{(l)}(x)-i\,k_l\,U_p^{(l)}(x)=\mathcal{O}\,(\,|x|^{-2}),\tag{2.35}$$

$$\frac{\partial}{\partial x_q}\,U_p^{(l)}(x)-i\,k_l\,\hat{x}_q\,U_p^{(l)}(x)=\mathcal{O}\,(\,|x|^{-2}),\tag{2.36}$$

$$\hat{x}_q=\frac{x_q}{|x|},\quad q=1,2,3,$$

where $U_\infty^{(l)}(\hat{x})$ is the so called *far field pattern*, cf., e.g., [**3**]

$$U_\infty^{(l)}(\hat{x}):=-\,\frac{1}{4\pi}\int\limits_{\partial\Omega^-}e^{-i\,k_l\,\hat{x}\cdot y}\,\Big\{\,\big[\,\partial_{n(y)}U^{(l)}(y)\big]^-\\ +i\,k_l\,\big(\hat{x}\cdot n(y)\big)\,\big[\,U^{(l)}(y)\big]^-\Big\}\,dS.$$

We remind the reader also about the celebrated Rellich-Vekua lemma stating that if $U_p^{(l)}$ solves equation (2.33) in Ω^- and

$$\lim_{R\to\infty}\int_{\Sigma_R}|U_p^{(l)}(x)|^2\,d\,\Sigma_R=0,$$

where Σ_R is the sphere centered at the origin and radius R, then $U_p^{(l)}=0$ in Ω^-. As a consequence we get that $U_\infty^{(l)}=0$ implies $U^{(l)}=0$ in Ω^-.

Note that in the Appendix the fundamental matrix $\Gamma(x-y,\sigma)$ for the operator $L(\partial,\sigma)$ satisfying the Sommerfeld-Kupradze radiation condition at infinity is constructed explicitly in terms of standard elementary functions, provided the conditions (2.31) are fulfilled.

2.5. *Formulation of boundary value problems.* Our main goal is to investigate the following exterior Dirichlet, Neumann and mixed BVPs of steady state oscillations.

Find a distributional solution $U \in [\,H^1_{\rm loc}(\Omega^-)\,]^6$ of the differential equation

$$L(\partial,\sigma)\,U = 0 \quad \text{in} \quad \Omega^- \tag{2.37}$$

satisfying the Sommerfeld-Kupradze radiation condition at infinity and one of the following boundary conditions:

(2.38) **Problem (D)$^-$:** $[\,U]^- = f$ on S,

(2.39) **Problem (N)$^-$:** $[\,T(\partial,n)U]^- = F$ on S,

(2.40) **Problem (M)$^-$:** $[\,U]^- = f_{_D}$ on S_D, $[\,T(\partial,n)\,U]^- = F_{_N}$ on S_N,

where

$$f \in [H^{1/2}(S)]^6, \qquad F \in [H^{-1/2}(S)]^6,$$
$$f_{_D} \in [H^{1/2}(S_D)]^6, \quad F_{_N} \in [H^{-1/2}(S_N)]^6,$$

and where S_D and S_N are two open, disjoint parts of S with $\overline{S_D} \cup \overline{S_N} = S$.

Here and in what follows we assume that $S = \partial\Omega^-$ is a *compact, piecewise smooth, simply connected* Lipschitz surface (if not otherwise stated). Moreover, we assume that both submanifolds S_D and S_N have a positive measure.

The Dirichlet type boundary conditions are understood in the usual trace sense, while the Neumann type boundary conditions are understood in the functional sense described in Remark 2.2.

Note that, instead of the above (weak) formulation, we can consider the *classical* one, when the surface S is C^1 smooth and the sought for radiating vector-function U is regular, i.e., $U \in [C^2(\Omega^-)]^6 \cap [C^1(\overline{\Omega^-})]^6$, all the boundary data are continuous, and all the conditions

(2.37)–(2.40) are understood in the usual classical (pointwise) sense. We recall that even for C^∞-smooth boundary S and C^∞-smooth boundary data f_D and f_N, a solution of the corresponding mixed BVP does not belong to the space $[C^{\alpha'}(\overline{\Omega^-})]^6$ with $\alpha' > 1/2$, in general. The smoothness is violated in a neighborhood of the curve $\partial S_D = \partial S_N$ (across which the types of boundary conditions change).

First we prove the following uniqueness

Theorem 2.4. *The homogeneous boundary value problems $(D)^-$, $(N)^-$ and $(M)^-$ have only the trivial solution.*

Proof. Let R be a sufficiently large positive number such that $\overline{\Omega^+} \subset B_R$, and set $\Omega_R^- := \Omega^- \cap B_R$, where B_R is a ball centered at the origin and radius R. Denote $\Sigma_R := \partial B_R$.

Let U be a radiating solution to one of the homogeneous boundary value problems $(D)^-$, $(N)^-$, or $(M)^-$.

Keeping in mind that $\overline{U}$ solves the equation (2.37) and applying Green's formula to the vectors U and $\overline{U}$ in the bounded domain Ω_R^- due to Remarks 2.1 and 2.2 we easily get

$$
\begin{aligned}
0 &= \int_{\Sigma_R} \{[\,T(\partial,\hat{x})\,U]\cdot[\,\overline{U}] - [\,U]\cdot[\,T(\partial,\hat{x})\,\overline{U}]\}\;d\,\Sigma_R \\
&= 2\,i\,\Im\Big\{\int_{\Sigma_R}[\,T(\partial,\hat{x})\,U]\cdot[\,\overline{U}]\,d\,\Sigma_R\Big\} \\
&= 2\,i\sum_{j=1}^{6}\Im\Big\{\int_{\Sigma_R}[\,T(\partial,\hat{x})\,U]_j\,\overline{U_j}\,d\,\Sigma_R\Big\}, \quad \hat{x} = \frac{x}{|x|}.
\end{aligned} \tag{2.41}
$$

Note that the integrals over the domain Ω_R^- and the surface $S = \partial\Omega^-$ (duality expressions) vanish in view of the homogeneity of the differential equations and the boundary conditions under consideration. Remark also that $\hat{x}$ is the exterior unit normal vector at the point $x \in \Sigma_R$.

Since U is radiating and R is sufficiently large, with the help of (2.3), (2.32) and (2.36) we derive for $x \in \Sigma_R$

$$
\begin{aligned}
(2.42)\quad & T(\partial,\hat{x})U(x) \\
& = T_0(\partial,\hat{x})U(x) + 2\,[\alpha\hat{x}\times\omega(x),\ \nu\hat{x}\times\omega(x)]^\top \\
& = \sum_{l=1}^{6}\Big\{ i\,k_l\,T_0(\hat{x},\hat{x})U^{(l)}(x) + 2\,[\alpha\hat{x}\times\omega^{(l)}(x),\ \nu\hat{x}\times\omega^{(l)}(x)]^\top\Big\} \\
& \quad + \mathcal{O}\,(R^{-2}) \\
& = \sum_{l=1}^{6}\Big\{ i\,k_l\,T_0(\hat{x},\hat{x})U^{(l)}_\infty(\hat{x}) + 2\,[\alpha\hat{x}\times\omega^{(l)}_\infty(\hat{x}),\ \nu\hat{x}\times\omega^{(l)}_\infty(\hat{x})]^\top\Big\} \\
& \quad \times \frac{\exp\{i\,k_l R\}}{R} + \mathcal{O}\,(R^{-2}).
\end{aligned}
$$

Comparing (2.2), (2.11) and (2.12), we see that

$$(2.43)\qquad T_0(\hat{x},\hat{x}) = L_0(\hat{x}).$$

With the help of (2.42), (2.43) and (2.34) from (2.41), we have

(2.44)

$$
\begin{aligned}
0 & = \sum_{j=1}^{6}\Im\Big\{\int_{\Sigma_R}[T(\partial,\hat{x})U]_j\,\overline{U_j}\,d\,\Sigma_R\Big\} \\
& = \Im\sum_{l,\,m=1}^{6}\int_{\Sigma_1}\exp\{i\,(k_l-k_m)R\}\Big\{ i\,k_l\,L_0(\hat{x})U^{(l)}_\infty(\hat{x})\cdot U^{(m)}_\infty(\hat{x}) \\
& \quad + 2\alpha\,[\hat{x}\times\omega^{(l)}_\infty(\hat{x})]\cdot u^{(m)}_\infty(\hat{x}) \\
& \quad + 2\nu\,[\hat{x}\times\omega^{(l)}_\infty(\hat{x})]\cdot \omega^{(m)}_\infty(\hat{x})\Big\}\,d\,\Sigma_1 + \mathcal{O}\,(R^{-1}).
\end{aligned}
$$

Let us integrate this equality with respect to R over the interval $[R_1,\,2R_1]$ and divide the result by R_1 (assuming that R_1 is sufficiently large). Keeping in mind that $k_l \neq k_m$ for $l \neq m$ we have

$$\frac{1}{R_1}\int_{R_1}^{2R_1}\exp\{i\,(k_l-k_m)R\}\,dR = \mathcal{O}\,(R_1^{-1})\quad \text{for}\quad l\neq m.$$

Taking into account these relations from (2.44) we finally get

$$
\begin{aligned}
\Im\sum_{l=1}^{6}\int_{\Sigma_1}\Big\{ & i\,k_l\,L_0(\hat{x})U^{(l)}_\infty(\hat{x})\cdot U^{(l)}_\infty(\hat{x}) + 2\alpha\,[\hat{x}\times\omega^{(l)}_\infty(\hat{x})]\cdot u^{(l)}_\infty(\hat{x}) \\
& + 2\nu\,[\hat{x}\times\omega^{(l)}_\infty(\hat{x})]\cdot\omega^{(l)}_\infty(\hat{x})\Big\}\,d\,\Sigma_1 = \mathcal{O}\,(R_1^{-1})\,.
\end{aligned}
$$

Pass to the limit, as $R \to \infty$, in the last equality to obtain

$$\Im \sum_{l=1}^{6} \int_{\Sigma_1} \Big\{ i\, k_l\, L_0(\hat{x}) U_\infty^{(l)}(\hat{x}) \cdot U_\infty^{(l)}(\hat{x}) + 2\alpha\, [\hat{x} \times \omega_\infty^{(l)}(\hat{x})] \cdot u_\infty^{(l)}(\hat{x}) \\ + 2\nu\, [\hat{x} \times \omega_\infty^{(l)}(\hat{x})] \cdot \omega_\infty^{(l)}(\hat{x}) \Big\}\, d\,\Sigma_1 = 0,$$

i.e.,

(2.45)

$$\sum_{l=1}^{6} \int_{\Sigma_1} \Big\{ k_l\, L_0(\hat{x}) U_\infty^{(l)}(\hat{x}) \cdot U_\infty^{(l)}(\hat{x}) + 2\alpha\, \Im \big\{ [\hat{x} \times \omega_\infty^{(l)}(\hat{x})] \cdot u_\infty^{(l)}(\hat{x}) \big\} \\ + 2\nu\, \Im \big\{ [\hat{x} \times \omega_\infty^{(l)}(\hat{x})] \cdot \omega_\infty^{(l)}(\hat{x}) \big\} \Big\}\, d\,\Sigma_1 = 0.$$

Our goal is to show that $U_\infty^{(l)}(\hat{x}) = 0$ for all $\hat{x} \in \Sigma_1$.

To this end let us remark that for arbitrary radiating solution U of the homogeneous differential equation (2.37) each summand $U^{(l)}$ in the representation (2.32) solves the same differential equation (2.37) since

$$U^{(l)} = \prod_{j=1\ (j \neq l)}^{6} \frac{1}{k_j^2 - k_l^2} \left(\Delta + k_j^2 \right) U.$$

Therefore, the equation

$$\begin{aligned} L(\partial, \sigma)\, U^{(l)}(x) &= L_0(\partial)\, U^{(l)}(x) + L_1(\partial)\, U^{(l)}(x) + L_2(\partial)\, U^{(l)}(x) \\ &= L_0(\partial)\, U^{(l)}(x) + \begin{bmatrix} 2\alpha \operatorname{curl} \omega^{(l)}(x) \\ 2\alpha \operatorname{curl} u^{(l)}(x) + 4\nu \operatorname{curl} \omega^{(l)}(x) \end{bmatrix} \\ &\quad + \begin{bmatrix} \varrho\, \sigma^2\, u^{(l)}(x) \\ (\mathcal{I}\, \sigma^2 - 4\alpha)\, \omega^{(l)}(x) \end{bmatrix} = 0 \end{aligned}$$

for sufficiently large $|x| = R$ yields

$$(i\, k_l)^2\, L_0(\hat{x})\, U_\infty^{(l)}(\hat{x}) + i\, k_l \begin{bmatrix} 2\alpha\, \hat{x} \times \omega_\infty^{(l)}(\hat{x}) \\ 2\alpha\, \hat{x} \times u_\infty^{(l)}(\hat{x}) + 4\nu\, \hat{x} \times \omega_\infty^{(l)}(\hat{x}) \end{bmatrix} \\ + \begin{bmatrix} \varrho\, \sigma^2\, u_\infty^{(l)}(\hat{x}) \\ (\mathcal{I}\, \sigma^2 - 4\alpha)\, \omega_\infty^{(l)}(\hat{x}) \end{bmatrix} = \mathcal{O}\,(R^{-1})$$

due to the radiation conditions (2.34)–(2.36). Passing to the limit as $R \to \infty$ and dividing the equality obtained by $i\,k_l$ we get

$$(i\,k_l)\,L_0(\hat{x})\,U_\infty^{(l)}(\hat{x}) + \begin{bmatrix} 2\alpha\,\hat{x}\times\omega_\infty^{(l)}(\hat{x}) \\ 2\alpha\,\hat{x}\times u_\infty^{(l)}(\hat{x}) + 4\nu\,\hat{x}\times\omega_\infty^{(l)}(\hat{x}) \end{bmatrix} - \frac{i}{k_l}\begin{bmatrix} \varrho\,\sigma^2\,u_\infty^{(l)}(\hat{x}) \\ (\mathcal{I}\,\sigma^2 - 4\alpha)\,\omega_\infty^{(l)}(\hat{x}) \end{bmatrix} = 0.$$

Multiply this equality by the vector $U_\infty^{(l)}(\hat{x})$ and apply the relations (2.29) and (2.30) to obtain

$$\begin{aligned}
(i\,k_l)\,L_0(\hat{x})\,&U_\infty^{(l)}(\hat{x})\cdot U_\infty^{(l)}(\hat{x}) + 2\alpha\,[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]u_\infty^{(l)}(\hat{x}) \\
&+ 2\alpha\,[\hat{x}\times u_\infty^{(l)}(\hat{x})]\cdot\omega_\infty^{(l)}(\hat{x}) \\
&+ 4\nu\,[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]\cdot\omega_\infty^{(l)}(\hat{x}) \\
&- \frac{i}{k_l}\,\{\varrho\,\sigma^2\,|\,u_\infty^{(l)}(\hat{x})\,|^2 + (\mathcal{I}\,\sigma^2 - 4\alpha)\,|\,\omega_\infty^{(l)}(\hat{x})\,|^2\} \\
= {}&(i\,k_l)\,L_0(\hat{x})\,U_\infty^{(l)}(\hat{x})\cdot U_\infty^{(l)}(\hat{x}) \\
&+ i\,4\alpha\,\Im\,\{[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]\cdot u_\infty^{(l)}(\hat{x})\} \\
&+ i\,4\nu\,\Im\,\{[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]\cdot\omega_\infty^{(l)}(\hat{x})\} \\
&- \frac{i}{k_l}\,\{\varrho\,\sigma^2\,|\,u_\infty^{(l)}(\hat{x})\,|^2 + (\mathcal{I}\,\sigma^2 - 4\alpha)\,|\,\omega_\infty^{(l)}(\hat{x})\,|^2\} = 0.
\end{aligned}$$

Whence it follows that

$$\begin{aligned}
2\Big\{\,&k_l\,L_0(\hat{x})\,U_\infty^{(l)}(\hat{x})\cdot U_\infty^{(l)}(\hat{x}) + 2\alpha\,\Im\,\{[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]\cdot u_\infty^{(l)}(\hat{x})\} \\
&+ 2\nu\,\Im\,\{[\hat{x}\times\omega_\infty^{(l)}(\hat{x})]\cdot\omega_\infty^{(l)}(\hat{x})\}\Big\} \\
= {}&k_l\,L_0(\hat{x})\,U_\infty^{(l)}(\hat{x})\cdot U_\infty^{(l)}(\hat{x}) \\
&+ \frac{1}{k_l}\,\{\varrho\,\sigma^2\,|\,u_\infty^{(l)}(\hat{x})\,|^2 + (\mathcal{I}\,\sigma^2 - 4\alpha)\,|\,\omega_\infty^{(l)}(\hat{x})\,|^2\}.
\end{aligned} \tag{2.46}$$

In view of (2.46) from (2.45) we have

$$\sum_{l=1}^{6}\int_{\Sigma_1}\Big\{\frac{k_l}{2}\,L_0(\hat{x})\,U_\infty^{(l)}(\hat{x})\cdot U_\infty^{(l)}(\hat{x}) + \frac{1}{2k_l}\,\{\varrho\,\sigma^2\,|\,u_\infty^{(l)}(\hat{x})\,|^2 + (\mathcal{I}\,\sigma^2 - 4\alpha)\,|\,\omega_\infty^{(l)}(\hat{x})\,|^2\}\Big\}\,d\,\Sigma_1 = 0,$$

which, due to (2.5) and positive definiteness of the matrix $L_0(\hat{x})$, yields $U^{(l)}_\infty(\hat{x}) = 0$, $l = \overline{1,6}$, on Σ_1. By the Rellich-Vekua lemma then $U^{(l)}(x) = 0$, $l = \overline{1,6}$, in Ω^-, and consequently, the vector U is identical zero in Ω^- in accordance with the representation (2.32). This completes the proof. $\square$

3. Integral representation of solutions. Properties of potentials.

3.1. *Integral representation formulae.* For simplicity, henceforward we assume, if not otherwise stated, that

$$S = \partial\Omega^{\pm} \in C^{k,\,\alpha'} \quad \text{with integer} \quad k \geq 2 \quad \text{and} \quad 0 < \alpha' \leq 1, \tag{3.1}$$

and $n(x)$ will stand for the outward unit normal vector to Ω^+ at the point $x \in S$.

Let $\Gamma(x-y,\sigma)$ be the radiating fundamental matrix of the operator $L(\partial,\sigma)$ whose explicit expression is given in the Appendix.

We introduce the generalized single and double layer potentials and the Newtonian type volume potential

$$V(\varphi)(x) = \int_S \Gamma\,(x-y,\sigma)\,\varphi(y)\,dS_y, \quad x \in \mathbf{R}^3 \setminus S, \tag{3.2}$$

$$W(\varphi)(x) = \int_S \left[T(\partial_y, n(y))\,\Gamma\,(y-x,\sigma)\right]^\top \varphi(y)\,dS_y, \quad x \in \mathbf{R}^3 \setminus S, \tag{3.3}$$

$$N_\Omega(\psi)(x) = \int_\Omega \Gamma\,(x-y,\sigma)\,\psi(y)\,dy, \quad x \in \mathbf{R}^3,$$

where $T(\partial,n)$ is the stress operator of the theory of hemitropic elasticity, see (2.1), $\varphi = (\varphi_1,\cdots,\varphi_6)^\top$ is a density vector-function defined on S, while a density vector-function $\psi = (\psi_1,\cdots,\psi_6)^\top$ is defined on $\Omega \in \{\Omega^+,\Omega^-\}$.

It can easily be checked that the potentials defined by (3.2) and (3.3) are radiating, C^∞-smooth in $\mathbf{R}^3 \setminus S$, and solve the homogeneous equations (2.37) for an arbitrary L_p-integrable vector function φ. The volume potential is radiating and solves the non-homogeneous equation $L(\partial,\sigma)\, N_{\Omega}(\psi)(x) = \psi(x)$ in $\mathbf{R}^3$ for $\psi \in [L_{p,\,comp}(\Omega)]^6$.

Applying Green's formula, see Remark 2.1, we can represent a solution of the steady state oscillation equations by means of the above introduced layer and volume potentials.

Theorem 3.1. *Let U be a regular vector of the class $[C^2(\overline{\Omega^+})]^6$. Then there holds the following integral representation formula*

(3.4)

$$W([U]^+)(x) - V([TU]^+)(x) + N_{\Omega^+}(L(\partial,\sigma)U)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+, \\ 0 & \text{for } x \in \Omega^-. \end{cases}$$

Note that by the standard limiting procedure this theorem can be extended to the case $U \in [H^1(\Omega^+)]^6$ with $L(\partial,\sigma)U \in [L_2(\Omega^+)]^6$, cf., e.g., [**6, 25, 26**].

Now we are in the position to prove the following

Theorem 3.2. *Let $U \in [H^1_{loc}(\overline{\Omega^-})]^6$ be a radiating solution of the homogeneous equation $L(\partial,\sigma)U(x) = 0$ in Ω^-. Then there holds the following integral representation formula*

$$-W([U]^-)(x) + V([TU]^-)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^-, \\ 0 & \text{for } x \in \Omega^+. \end{cases} \tag{3.5}$$

Proof. Let R be a sufficiently large positive number such that $\overline{\Omega^+} \subset B_R$, where as above B_R is the ball of radius R centered at the origin. Denote $\Omega^-_R := \Omega^- \cap B_R$.

Let $x \in \Omega^-$ be an arbitrary point, and choose R such that $x \in \Omega^-_R$. Write the integral representation formula (3.4) for $U(x)$ in the bounded domain Ω^-_R,

$$U(x) = -W([U]^-)(x) + V([TU]^-)(x) + \Psi(x,R) \tag{3.6}$$

with

$$\Psi(x,R) := \int_{\Sigma_R} \Big\{ \big[T(\partial_y,\hat{y})\,\Gamma\,(y-x,\sigma)\big]^\top U(y) - \Gamma\,(x-y,\sigma)\,T(\partial_y,\hat{y})U(y) \Big\}\, d\,\Sigma_R, \tag{3.7}$$

where Σ_R is the boundary of B_R and $\hat{y} = y/|y|$ is the outward normal to Σ_R.

Further, let

$$\widetilde{U}(x) := U(x) + W([U]^-)(x) - V([TU]^-)(x).$$

From (3.6) we then have $\widetilde{U}(x) = \Psi(x,R)$, $x \in \Omega_R^-$. Note that the lefthand side expression $\widetilde{U}(x)$ does not depend on R.

Let us integrate the last equality with respect to R over the interval $(R_1, 2R_1)$ and divide the result by R_1 where R_1 is a sufficiently large number. We get

$$\widetilde{U}(x) = \frac{1}{R_1} \int_{R_1}^{2R_1} \Psi(x,R)\, dR. \tag{3.8}$$

In what follows we show that for a radiating solution U, the righthand side expression in (3.8) tends to zero as $R_1 \to 0$.

To this end, note that, since U and $\Gamma(\cdot,\sigma)$ are radiating, for a fixed x and sufficiently large $|y|$ we have (see the Appendix)

$$T(\partial_y,\hat{y})\, U(y) = \sum_{j=1}^{6} \{i\, k_j\, T_0(\hat{y},\hat{y})\, U^{(j)}(y) + \mathcal{A}(\hat{y})\, U^{(j)}(y)\} + \mathcal{O}(R^{-2}),$$

$$\begin{aligned} T(\partial_y,\hat{y})\, \Gamma\, (y-x,\sigma) = \sum_{j=1}^{6} \Big\{ & i\, k_j\, T_0(\hat{y},\hat{y})\, \Gamma^{(j)}(y-x,\sigma) \\ & + \mathcal{A}(\hat{y})\, \Gamma^{(j)}(y-x,\sigma) \Big\} + \mathcal{O}(R^{-2}), \end{aligned}$$

where, see (2.3) and (2.8)

$$\mathcal{A}(\hat{y}) := \begin{bmatrix} [\,0\,]_{3\times 3} & 2\alpha\, R(\hat{y}) \\ [\,0\,]_{3\times 3} & 2\nu\, R(\hat{y}) \end{bmatrix}_{6\times 6}.$$

Evidently,

$$\mathcal{A}(\hat{y})\, U^{(j)}(y) = 2\, [\alpha\, \hat{y} \times \omega^{(j)}(y),\, \nu\, \hat{y} \times \omega^{(j)}(y)]^\top.$$

Therefore, from (3.11) and (3.7) it follows that

$$(3.9)\quad \widetilde{U}(x) = \frac{1}{R_1}\int_{R_1}^{2R_1} dR \int_{\Sigma_R} \sum_{j,q=1}^{6} \Big\{ \Big[\Big(i\,k_j\, T_0(\hat{y},\hat{y}) + \mathcal{A}(\hat{y}) \Big) \widetilde{\Gamma}^{(j)}(y-x,\sigma) \Big]^\top U^{(q)}(y) - [\widetilde{\Gamma}^{(j)}(y-x,\sigma)]^\top [i\,k_q\, T_0(\hat{y},\hat{y})\, U^{(q)}(y) + \mathcal{A}(\hat{y})\, U^{(q)}(y)] \Big\}\, d\,\Sigma_R + \mathcal{O}(R_1^{-1}).$$

To show that the righthand side in (3.9) tends to zero as $R_1 \to \infty$, it suffices to prove that

$$(3.10)\quad \psi_{jq}(R_1) := \frac{1}{R_1}\int_{R_1}^{2R_1} dR \int_{\Sigma_1} h^{(j)}(R\hat{y})\, g^{(q)}(R\hat{y})\, R^2\, d\,\Sigma_1 \to 0$$

as $R_1 \to \infty$, where

$$h^{(j)}(R\hat{y}) = \mathcal{O}(R^{-1}), \quad \partial_R\, h^{(j)}(R\hat{y}) - i\,k_j\, h^{(j)}(R\hat{y}) = \mathcal{O}(R^{-2}),$$
$$g^{(q)}(R\hat{y}) = \mathcal{O}(R^{-1}), \quad \partial_R\, g^{(q)}(R\hat{y}) - i\,k_q\, g^{(q)}(R\hat{y}) = \mathcal{O}(R^{-2}),$$
$$\partial_R := \partial/\partial R.$$

Note that $h^{(j)}(R\hat{y}) := \Gamma^{(j)}_{ps}(R\hat{y}-x,\sigma)$ and $g^{(q)}(R\hat{y}) := U^{(q)}_m(R\hat{y})$ satisfy the above relations due to the radiation conditions.

Taking into account that $k_j + k_q > 0$ we get

$$\begin{aligned} h^{(j)}(R\hat{y})\, g^{(q)}(R\hat{y}) &= \frac{1}{i(k_j+k_q)} \Big[i\,k_j\, h^{(j)}(R\hat{y})\, g^{(q)}(R\hat{y}) + h^{(j)}(R\hat{y})\, i\,k_q\, g^{(q)}(R\hat{y}) \Big] \\ &= \frac{1}{i(k_j+k_q)} \Big[g^{(q)}(R\hat{y})\, \partial_R h^{(j)}(R\hat{y}) + h^{(j)}(R\hat{y})\, \partial_R g^{(q)}(R\hat{y}) \Big] \\ &\quad + \mathcal{O}(R^{-3}). \end{aligned}$$

Therefore from (3.10) with the help of the integration by parts formula we derive

$$\begin{aligned}
\psi_{jq}(R_1) &= \frac{1}{R_1}\int_{\Sigma_1} d\,\Sigma \int_{R_1}^{2R_1} \frac{R^2}{i(k_j+k_q)} \\
&\quad \times \frac{\partial}{\partial R}\Big[h^{(j)}(R\hat{y})\,g^{(q)}(R\hat{y})\Big]\,dR + \mathcal{O}(R_1^{-1}) \\
&= \frac{1}{i(k_j+k_q)R_1}\int_{\Sigma_1}\bigg\{\Big[R^2\,h^{(j)}(R\hat{y})\,g^{(q)}(R\hat{y})\Big]_{R_1}^{2R_1} \\
&\quad - \int_{R_1}^{2R_1} h^{(j)}(R\hat{y})\,g^{(q)}(R\hat{y})\,2R\,dR\bigg\}\,d\,\Sigma_1 \\
&= \frac{1}{i(k_j+k_q)R_1}\int_{\Sigma_1}\mathcal{O}(1)\,d\,\Sigma_1 = \mathcal{O}(R_1^{-1}) \to 0 \\
&\quad \text{as} \quad R_1 \longrightarrow +\infty.
\end{aligned}$$

Thus $\psi_{jq}(R_1) \to 0$ as $R_1 \to +\infty$, which shows that the righthand side in (3.9) tends to zero as $R_1 \to +\infty$. In turn this yields $\widetilde{U}(x) = 0$, whence the proof of the equality (3.5) follows for $x \in \Omega^-$. The proof for the case $x \in \Omega^+$ may be verbatim performed. $\square$

3.2. *Properties of potentials and boundary pseudodifferential operators.* The jump and mapping properties of the above introduced single and double layer potentials and the corresponding boundary integral (pseudodifferential) operators in the Hölder ($C^{k+\alpha'}$), Sobolev-Slobodetski (W_p^s), Bessel potential (H_p^s) and Besov ($B_{p,q}^s$) spaces can be established by standard methods, see, e.g., [**6, 8, 9, 11, 19, 26, 28**–**33**]. We remark only that the layer potentials corresponding to the fundamental matrices with different values of the parameter σ (σ_1 and σ_2 say) have the same smoothness properties and possess the same jump relations, since the entries of the difference of the fundamental matrices $\Gamma(x,\sigma_1) - \Gamma(x,\sigma_2)$ are bounded functions in $\mathbf{R}^3$ and their derivatives of order m have a singularity of type $\mathcal{O}(|x|^{-m})$ in a neighborhood of the origin. Moreover, the boundary integral operators generated by the single layer potentials (respectively, by the double layer potentials) constructed by the kernels $\Gamma(x,\sigma_1)$ and $\Gamma(x,\sigma_2)$ differ by a compact perturbations. Therefore, using the word-for-word arguments given in [**33**] we can prove the following theorems concerning the above introduced layer potentials.

Theorem 3.3. *Let S, k, and α' be as in* (3.1), $0 < \gamma' < \alpha'$, *and let* $m \le k-1$ *be an integer. Then, the operators*

$$V : [C^{m,\gamma'}(S)]^6 \to [C^{m+1,\gamma'}(\overline{\Omega^{\pm}})]^6, \qquad W : [C^{m,\gamma'}(S)]^6 \to [C^{m,\gamma'}(\overline{\Omega^{\pm}})]^6 \tag{3.11}$$

are bounded.

For any $g \in [C^{0,\gamma'}(S)]^6$, $h \in [C^{1,\gamma'}(S)]^6$, *and any* $x \in S$

$$[V(g)(x)]^{\pm} = V(g)(x) = \mathcal{H}\, g(x), \tag{3.12}$$

$$[T(\partial_x, n(x))\, V(g)(x)]^{\pm} = \left[\mp 2^{-1} I_6 + \mathcal{K}\right] g(x), \tag{3.13}$$

$$[W(g)(x)]^{\pm} = \left[\pm 2^{-1} I_6 + \mathcal{K}^*\right] g(x), \tag{3.14}$$

$$[T(\partial_x, n(x))\, W(h)(x)]^{+} = [T(\partial_x, n(x))\, W(h)(x)]^{-} = \mathcal{L}\, h(x), \tag{3.15}$$

where

$$\mathcal{H}\, g(x) := \int_S \Gamma(x-y,\sigma)\, g(y)\, dS_y, \tag{3.16}$$

$$\mathcal{K}\, g(x) := \int_S [\, T(\partial_x, n(x))\, \Gamma(x-y,\sigma)\,]\, g(y)\, dS_y, \tag{3.17}$$

$$\mathcal{K}^*\, g(x) := \int_S [\, T(\partial_y, n(y))\, \Gamma(y-x,\sigma)\,]^{\top}\, g(y)\, dS_y, \tag{3.18}$$

$$\mathcal{L}\, h(x) := \lim_{\Omega^{\pm} \ni z \to x \in S} T(\partial_z, n(x)) \int_S [\, T(\partial_y, n(y))\, \Gamma(y-z,\sigma)\,]^{\top}\, h(y)\, dS_y. \tag{3.19}$$

It can easily be shown that the operators $\mathcal{K}$ and $\mathcal{K}^*$ are mutually adjoint singular integral operators, $\mathcal{H}$ is a smoothing (weakly singular) integral operator, while $\mathcal{L}$ is a singular integro-differential operator. For

C^∞-smooth surfaces all these operators can be treated as pseudodifferential operators on manifolds.

Theorem 3.4. *Let S be as in* (3.1). *The operators V and W can be extended by continuity to the bounded mappings*

$$\begin{aligned} V &: [H^{-1/2}(S)]^6 \longrightarrow [H^1(\Omega^+)]^6 \qquad \Big[[H^{-1/2}(S)]^6 \longrightarrow [H^1_{\mathrm{loc}}(\Omega^-)]^6\Big], \\ W &: [H^{1/2}(S)]^6 \longrightarrow [H^1(\Omega^+)]^6 \qquad \Big[[H^{1/2}(S)]^6 \longrightarrow [H^1_{\mathrm{loc}}(\Omega^-)]^6\Big]. \end{aligned}$$

The jump relations (3.12)–(3.15) *on S remain valid for the extended operators in the corresponding function spaces.*

Theorem 3.5. *Let S, k, α', γ' and m be as in Theorem* 3.3. *Then the operators*

$$\mathcal{H} : [C^{m,\gamma'}(S)]^6 \longrightarrow [C^{m+1,\gamma'}(S)]^6, \tag{3.20}$$

$$: [H^{-1/2}(S)]^6 \longrightarrow [H^{1/2}(S)]^6, \tag{3.21}$$

$$\mathcal{K} : [C^{m,\gamma'}(S)]^6 \longrightarrow [C^{m,\gamma'}(S)]^6, \tag{3.22}$$

$$: [H^{-1/2}(S)]^6 \longrightarrow [H^{-1/2}(S)]^6, \tag{3.23}$$

$$\mathcal{K}^* : [C^{m,\gamma'}(S)]^6 \longrightarrow [C^{m,\gamma'}(S)]^6, \tag{3.24}$$

$$: [H^{1/2}(S)]^6 \longrightarrow [H^{1/2}(S)]^6, \tag{3.25}$$

$$\mathcal{L} : [C^{m,\gamma'}(S)]^6 \longrightarrow [C^{m-1,\gamma'}(S)]^6, \tag{3.26}$$

$$: [H^{1/2}(S)]^6 \longrightarrow [H^{-1/2}(S)]^6 \tag{3.27}$$

are bounded.

Moreover,

(i) *the principal homogeneous symbol matrices of the operators $\pm 2^{-1}I_6 + \mathcal{K}$ and $\pm 2^{-1}I_6 + \mathcal{K}^*$ are nondegenerate, while the principal homogeneous symbol matrices of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive definite*;

(ii) *the operators $\mathcal{H}$, $\pm 2^{-1}I_6 + \mathcal{K}$, $\pm 2^{-1}I_6 + \mathcal{K}^*$ and $\mathcal{L}$ are elliptic pseudodifferential operators* (*of order* -1, 0, 0, *and* 1, *respectively*) *with zero index*;

(iii) *the following equalities hold in appropriate function spaces*:

$$\begin{aligned} &\mathcal{K}^*\,\mathcal{H} = \mathcal{H}\,\mathcal{K}, && \mathcal{L}\,\mathcal{K}^* = \mathcal{K}\,\mathcal{L}, \\ &\mathcal{H}\,\mathcal{L} = -4^{-1}\,I_6 + [\,\mathcal{K}^*\,]^2, && \mathcal{L}\,\mathcal{H} = -4^{-1}\,I_6 + \mathcal{K}^2; \end{aligned} \tag{3.28}$$

(iv) *the operators* $\mathcal{K}$ *and* $\mathcal{K}^*$ (*acting between the Bessel potential spaces as in* (3.23) *and* (3.25)) *are mutually adjoint with respect to the duality brackets* (2.24), *while the operators* $\mathcal{H}$ *and* $\mathcal{L}$ (*acting between the Bessel potential spaces as in* (3.21) *and* (3.27)) *are self-adjoint.*

The next assertion is a direct consequence of the general theory of elliptic pseudodifferential operators on smooth manifolds without boundary, see, e.g., [**7, 11, 13, 14, 40, 42, 43**, and the references therein].

Theorem 3.6. *Let* V, W, $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}^*$ *and* $\mathcal{L}$ *be as in Theorems* 3.3, 3.4 *and* 3.5, *and let* $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $S \in C^\infty$. *The layer potential operators* (3.11) *and the boundary integral* (*pseudodifferential*) *operators* (3.20)–(3.27) *can be extended continuously to the following bounded operators*

$$\begin{aligned} V &: [B^s_{p,p}(S)]^6 \longrightarrow [H^{s+1+(1/p)}_p(\Omega^+)]^6, \\ &: \big[\,[B^s_{p,p}(S)]^6 \longrightarrow [H^{s+1+(1/p)}_{p,\mathrm{loc}}(\Omega^-)]^6\,\big], \\ &: [B^s_{p,q}(S)]^6 \longrightarrow [B^{s+1+(1/p)}_{p,q}(\Omega^+)]^6, \\ &: \big[\,[B^s_{p,q}(S)]^6 \longrightarrow [B^{s+1+(1/p)}_{p,q,\mathrm{loc}}(\Omega^-)]^6\,\big], \\ W &: [B^s_{p,p}(S)]^6 \longrightarrow [H^{s+(1/p)}_p(\Omega^+)]^6, \\ &: \big[\,[B^s_{p,p}(S)]^6 \longrightarrow [H^{s+(1/p)}_{p,\,loc}(\Omega^-)]^6\,\big], \\ &: [B^s_{p,q}(S)]^6 \longrightarrow [B^{s+(1/p)}_{p,q}(\Omega^+)]^6, \\ &: \big[\,[B^s_{p,q}(S)]^6 \longrightarrow [B^{s+(1/p)}_{p,q,\mathrm{loc}}(\Omega^-)]^6\,\big], \end{aligned}$$

(3.29)
$$\mathcal{H} : [H^s_p(S)]^6 \longrightarrow [H^{s+1}_p(S)]^6 \quad \big[\,[B^s_{p,q}(S)]^6 \longrightarrow [B^{s+1}_{p,q}(S)]^6\big],$$

(3.30)
$$\pm 2^{-1} I_6 + \mathcal{K} : [H^s_p(S)]^6 \longrightarrow [H^s_p(S)]^6 \quad \big[\,[B^s_{p,q}(S)]^6 \longrightarrow [B^s_{p,q}(S)]^6\big],$$

(3.31)

$$\pm 2^{-1} I_6 + \mathcal{K}^* : [H_p^s(S)]^6 \longrightarrow [H_p^s(S)]^6 \quad \big[\,[B_{p,q}^s(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6\big],$$

(3.32)

$$\mathcal{L} : [H_p^{s+1}(S)]^6 \longrightarrow [H_p^s(S)]^6 \quad \big[\,[B_{p,q}^{s+1}(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6\big].$$

The jump relations (3.12)–(3.15) *remain valid for arbitrary* $g \in [B_{p,q}^s(S)]^6$ *with* $s \in \mathbf{R}$ *if the limiting values* (*traces*) *on* S *are understood in the sense described in* [**40**].

The null-spaces of the operators (3.29)–(3.32) *are invariant with respect to* p, q, *and* s.

3.3. *Some results from the theory of pseudodifferential equations on manifolds with boundary.* In this subsection we shall present some results from the theory of elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which will be the main tools for proving existence theorems for the mixed problems. They can be found in [**7, 11, 13, 41, 42**].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n-dimensional, non self-intersecting, two-sided manifold with boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathcal{A}$ be a strongly elliptic $m \times m$ matrix pseudodifferential operator of order $r \in \mathbf{R}$ on $\overline{\mathcal{M}}$. Denote by $\sigma_{\mathcal{A}}(x,\xi)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}$ in some local coordinate system ($x \in \mathcal{M}$, $\xi \in \mathbf{R}^n \setminus \{0\}$) and associate with $\sigma_{\mathcal{A}}$ the $m \times m$ matrix function

$$\mathcal{A}_{0\eta}(x,\xi) = |\xi|^{-r}\, \sigma_{\mathcal{A}}(x, |\xi'|\eta, \xi_n), \tag{3.33}$$

where $\eta \in \mathbf{R}^{n-1}$ with $|\eta| = 1$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$.

It is known that the matrix $\mathcal{A}_{0\eta}$ given by (3.33) admits the factorization

$$\mathcal{A}_{0\eta}(x,\xi) = \mathcal{A}_\eta^-(x,\xi)\, D(\eta,x,\xi)\, \mathcal{A}_\eta^+(x,\xi) \quad \text{for} \quad x \in \partial\mathcal{M},$$

where $[\mathcal{A}_\eta^-(x,\xi)]^{\pm 1}$ and $[\mathcal{A}_\eta^+(x,\xi)]^{\pm 1}$ are matrices, which are homogeneous of degree 0 in ξ and admit analytic bounded continuations with respect to ξ_n into the lower and upper complex half-planes, respectively; $D(\eta,x,\xi)$ is a bounded lower triangular matrix with entries of the form

$$\left(\frac{\xi_n - i\,|\xi'|}{\xi_n + i\,|\xi'|}\right)^{\delta_j(x)}, \quad j = 1, \dots, m,$$

on the main diagonal; here

$$\delta_j(x) = (2\pi\, i)^{-1} \ln \lambda_j(x), \quad j = 1, \dots, m,$$

where $\lambda_1(x), \dots, \lambda_m(x)$ are the eigenvalues of the matrix

$$\tilde{\sigma}_{\mathcal{A}}(x) = [\sigma_{\mathcal{A}}(x, 0, \cdots, 0, -1)]^{-1}\, [\sigma_{\mathcal{A}}(x, 0, \cdots, 0, +1)].$$

The branch in the logarithmic function is chosen with regard to the inequality

$$1/p - 1 < \Re\, \delta_j(x) < 1/p, \quad j = 1, \dots, m.$$

The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Note that if $\sigma_{\mathcal{A}}(x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbf{R}^n \setminus \{0\}$, then

$$\Re\, \delta_j(x) = 0 \quad \text{for} \quad j = 1, \dots, m,$$

since, in this case, the eigenvalues of the matrix $\tilde{\sigma}_{\mathcal{A}}(x)$ are positive numbers for any $x \in \overline{\mathcal{M}}$. The Fredholm properties of such operators are characterized by the following assertion, see, e.g., [**7, 42**].

Lemma 3.7. *Let $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le q \le \infty$, and let $\mathcal{A}$ be a strongly elliptic pseudodifferential operator of order $r \in \mathbf{R}$ having a positive definite principal homogeneous symbol matrix, i.e.,*

$$\sigma_{\mathcal{A}}(x, \xi)\, \zeta \cdot \zeta \ge c_0\, |\zeta|^2 \quad \textit{for} \quad x \in \overline{\mathcal{M}},\ \xi \in \mathbf{R}^n$$
$$\textit{with} \quad |\xi| = 1, \quad \textit{and} \quad \zeta \in \mathbf{C}^m,$$

where c_0 is a positive constant.

Then the operators

$$\mathcal{A} : [\tilde{H}^s_p(\mathcal{M})]^m \to [H^{s-r}_p(\mathcal{M})]^m, \tag{3.34}$$

$$: [\tilde{B}^s_{p,q}(\mathcal{M})]^m \to [B^{s-r}_{p,q}(\mathcal{M})]^m, \tag{3.35}$$

are Fredholm with zero index if and only if

$$\frac{1}{p} - 1 < s - \frac{r}{2} < \frac{1}{p}. \tag{3.36}$$

Moreover, the null-spaces and indices of the operators (3.34)*, respectively* (3.35)*, are the same provided* p *and* s *satisfy the inequality* (3.36) *and* $q \in [1, +\infty]$.

4. Existence and regularity results.

4.1. *Problem* $(D)^-$. We look for a solution to Problem $(D)^-$ in the form

$$U(x) = W(g)(x) + i\, V(g)(x), \quad x \in \Omega^-, \tag{4.1}$$

where W and V are double and single layer potentials respectively and $g = (g_1, g_2, \dots, g_6)^\top : S \to \mathbf{C}^6$ is a sought for vector-function. Due to Theorem 3.3 the boundary condition (2.38) leads then to the integral equation

$$\big\{ -2^{-1} I_6 + \mathcal{K}^* + i\,\mathcal{H} \big\}\, g = f \quad \text{on} \quad S, \tag{4.2}$$

where $\mathcal{K}^*$ and $\mathcal{H}$ are defined as in Theorem 3.3, see (3.16) and (3.18).

We assume that S, k, α', γ' and m are as in Theorem 3.3, and either

$$f \in [H^{1/2}(S)]^6 \quad \text{and} \quad g \in [H^{1/2}(S)]^6,$$

or

$$f \in [C^{m,\gamma'}(S)]^6 \quad \text{and} \quad g \in [C^{m,\gamma'}(S)]^6.$$

Lemma 4.1. *Let* S, k *and* α' *be as in* (3.1)*. The operator*

$$\mathcal{D} := -2^{-1} I_6 + \mathcal{K}^* + i\,\mathcal{H} \,:\, [H^{1/2}(S)]^6 \longrightarrow [H^{1/2}(S)]^6 \tag{4.3}$$

is invertible.

Proof. The normally solvable singular integral operator (elliptic pseudodifferential operator of order 0) (4.3) is Fredholm with zero index due to Theorem 3.5. Therefore, for its invertibility we need to show that the corresponding null-space is trivial. To this end let us prove that the adjoint operator

$$-2^{-1} I_6 + \mathcal{K} + i\,\mathcal{H} \,:\, [H^{-1/2}(S)]^6 \longrightarrow [H^{-1/2}(S)]^6$$

is injective, see Theorem 3.5 (iv). Let $h \in [H^{-1/2}(S)]^6$ be a solution to the corresponding homogeneous equation

$$\big[- 2^{-1} I_6 + \mathcal{K} + i\,\mathcal{H} \big]\, h = 0 \quad \text{on} \quad S. \tag{4.4}$$

Consider the single layer potential $U_0(x) := V(h)(x)$, $x \in \mathbf{R}^3 \backslash S$. Clearly $U_0 \in [H^1_{\mathrm{loc}}(\mathbf{R}^3)]^6$ and satisfies the Sommerfeld-Kupradze radiation conditions. Equation (4.4) corresponds to the boundary value condition

$$[\,T(\partial, n) U_0]^+ + i\,[\,U_0]^+ = 0 \quad \text{on} \quad S. \tag{4.5}$$

Applying Green's identity (2.23) in Ω^+ with $U' = U = U_0$ and taking into consideration (4.5), we get

$$\begin{aligned} i\,\Big\langle [\,U_0]^+, \overline{[\,U_0\,]^+} \Big\rangle_S &= i \int_S |\,[\,U_0]^+\,|^2\, dS \\ &= \int_{\Omega^+} \big[E(U_0, \overline{U_0}) - \varrho\sigma^2 |u_0|^2 - (\mathcal{I}\sigma^2 - 4\alpha)|\omega_0|^2 \big]\; dx. \end{aligned}$$

Since the righthand side expression is real, we conclude that $[U_0]^+_S = 0$, and consequently $[T\,U_0]^+_S = 0$ due to (4.5). Therefore by the integral representation formula (3.4), see Theorem 3.1, it follows that $U_0(x) = 0$ in Ω^+. Further, using the continuity property of the single layer potential we have $[U_0]^+_S = [U_0]^-_S = 0$. Since U_0 is radiating, by the uniqueness Theorem 2.4, U_0 vanishes in Ω^-. By the jump formulas then we have $[T\,U_0]^-_S - [T\,U_0]^+_S = h = 0$. Thus, equation (4.4) has only the trivial solution, which implies that the null-space of the operator (4.3) is trivial. This completes the proof. □

By Theorems 3.3, 3.5, 3.6 and Lemma 4.1, the well-known imbedding theorems and the interpolation properties of the function spaces under consideration, we arrive at the following corollaries.

Corollary 4.2. *Let S, k, α', γ' and m be as in Theorem* 3.3. *The operator*

$$\mathcal{D} = -\,2^{-1} I_6 + \mathcal{K}^* + i\,\mathcal{H} : [C^{m,\,\gamma'}(S)]^6 \longrightarrow [C^{m,\,\gamma'}(S)]^6$$

is invertible.

Corollary 4.3. *Let $S \in C^{\infty}$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le q \le \infty$. The operators*

$$\mathcal{D} = -\,2^{-1}\, I_6 + \mathcal{K}^* + i\,\mathcal{H} \,:\, [H^s_p(S)]^6 \longrightarrow [H^s_p(S)]^6$$
$$:\, [B^s_{p,q}(S)]^6 \longrightarrow [B^s_{p,q}(S)]^6$$

are invertible.

These assertions lead to the following existence results.

Theorem 4.4. (i) *Let S, k, and α' be as in* (3.1)*. The exterior Dirichlet problem $(D)^-$ is uniquely solvable for arbitrary boundary data $f \in [H^{1/2}(S)]^6$, and the solution $U \in [H^1_{\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ can be represented in the form* (4.1) *where the density vector $g \in [H^{1/2}(S)]^6$ solves the integral equation* (4.2)*.*

Moreover, in addition,

(ii) *if S, k, α', γ' and m are as in Theorem* 3.3 *and $f \in [C^{m,\gamma'}(S)]^6$, then the density vector g belongs to the space $[C^{m,\gamma'}(S)]^6$ and, consequently, the solution U represented in the form* (4.1) *belongs to the space $[C^{m,\gamma'}(\overline{\Omega^-})]^6 \cap SK(\Omega^-)$;*

(iii) *if $S \in C^{\infty}$, $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbf{R}$, and $f \in [B^{s-(1/p)}_{p,q}(S)]^6$, then the density vector g belongs to the space $[B^{s-(1/p)}_{p,q}(S)]^6$ and, consequently, the solution U represented in the form* (4.1) *belongs to the space $[B^s_{p,q,\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$.*

4.2. *Problem $(N)^-$.* We look for a solution to Problem $(N)^-$ again in the form (4.1). Due to Theorem 3.3 the boundary condition (2.39) leads then to the integral equation

$$\big\{\, \mathcal{L} + i\,\big[2^{-1}\, I_6 + \mathcal{K}\big]\,\big\}\, g = F \quad \text{on} \quad S, \tag{4.6}$$

where $\mathcal{L}$ and $\mathcal{K}$ are defined as in Theorem 3.3, see (3.17) and (3.19).

Here we assume again that S, k, α', γ' and $m \ge 1$ are as in Theorem 3.3, and either

$$F \in [H^{-1/2}(S)]^6 \quad \text{and} \quad g \in [H^{1/2}(S)]^6,$$

or

$$F \in [C^{m-1,\gamma'}(S)]^6 \quad \text{and} \quad g \in [C^{m,\gamma'}(S)]^6.$$

As a first step we prove the following invertibility result.

Lemma 4.5. *Let S, k, and α' be as in* (3.1). *The operator*

$$\mathcal{N} := \mathcal{L} + i\left[2^{-1} I_6 + \mathcal{K}\right] \,:\, [H^{1/2}(S)]^6 \longrightarrow [H^{-1/2}(S)]^6 \tag{4.7}$$

is invertible.

Proof. The elliptic pseudodifferential operator of order 1 (singular integro-differential operator) (4.7) is Fredholm with zero index due to Theorem 3.5. Therefore the invertibility of (4.7) follows from its injectivity.

To show that the kernel (null-space) of the operator (4.7) is trivial, we consider the homogeneous equation

$$\left\{\mathcal{L} + i\left[2^{-1} I_6 + \mathcal{K}\right]\right\} g = 0 \quad \text{on} \quad S.$$

Let $g_0 \in [H^{1/2}(S)]^6$ be some solution of this equation and $U_0(x) := W(g_0)(x) + i\,V(g_0)(x)$, $x \in \mathbf{R}^3 \setminus S$. It is easy to see that then $U_0 \in [H^1_{\text{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ is a solution to the homogeneous BVP $(N)^-$ in Ω^-. Therefore $U_0 = 0$ in Ω^- due to uniqueness Theorem 2.4. Whence, $[U_0]^-_S = \mathcal{D}\,g_0 = 0$ and consequently $g_0 = 0$ on S in accordance with Lemma 4.1. This completes the proof. $\square$

Quite similarly as above, by Theorems 3.3, 3.5, 3.6 and Lemma 4.5, the well-known imbedding theorems and the interpolation properties of the function spaces under consideration we immediately arrive at the following corollaries.

Corollary 4.6. *Let S, k, α', γ' and $m \geq 1$ be as in Theorem* 3.3. *The operator*

$$\mathcal{N} = \mathcal{L} + i\left[2^{-1} I_6 + \mathcal{K}\right] \,:\, [C^{m,\gamma'}(S)]^6 \longrightarrow [C^{m-1,\gamma'}(S)]^6 \tag{4.8}$$

is invertible.

Corollary 4.7. *Let $S \in C^{\infty}$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le q \le \infty$. The operators*

$$\begin{aligned} \mathcal{N} = \mathcal{L} + i\left[2^{-1}\, I_6 + \mathcal{K}\right] \,:\, & [H_p^{s+1}(S)]^6 \longrightarrow [H_p^s(S)]^6 \\ :\, & [B_{p,q}^{s+1}(S)]^6 \longrightarrow [B_{p,q}^s(S)]^6 \end{aligned} \tag{4.9}$$

are invertible.

From invertibility of the operators (4.7), (4.8) and (4.9), the following existence results follow directly.

Theorem 4.8. (i) *Let S, k, and α' be as in* (3.1)*. The exterior Neumann problem $(N)^-$ is uniquely solvable for arbitrary boundary data $F \in [H^{-1/2}(S)]^6$, and the solution $U \in [H^1_{\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ can be represented in the form* (4.1) *where the density vector $g \in [H^{1/2}(S)]^6$ solves the integral equation* (4.6)*.*

Moreover, in addition,

(ii) *if S, k, α', γ', $m \ge 1$ are as in Theorem* 3.3 *and $F \in [C^{m-1,\gamma'}(S)]^6$, then the density vector g belongs to the space $[C^{m,\gamma'}(S)]^6$ and, consequently, the solution U represented in the form* (4.1) *belongs to the space $[C^{m,\gamma'}(\overline{\Omega^-})]^6 \cap SK(\Omega^-)$;*

(iii) *if $S \in C^{\infty}$, $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbf{R}$, and $F \in [B_{p,q}^{s-1-(1/p)}(S)]^6$, then the density vector g belongs to the space $[B_{p,q}^{s-(1/p)}(S)]^6$ and, consequently, the solution U represented in the form* (4.1) *belongs to the space $[B^s_{p,q,\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$.*

4.3. *Steklov-Poincaré type relations.* From the results of the previous subsections, it follows that an arbitrary radiating solution vector $U \in [B^s_{p,q,\,loc}(\Omega^-)]^6$ to the homogeneous oscillation equations (2.37) can be represented by the two equivalent formulas with the help of the corresponding Cauchy data,

$$\begin{aligned} U(x) &= W(\mathcal{D}^{-1}\,[U]^-)(x) + i\,V(\mathcal{D}^{-1}\,[U]^-)(x), \quad x \in \Omega^-, \\ U(x) &= W(\mathcal{N}^{-1}\,[TU]^-)(x) + i\,V(\mathcal{N}^{-1}\,[TU]^-)(x), \quad x \in \Omega^-, \end{aligned} \tag{4.10}$$

since the operators $\mathcal{D}$ and $\mathcal{N}$ are invertible in the corresponding function spaces. These representations lead to the formulas relating the

Cauchy data of the vector U on S,

$$[TU]^- = \mathcal{N}\,\mathcal{D}^{-1}\,[U]^- \quad \text{and} \quad [U]^- = \mathcal{D}\,\mathcal{N}^{-1}\,[TU]^- \quad \text{on} \quad S.$$

These equations are called the *Steklov-Poincaré type relations*, which are equivalent to the equality $\mathcal{D}^{-1}\,[U]^- = \mathcal{N}^{-1}\,[TU]^-$ on S.

Lemma 4.9. *Let s, p, q and S be as in Theorems* 4.4 (iii)*. If two vectors $g \in [B^{s-(1/p)}_{p,q}(S)]^6$ and $h \in [B^{s-1-(1/p)}_{p,q}(S)]^6$ are related by the equation $\mathcal{N}^{-1}\,h = \mathcal{D}^{-1}\,g$ on S, then g and h are Cauchy data on S of some radiating solution U of the homogeneous steady state oscillation equations in Ω^-, namely, $g = [U]^-$ and $h = [TU]^-$ on S.*

Proof. It follows directly from Theorems 4.4 and 4.8, and Corollaries 4.3 and 4.7. $\square$

Lemma 4.10. *Let $S \in C^\infty$, $s \in \mathbf{R}$, $1 < p < \infty$, $1 \le q \le \infty$. The Steklov-Poincaré operators*

$$\begin{aligned} \mathcal{P} := -\mathcal{N}\,\mathcal{D}^{-1} &: [H^s_p(S)]^6 \longrightarrow [H^{s-1}_p(S)]^6 \\ &: [B^s_{p,q}(S)]^6 \longrightarrow [B^{s-1}_{p,q}(S)]^6 \end{aligned} \tag{4.11}$$

are elliptic invertible pseudodifferential operators of order 1 *with positive definite principal homogeneous symbol matrix. Moreover, for $s = 1/2$ and $p = q = 2$ the operator $\mathcal{P}$ is self-adjoint.*

Proof. The ellipticity and invertibility of the operators (4.11) follow from Theorem 3.5 and Corollaries 4.3 and 4.7. The fact that for $s = 1/2$ and $p = q = 2$ the operator $\mathcal{P}$ is self-adjoint follows from the equation

$$\begin{aligned} &\big[-2^{-1}\,I_6 + \mathcal{K} + i\,\mathcal{H}\big]\,\big[\mathcal{L} + i\,\big(2^{-1}\,I_6 + \mathcal{K}\big)\big] \\ &\qquad = \big[\mathcal{L} + i\,\big(2^{-1}\,I_6 + \mathcal{K}^*\big)\big]\,\big[-2^{-1}\,I_6 + \mathcal{K}^* + i\,\mathcal{H}\big], \end{aligned}$$

which is a ready consequence of the intertwining identities (3.28).

Further, denote by $\sigma(\mathcal{A}; x; \xi)$ the principal homogeneous symbol matrix of a pseudodifferential operator $\mathcal{A}$ on the manifold S. Here $x \in S$ and $\xi \in \mathbf{R}^2 \setminus \{0\}$. It is evident that

$$\sigma(\mathcal{P}; x; \xi) = -\sigma(\mathcal{L}; x; \xi)\ \big[\sigma(-2^{-1}I + \mathcal{K}^*; x; \xi)\big]^{-1}.$$

Using again the identities (3.28) and Theorem 3.5, we derive

$$\begin{aligned}\sigma(\mathcal{P};x;\xi) &= -\left[\sigma(\mathcal{H};x;\xi)\right]^{-1}\left[\sigma(2^{-1}I+\mathcal{K}^*;x;\xi)\right]\\ &= -\left[\sigma(2^{-1}I+\mathcal{K};x;\xi)\right]\left[\sigma(\mathcal{H};x;\xi)\right]^{-1}.\end{aligned}$$

Applying the word for word arguments of the proof of Lemma 7.14 in [**33**] we can establish that the principal homogeneous symbol matrix of the pseudodifferential operator $\mathcal{P}$ is positive definite. □

4.4. *Mixed BVP.* In this subsection we consider the mixed BVP $(M)^-$, see (2.37) and (2.40), and assume that

$$\begin{gathered} f_{D} \in [H^{1/2}(S_D)]^6, \quad F_{N} \in [H^{-1/2}(S_N)]^6, \quad S \in C^\infty,\\ \partial S_D = \partial S_N \in C^\infty. \end{gathered} \tag{4.12}$$

Moreover, for simplicity, throughout this subsection we provide that $S \in C^\infty$.

Denote by $f^{(e)}$ some fixed extension of the vector-function f_D from S_D onto the whole of S preserving the function space,

$$\begin{gathered} f^{(e)} \in [H^{1/2}(S)]^6, \quad r_{S_D} f^{(e)} = f_D,\\ \| f^{(e)} \|_{[H^{1/2}(S)]^6} \leq C_0 \| f_D \|_{[H^{1/2}(S_D)]^6}, \end{gathered}$$

where C_0 is some positive constant independent of f_D (concerning the boundedness of extension operators see, e.g., [**44**, Chapter 4, Section 4.2], [**26**, Appendix A].

Evidently, every extension f of f_D onto S which preserves the function space can now be represented as

$$f = f^{(e)} + \varphi \quad \text{with} \quad \varphi \in [\tilde{H}^{1/2}(S_N)]^6.$$

In accordance with (4.10) we can look for a solution to the mixed BVP in the form

$$U(x) = W\left(\mathcal{D}^{-1}\,(f^{(e)}+\varphi)\right)(x) + i\,V\left(\mathcal{D}^{-1}\,(f^{(e)}+\varphi)\right)(x), \tag{4.13}$$

where $\varphi \in [\tilde{H}^{1/2}(S_N)]^6$ is an unknown vector function.

It is easy to check that the Dirichlet condition on S_D in (2.40) is satisfied automatically. It remains only to satisfy the Neumann condition on S_N which leads to the pseudodifferential equation for the unknown vector function φ

$$\mathcal{N}\,\mathcal{D}^{-1}\,(f^{(e)}+\varphi)=F_N \quad \text{on} \quad S_N. \tag{4.14}$$

Equation (4.14) can be rewritten as

$$r_{_{S_N}}\,\mathcal{P}\,\varphi=F^{(0)} \quad \text{on} \quad S_N, \tag{4.15}$$

where $\mathcal{P}=-\mathcal{N}\,\mathcal{D}^{-1}$ and

$$F^{(0)}:=-F_N-r_{_{S_N}}\,\mathcal{P}f^{(e)}\in[H^{-1/2}(S_N)]^6.$$

Lemma 4.11. *The operators*

(4.16)
$$r_{_{S_N}}\,\mathcal{P}\,:\,[\tilde{H}^s_p(S_N)]^6\longrightarrow[H^{s-1}_p(S_N)]^6 \quad \Big[\tilde{B}^s_{p,q}(S_N)]^6\longrightarrow[B^{s-1}_{p,q}(S_N)]^6\Big],$$

are invertible if and only if

$$\frac{1}{p}-\frac{1}{2}<s<\frac{1}{p}+\frac{1}{2}. \tag{4.17}$$

Moreover, the operators (4.16) *have the trivial null-spaces and zero indices, for all values of the parameter* $q\in[1,+\infty]$, *provided* p *and* s *satisfy the inequality* (4.17).

Proof. The mapping properties (4.16) follow from Corollaries 4.3 and 4.7. Further, from Lemmas 3.7 and 4.10 it follows that the operators (4.16) are Fredholm with zero index.

To prove the invertibility we first consider the particular case $p=2$, $s=1/2$, and $q=2$, and show that the null-space of the operator

(4.18)
$$r_{_{S_N}}\,\mathcal{P}\,:\,[\tilde{H}^{1/2}_2(S_N)]^6=[\tilde{B}^{1/2}_{2,2}(S_N)]^6\longrightarrow[H^{-1/2}_2(S_N)]^6=[B^{-1/2}_{2,2}(S_N)]^6$$

is trivial, i.e., the equation

$$r_{S_N}\,\mathcal{P}\,\varphi = 0 \quad \text{on} \quad S_N \tag{4.19}$$

admits only the trivial solution ($\varphi = 0$) in the space $[\tilde{H}_2^{1/2}(S_N)]^6$.

Indeed, let $\varphi \in [\tilde{H}_2^{1/2}(S_N)]^6$ be any solution of the homogeneous equation (4.19). It is evident that, due to Theorem 3.4, the vector

$$U(x) = W\left(\mathcal{D}^{-1}\,\varphi\right)(x) + i\,V\left(\mathcal{D}^{-1}\,\varphi\right)(x), \quad x \in \Omega^-,$$

is radiating, belongs to the space $[H^1_{2,\mathrm{loc}}(\Omega^-)]^6 = [W^1_{2,\mathrm{loc}}(\Omega^-)]^6$, and solves the homogeneous mixed BVP. Therefore, $U(x) = 0$ for $x \in \Omega^-$, due to the uniqueness Theorem 2.4. The evident equation $[U]_S^- = \varphi = 0$ on S immediately implies that the operator (4.18) is injective. In accordance with Lemma 4.10 the principal homogeneous symbol matrix of $\mathcal{P}$ is positive definite. In view of Lemma 3.7 the operator (4.18) is then Fredholm with zero index. Consequently, from the injectivity of (4.18) its invertibility follows. Now Lemma 3.7 completes the proof. □

As an immediate consequence of Lemma 4.11 and the uniqueness Theorem 2.4 we have the following existence result.

Theorem 4.12. *Let the conditions* (4.12) *be fulfilled. Then the mixed* BVP $(M)^-$ *has a unique solution in the class* $[H^1_{\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ *representable in the form of* (4.13) *where* $\varphi \in [\tilde{H}^{1/2}(S_N)]^6$ *is defined by the uniquely solvable pseudodifferential equation* (4.15).

The solution U and the vector $f^{(e)} + \varphi$ do not depend on the form of the extension operator.

In turn this theorem yields

Corollary 4.13. *Let* $4/3 < p < 4$ *and*

$$f_D \in [B^{1-1/p}_{p,p}(S_D)]^6, \qquad F_N \in [B^{-1/p}_{p,p}(S_N)]^6. \tag{4.20}$$

Then the mixed problem $(M)^-$ *has a unique radiating solution* $U \in [W^1_{p,\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ *which is representable in the form of* (4.13), *where* $f^{(e)} \in [B^{1-1/p}_{p,p}(S)]^6$ *is some fixed extension of the vector function* f_D *from* S_D *onto* S *preserving the function space and* $\varphi \in [\tilde{B}^{1-1/p}_{p,p}(S_N)]^6$ *is defined by the uniquely solvable pseudodifferential equation*

$$\begin{gathered} r_{S_N}\,\mathcal{P}\,\varphi = F^{(0)} \quad \text{on} \quad S_N \quad \text{with} \\ F^{(0)} := -F_N - r_{S_N}\,\mathcal{P}\,f^{(e)} \in [B^{-1/p}_{p,p}(S_N)]^6. \end{gathered} \tag{4.21}$$

Proof. First we note that, in accordance with Lemma 3.7, equation (4.21) is uniquely solvable for $s = 1 - 1/p$ with $4/3 < p < 4$. The restriction for p follows from the inequality (4.17). This implies that Problem $(M)^-$ is solvable in the space $[W^1_{p,\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$ with $p \in (4/3, 4)$.

Next we show the uniqueness of solution in the space $[W^1_{p,\mathrm{loc}}(\Omega^-)]^6$ for arbitrary $p \in (4/3, 4)$ (for $p = 2$ it has been proved in Theorem 2.4). Let $U \in [W^1_{p,\mathrm{loc}}(\Omega^-)]^6$ be some radiating solution of the homogeneous mixed BVP $(M)^-$. Clearly, then

$$[U]^- \in [\tilde{B}^{1-1/p}_{p,p}(S_N)]^6. \tag{4.22}$$

Due to the results mentioned in subsection 4.3 we have the representation

$$U(x) = W(\mathcal{D}^{-1}\,[U]^-)(x) + i\,V(\mathcal{D}^{-1}\,[U]^-)(x), \quad x \in \Omega^-.$$

Since U satisfies the homogeneous Neumann condition on S_N, we have $r_{S_N}\,\mathcal{P}\,[U]^- = 0$ on S_N, whence $[U]^- = 0$ on S follows due to the inclusion (4.22), Lemma 4.11, and the inequality $4/3 < p < 4$. Therefore, $[U]^- = 0$ on S and consequently $U = 0$ in Ω^-. $\square$

Further we prove the main regularity result for a solution to the mixed problem $(M)^-$.

Theorem 4.14. *Let the conditions* (4.20) *and*

(4.23)

$$4/3 < p < 4, \quad 1 < t < \infty, \quad 1 \le q \le \infty, \quad 1/t - 1/2 < s < 1/t + 1/2,$$

be fulfilled, and let $U \in [W_p^1(\Omega^-)]^6 \cap SK(\Omega^-)$ *be the unique radiating solution to the mixed problem* $(M)^-$ *represented by the formula* (4.13) *with* φ *satisfying equation* (4.21).

In addition to (4.20),

i) *if*

$$f_D \in [B_{t,t}^s(S_D)]^6, \quad F_N \in [B_{t,t}^{s-1}(S_N)]^6, \tag{4.24}$$

then

$$U \in [H_{t,\mathrm{loc}}^{s+1/t}(\Omega^-)]^6 \cap SK(\Omega^-); \tag{4.25}$$

ii) *if*

$$f_D \in [B_{t,q}^s(S_D)]^6, \quad F_N \in [B_{t,q}^{s-1}(S_N)]^6, \tag{4.26}$$

then

$$U \in [B_{t,q,\mathrm{loc}}^{s+1/t}(\Omega^-)]^6 \cap SK(\Omega^-); \tag{4.27}$$

iii) *if*

$$f_D \in [C^{\alpha'}(S_D)]^6, \quad F_N \in [B_{\infty,\infty}^{\alpha'-1}(S_N)]^6, \quad \alpha' > 0, \tag{4.28}$$

then

$$U \in \Big[\bigcap_{\beta' < \alpha''} [C^{\beta'}(\overline{\Omega^-})]^6 \Big] \bigcap SK(\Omega^-) \quad \text{with} \quad \alpha'' := \min\{\alpha', 1/2\}.$$

Proof. Applying Corollary 4.13, Lemma 4.11, the inclusions (4.20) and (4.24), respectively (4.26) along with the inequalities (4.23), we conclude from (4.24) that $\varphi \in [\tilde{B}_{t,t}^s(S_N)]^6$, respectively $\varphi \in [\tilde{B}_{t,q}^s(S_N)]^6$, since $F_0 \in [B_{t,t}^{s-1}(S_N)]^6$, respectively $F_0 \in [B_{t,q}^{s-1}(S_N)]^6$.

Note that $f^{(e)} \in [B_{t,t}^s(S)]^6$, respectively $f^{(e)} \in [B_{t,q}^s(S)]^6$, is some extension of the vector f_D onto the whole of S. Therefore, by Theorem 3.6 and the representation formula (4.13) the inclusion (4.25), respectively (4.27), follows.

To prove (iii) we use the following embeddings, see, e.g., [**44**],

$$
\begin{aligned}
C^{\alpha'}(\mathcal{S}) = B^{\alpha'}_{\infty,\infty}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,1}(\mathcal{S}) \subset B^{\alpha'-\varepsilon'}_{\infty,q}(\mathcal{S}) \\
\subset B^{\alpha'-\varepsilon'}_{t,q}(\mathcal{S}) \subset C^{\alpha'-\varepsilon'-k/t}(\mathcal{S}),
\end{aligned} \tag{4.29}
$$

where ε' is an arbitrary small positive number, $\mathcal{S} \subset \mathbf{R}^3$ is a compact k-dimensional, $k = 2, 3$, smooth manifold with smooth boundary, $1 \le q \le \infty$, $1 < t < \infty$ and $\alpha' - \varepsilon' - k/t > 0$, α', and $\alpha' - \varepsilon' - k/t$ are not integers. From (4.28) and the embeddings (4.29), the condition (4.27) follows with any $s \le \alpha' - \varepsilon'$.

Bearing in mind (4.23) and taking t sufficiently large and ε' sufficiently small, we may put $s = \alpha' - \varepsilon'$ if

$$
1/t - 1/2 < \alpha' - \varepsilon' < 1/t + 1/2, \tag{4.30}
$$

and $s \in (1/t - 1/2, 1/t + 1/2)$ if

$$
1/t + 1/2 < \alpha' - \varepsilon'. \tag{4.31}
$$

By (4.27) the solution U belongs then to $[B^{s+1/t}_{t,q,\mathrm{loc}}(\Omega^-)]^6$ with $s + 1/t = \alpha' - \varepsilon' + 1/t$ if (4.30) holds, and with $s + 1/t \in (2/t - 1/2, 2/t + 1/2)$ if (4.31) holds. In the last case we can take $s + 1/t = 2/t + 1/2 - \varepsilon'$. Therefore, we have either $U \in [B^{\alpha'-\varepsilon'+1/t}_{t,q,\mathrm{loc}}(\Omega^-)]^6$, or $U \in [B^{1/2+2/t-\varepsilon'}_{t,q,\mathrm{loc}}(\Omega^-)]^6$ in accordance with the inequalities (4.30) and (4.31). The last embedding in (4.29), with $k = 3$, yields that either $U \in [C^{\alpha'-\varepsilon'-2/t}(\overline{\Omega^-})]^6$, or $U \in [C^{1/2-\varepsilon'-1/t}(\overline{\Omega^-})]^6$ which leads to the inclusion

$$
U \in [C^{\alpha''-\varepsilon'-2/t}(\overline{\Omega^-})]^6, \tag{4.32}
$$

where $\alpha'' := \min\{\alpha', 1/2\}$. Since t is sufficiently large and ε' is sufficiently small, the embedding (4.32) completes the proof. □

4.5. *Some remarks concerning the* BVP*s in Lipschitz domains.* Here we discuss some results concerning the above considered BVPs in the W^1_2-weak setting (see the formulation of the BVPs in subsection 2.5) which remain valid for domains with Lipschitz boundaries. We recall that all the boundary conditions under consideration are understood in the usual or generalized trace sense.

Using the above established results, Remark 5.1 (see the Appendix), and the results obtained in subsections 7.1 and 7.2 in [**34**] it can be shown that if the boundary $S = \partial\Omega^-$ is Lipschitz then the uniqueness Theorem 2.4, the integral representation formulas (3.4) and (3.5), mapping properties of potentials described in Theorem 3.4, properties of the boundary integral operators (3.21), (3.23), (3.25) and (3.27), also formulas (3.28) still hold true. As a consequence we easily derive that Lemmas 4.1 and 4.5 about the invertibility of the operators $\mathcal{D}$ and $\mathcal{N}$, and the existence results concerning the Dirichlet and Neumann problems, Theorems 4.4 (i) and 4.8 (i) are valid for Lipschitz domains as well.

Concerning equation (4.15) which corresponds to the mixed boundary value problem $(M)^-$, see (2.40), we have the following existence result in the case of Lipschitz domains.

Lemma 4.15. *Let $S = \partial\Omega^-$ be a Lipschitz surface, and let $S_N \subset S$ be a proper part of S with Lipschitz boundary ∂S_N. The operator*

$$r_{S_N} \mathcal{P} = r_{S_N} \left[-\mathcal{N} \mathcal{D}^{-1} \right] : [\widetilde{H}_2^{1/2}(S_N)]^6 \to [H_2^{-1/2}(S_N)]^6 \tag{4.33}$$

is invertible.

Proof. First of all let us remark that the operator $\mathcal{P} : [H_2^{1/2}(S)]^6 \to [H_2^{-1/2}(S)]^6$ is bounded due to Lemmas 4.1 and 4.5. Consequently, the sesquilinear form $\langle r_{S_N} \mathcal{P} \varphi , \psi \rangle_{S_N}$ defined on $[\tilde{H}_2^{1/2}(S_N)]^6 \times [\tilde{H}_2^{1/2}(S_N)]^6$ is bounded. Here the symbol $\langle \cdot , \cdot \rangle_{S_N}$ again denotes the duality between the mutually adjoint spaces $[H_2^{-1/2}(S_N)]^6$ and $[\tilde{H}_2^{1/2}(S_N)]^6$.

Next, let us prove that the null-space of the operator (4.33) is trivial. Let, $\varphi \in [\tilde{H}_2^{1/2}(S_N)]^6$ and $r_{S_N} \mathcal{P} \varphi = 0$ on S_N. Then it follows that the radiating vector

$$U = W\left(\mathcal{D}^{-1} \varphi\right) + i\, V\left(\mathcal{D}^{-1} \varphi\right) \in [H^1_{2,\mathrm{loc}}(\Omega^-)]^6 \cap SK(\Omega^-)$$

solves the homogeneous mixed boundary value problem (2.40) in Ω^- (with $f_D = 0$ on $S_D = S \setminus \overline{S_N}$ and $F_N = 0$ on S_N). Due to the uniqueness Theorem 2.4 we have $U = 0$ in Ω^-. This implies that $[U]^-_S = \varphi = 0$ on S. Thus the operator (4.33) is injective.

Further, we show that the following Gårding type inequality

$$\Big\langle r_{S_N} \mathcal{P}\,\varphi\,,\,\overline{\varphi}\Big\rangle_{S_N} \geq\; a_1\,\|\varphi\|_{[\tilde{H}_2^{1/2}(S)]^6} - a_2\,\|\varphi\|_{[H_2^0(S)]^6} \tag{4.34}$$

holds for arbitrary $\varphi \in [\widetilde{H}_2^{1/2}(S_N)]^6$ with positive constants a_j ($j = 1,2$) independent of φ. To this end, we apply Green's formula (2.25) to the vectors

$$U(x) := W\left(\mathcal{D}^{-1}\,\varphi\right)(x) + i\,V\left(\mathcal{D}^{-1}\,\varphi\right)(x) \quad \text{and} \quad U'(x) := \chi(x)\,U(x),$$

where φ is an arbitrary vector from the space $[\tilde{H}_2^{1/2}(S_N)]^6$ and $\chi \in C^\infty_{comp}(\mathbf{R}^n)$ is a fixed cut-off real function with $\chi(x) = 1$ in a spatial neighborhood of the boundary S. We arrive at the equality

$$\begin{aligned}\int_{\Omega_0^-}\left[-L_2(\partial)U\cdot\chi U + E(U,\chi\overline{U})\right]\,dx &= -\left\langle [TU]^-, [\overline{U}]^-\right\rangle_{\partial\Omega^-}\\ &= \Big\langle r_{S_N}\,\mathcal{P}\,\varphi,\overline{\varphi}\Big\rangle_{S_N},\end{aligned} \tag{4.35}$$

where $\Omega_0^- = \Omega^- \cap \operatorname{supp}\chi$ is a bounded region and $L_2(\partial)$ is given by (2.14). With the help of (2.22), the trace lemma, and the inequality, cf., e.g., [**26**, Chapter 6, Theorem 6.12 and Exercise 6.4]

$$||\,U\,||^2_{[H_2^0(\Omega_0^-)]^6} \leq c_1\,||\,\varphi\,||^2_{[H_2^{-1/2}(\partial\Omega^-)]^6},$$

from (4.35) we easily derive

$$\begin{aligned}\Big\langle r_{S_N}\,\mathcal{P}\,\varphi\,,\,\overline{\varphi}\Big\rangle_{S_N} &\geq c_2\,||\,U\,||^2_{[H_2^1(\Omega_0^-)]^6} - c_3\,||\,U\,||^2_{[H_2^0(\Omega_0^-)]^6}\\ &\geq c_4\,||\,[U]^-\,||^2_{[H_2^{1/2}(\partial\Omega^-)]^6} - c_5\,||\,\varphi\,||^2_{[H_2^{-1/2}(\partial\Omega^-)]^6}\\ &\geq c_4\,||\,\varphi\,||^2_{[H_2^{1/2}(\partial\Omega^-)]^6} - c_5\,||\,\varphi\,||^2_{[H_2^0(\partial\Omega^-)]^6},\end{aligned}$$

where all the constants $c_k > 0$ are independent of φ. This proves the inequality (4.34).

Due to the well-known results based on the Lax-Milgram lemma we conclude that the operator (4.33) is Fredholm with zero index (see, e.g., [**26**, Chapter 2, Theorem 2.32]). Therefore the injectivity of the operator (4.33) implies its invertibility. □

From Lemma 4.15 it follows immediately that Theorem 4.12 holds true for Lipschitz domains as well.

Appendix

5. Fundamental matrices. Here we give an explicit expression of the radiating fundamental matrix $\Gamma(x,\sigma)$ of the differential equations (in the distributional sense) of steady state oscillations

$$L(\partial,\sigma)\,\Gamma(x,\sigma)=\delta(x)\,I_6.$$

If we follow the approach employed in Section 3 of the reference [**33**] and take into consideration the conditions (2.31), we arrive at the following formula

$$\Gamma(x,\sigma)=\sum_{j=1}^{6}\Gamma^{(j)}(x,\sigma), \tag{5.1}$$

where

$$\Gamma^{(j)}(x,\sigma)=-\frac{a_j}{4\pi\,d_1^2\,d_2}\begin{bmatrix} L^{(4)}(\partial,\sigma)\,M(\partial) & -L^{(2)}(\partial,\sigma)\,M(\partial)\\ -L^{(2)}(\partial,\sigma)\,M(\partial) & L^{(1)}(\partial,\sigma)\,M(\partial)\end{bmatrix}\frac{e^{ik_j|x|}}{|x|},$$

d_1 and d_2 are given by (2.20), k_j, $j=\overline{1,6}$, are the roots of the characteristic equation introduced in subsection 2.4 and satisfy the conditions (2.31), the numbers a_j, $j=\overline{1,6}$ are defined by the system of linear algebraic equations

$$\sum_{j=1}^{6}k_j^{2\ell}\,a_j=0,\quad \ell=\overline{0,4},\qquad \sum_{j=1}^{6}k_j^{10}\,a_j=1; \tag{5.2}$$

the 3×3 matrix differential operators $L^{(j)}(\partial,\sigma)$ are given by (2.7) and

$$\begin{aligned} M(\partial) &:= a(\partial)\,[a(\partial)-b(\partial)\,\Delta]\,I_3+\left[a(\partial)\,b(\partial)+[c(\partial)]^2\right]\,Q(\partial)\\ &\quad + c(\partial)\,[a(\partial)-b(\partial)\,\Delta]\,R(\partial)\end{aligned}$$

with

$$\begin{aligned}
a(\partial) &= [(\mu+\alpha)(\gamma+\varepsilon)-(\kappa+\nu)^2]\,\Delta\,\Delta + [\,(\mu+\alpha)\,(\mathcal{I}\,\sigma^2-4\alpha)\\
&\quad + (\gamma+\varepsilon)\,\varrho\,\sigma^2+4\alpha^2]\,\Delta + \varrho\,\sigma^2\,(\mathcal{I}\sigma^2-4\alpha),\\
b(\partial) &= -[(\mu+\alpha)(\beta+\gamma-\varepsilon)+(\lambda+\mu-\alpha)(\beta+2\gamma)-(\delta+\kappa-\nu)^2\\
&\quad - 2(\kappa+\nu)(\delta+\kappa-\nu)]\,\Delta - [(\beta+\gamma-\varepsilon)\,\varrho\,\sigma^2\\
&\quad + (\lambda+\mu-\alpha)(\mathcal{I}\,\sigma^2-4\alpha)-4\alpha^2],\\
c(\partial) &= 4[\alpha(\kappa+\nu)-\nu(\mu+\alpha)]\,\Delta - 4\nu\,\varrho\,\sigma^2.
\end{aligned}$$

This representation shows that the entries of the matrix $\Gamma^{(j)}(x,\sigma)$ and its derivatives satisfy the Sommerfeld radiation conditions at infinity,

$$\left.\begin{aligned}
\frac{\partial}{\partial|x|}\,\Gamma^{(j)}_{pq}(x,\sigma) - i\,k_j\,\Gamma^{(j)}_{pq}(x,\sigma) = \mathcal{O}\,(|x|^{-2}),\\
\frac{\partial}{\partial x_l}\,\Gamma^{(j)}_{pq}(x,\sigma) - i\,k_j\,\frac{x_l}{|x|}\,\Gamma^{(j)}_{pq}(x,\sigma) = \mathcal{O}\,(|x|^{-2})
\end{aligned}\right. \quad \text{as} \quad |x|\to+\infty.$$

These asymptotic equalities can be differentiated many times with respect to the variable x.

In view of (5.2) we have

$$\begin{aligned}
\sum_{j=1}^{6}\frac{a_j}{|x|}\,e^{i\,k_j|x|} &= \sum_{q\in\{1,3,5,7,9\}}\left\{\frac{|x|^{q-1}\,i^q}{q!}\sum_{j=1}^{6}a_j\,k_j^q\right\} - \frac{|x|^9}{10!}\\
&\quad + \sum_{q=11}^{\infty}\sum_{j=1}^{6}\frac{|x|^{q-1}\,i^q\,a_j\,k_j^q}{q!}
\end{aligned}$$

which yields that the fundamental solution (5.1) has a singularity of type $\mathcal{O}(|x|^{-1})$ in a neighborhood of the origin, since the entries of the matrix $L^{(j)}(\partial,\sigma)\,M(\partial)$, $j=\overline{1,4}$ are differential operators of order 10. One can also show that $\Gamma(-x,\sigma)=[\Gamma(x,\sigma)]^\top$.

Denote by $\Gamma_0(x)$ the fundamental matrix of the operator $L_0(\partial)$, see (2.11) and (2.12), i.e., $\Gamma_0(x)$ solves the equation (in the distributional sense) $L_0(\partial)\,\Gamma_0(x)=\delta(x)\,I_6$. The explicit form of $\Gamma_0(x)$ is given in [**33**],

$$\begin{aligned}
&\Gamma_0(x)\\
&= -\frac{1}{8\pi d_1 d_2|x|}\left\{\begin{bmatrix} d_3\,I_3 & -d_4\,I_3\\ -d_4\,I_3 & d_5\,I_3\end{bmatrix} - \frac{1}{|x|^2}\begin{bmatrix} d_6\,Q(x) & -d_7\,Q(x)\\ -d_7\,Q(x) & d_8\,Q(x)\end{bmatrix}\right\},
\end{aligned}$$

where

$$d_3 := d_2\,(\gamma+\varepsilon) + d_1\,(\beta+2\gamma), \quad d_4 := d_2\,(\kappa+\nu) + d_1\,(\delta+2\kappa),$$
$$d_5 := d_2\,(\mu+\alpha) + d_1\,(\lambda+2\mu), \quad d_6 := d_1\,(\beta+2\gamma) - d_2\,(\gamma+\varepsilon),$$
$$d_7 := d_1\,(\delta+2\kappa) - d_2\,(\kappa+\nu), \quad d_8 := d_1\,(\lambda+2\mu) - d_2\,(\mu+\alpha).$$

Clearly, the entries of the matrix $\Gamma_0(x)$ are homogeneous functions of order -1. Moreover, in a neighborhood of the origin (i.e., for small $|x|$) the following relations

$$\partial^\alpha\,[\Gamma(x,\sigma) - \Gamma_0(x)] = \mathcal{O}\,(|x|^{-|\alpha|}) \tag{5.3}$$

hold for an arbitrary multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. This implies that the $\Gamma_0(x)$ is the principal singular part of the matrix $\Gamma(x,\sigma)$, cf. [**33**]. Denote by $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{K}_0^*$ and $\mathcal{L}_0$ the boundary pseudodifferential operators generated by the single and double layer potentials constructed with the help of the matrix $\Gamma_0(x)$ and the boundary operator $T_0(\partial, n)$, cf. (3.16)–(3.19),

$$V_0(\varphi)(x) = \int_S \Gamma_0(x-y)\,\varphi(y)\,dS_y, \quad x \in \mathbf{R}^3 \setminus S,$$
$$W_0(\varphi)(x) = \int_S [\,T_0(\partial_y, n(y))\,\Gamma_0(y-x)\,]^\top\,\varphi(y)\,dS_y, \quad x \in \mathbf{R}^3 \setminus S,$$

where $T_0(\partial, n)$ is the principal homogeneous part of the stress operator $T(\partial, n)$, see (2.1) and (2.2).

Remark 5.1. With the help of relation (5.3) it can easily be shown that the operators $\mathcal{H}_0$, $\mathcal{K}_0$, $\mathcal{K}_0^*$ and $\mathcal{L}_0$ have the same principal homogeneous symbol matrices, the same mapping and Fredholm properties as the respective operators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{K}^*$ and $\mathcal{L}$. The differences $\mathcal{H}-\mathcal{H}_0$, $\mathcal{K}-\mathcal{K}_0$, $\mathcal{K}^*-\mathcal{K}_0^*$ and $\mathcal{L}-\mathcal{L}_0$ acting between the corresponding function spaces involved in (3.29)–(3.32) are compact (smoothing) operators. Note that this is also valid for complex values of the parameter σ.

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Department of Mathematics, Georgian Technical University, 77 M. Kostava, Tbilisi 0175, Republic of Georgia
E-mail address: `natrosh@hotmail.com`

Department of Mathematics, Georgian Technical University, 77 M. Kostava, Tbilisi 0175, Republic of Georgia

Department of Mathematics, Georgian Technical University, 77 M. Kostava, Tbilisi 0175, Republic of Georgia