JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 17, Number 1, Spring 2005

A NEWTON-LIKE ITERATIVE PROCESS FOR THE NUMERICAL SOLUTION OF FREDHOLM NONLINEAR INTEGRAL EQUATIONS

M.A. HERNÁNDEZ AND M.A. SALANOVA

ABSTRACT. In this paper, we give a semi-local convergence result for an iterative process of Newton-Kantorovich-type to solve nonlinear integral equations of Fredholm type and second kind. We also illustrate with several examples the technique for constructing a functional sequence that approaches solutions.

1. Introduction. Consider a nonlinear integral equation of Fredholm type and second kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)H(\phi(t))\,dt, \quad x \in [a,b], \tag{1}$$

where λ is a real number, the kernel $K(x,t)$ is a continuous function in $[a,b] \times [a,b]$, $H : \mathbf{R} \to \mathbf{R}$ is a differentiable real function, and $f(x)$ is a given continuous real function defined in $[a,b]$. When H is a linear function, there are several numerical methods to approximate the solution of this type of equations [**9, 10**]. If H is nonlinear, the more usual numerical procedures, such as Nyström, Galerkin, collocation, etc., methods [**4, 6**] have the following two principal aspects. Firstly, equation (1) is discretized and the associated nonlinear finite system is solved by applying some methods to approximate the solutions. Next, by interpolation, we obtain the approximation of the solution. Our technique is different because we apply directly an iterative method to integral equation (3). So, in this way we obtain approximations to the solution.

Key words and phrases. Fredholm integral equation, Iterative method, Newton-like method.
AMS *Mathematics Subject Classification.* 45B05, 47H15, 65J15.
Research supported in part by the University of La Rioja, Grant API-03/15, and DGES, Grant BFM2002-00222.
Received by the editors on November 8, 2003, and in revised form on January 7, 2004.

Firstly, we make a point-to-point iteration to solve this type of equation. The technique consists of writing equation (1) in the form:

$$F(\phi) = 0, \tag{2}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator defined by

$$F(\phi)(x) = \phi(x) - f(x) - \lambda \int_a^b K(x,t)H(\phi(t))\,dt, \tag{3}$$

and $X = Y = C([a,b])$ is the space of continuous functions on the interval $[a,b]$, equipped with the max-norm:

$$\|\phi\| = \max_{x\in[a,b]} |\phi(x)|, \quad \phi \in X.$$

Then we consider a point-to-point method, $\phi_{n+1} = G(\phi_n)$, to obtain approximations to a solution of equation (1). The Picard iteration and the Newton-Kantorovich method [**1, 8**] are the more used processes to solve this kind of problem.

The Picard iteration is given by the following algorithm:

$$\phi_{n+1} = G(\phi_n) = \phi_n - F(\phi_n), \quad n \geq 0,$$

where the starting point $\phi_0 \in C[a,b]$ is given. It is well known [**7**] that Picard's iteration converges if G is a contractive operator, but this condition is very restrictive and its application is then very difficult to satisfy.

The Newton-Kantorovich iteration [**3, 7**] is given by

$$\phi_0 \in C[a,b], \quad \phi_{n+1} = G(\phi_n) = \phi_n - [F'(\phi_n)]^{-1}F(\phi_n), \quad n \geq 0.$$

Note that this method requires one to calculate the operator $[F'(\phi)]^{-1}$ in each step. This can make the use of this method difficult. When the kernel of integral equation (1) is degenerated, it is simple to obtain the operator $[F'(\phi)]^{-1}$.

For instance, if $K(x,t) = \alpha(x)\beta(t)$ and we denote $F'(\phi) = F'_\phi$, it follows that:

$$F'_\phi[\psi](x) = \psi(x) - \lambda\alpha(x) \int_a^b \beta(t)H'(\phi(t))\psi(t)\,dt, \quad x \in [a,b], \quad \psi \in X.$$

If we denote $F'_\phi[\psi](x) = \theta(x)$, then $[F'_\phi]^{-1}[\theta](x) = \psi(x)$ and $\psi(x) = \theta(x) + \lambda\alpha(x)J_\psi$, where

$$J_\psi = \int_a^b \beta(t)H'(\phi(t))\psi(t)\,dt$$

is a value that can be calculated independently of ψ. This can be done in the following form: firstly, we multiply the equality

$$\psi(x) - \lambda\alpha(x)\int_a^b \beta(t)H'(\phi(t))\psi(t)\,dt = \theta(x)$$

by $\beta(x)H'(\phi(x))$; then, we integrate in the x variable the equality obtained to get

$$J_\psi - \lambda\int_a^b \beta(x)H'(\phi(x))\alpha(x)\,dx\,J_\psi = \int_a^b \beta(x)H'(\phi(x))\theta(x)\,dx.$$

If we denote

$$A = \int_a^b \beta(x)H'(\phi(x))\alpha(x)\,dx \quad \text{and} \quad B = \int_a^b \beta(x)H'(\phi(x))\theta(x)\,dx,$$

we obtain the equation $(1-\lambda A)J_\psi = B$, and therefore

$$J_\psi = \frac{B}{1-\lambda A}.$$

Thus, we can find the inverse of F'_ϕ:

$$[F'_\phi]^{-1}[\theta](x) = \theta(x) + \lambda\alpha(x)\,\frac{B}{1-\lambda A}.$$

Then, in a general situation, when $K(x,t)$ is not degenerated, we look for an approximation of this kernel by another degenerated one $\widetilde{K}(x,t)$ and we approximate

$$F'_\phi[\psi](x) = \psi(x) - \lambda\int_a^b K(x,t)H'(\phi(t))\psi(t)\,dt, \quad x\in[a,b], \quad \psi\in X,$$

by

$$A_\phi[\psi](x) = \psi(x) - \lambda \int_a^b \widetilde{K}(x,t) H'(\phi(t))\psi(t)\,dt, \qquad x \in [a,b], \quad \psi \in X, \tag{4}$$

then we can obtain A_ϕ^{-1}.

For example, in Section 3, we study this situation when we take

$$\widetilde{K}(x,t) = \sum_{j=1}^{m} \alpha_j(x)\beta_j(t)$$

by means of the Taylor formula of $K(x,t)$.

So, in this paper, we consider a Newton-type method [**2**] given by the following algorithm:

$$\phi_{n+1} = G(\phi_n) = \phi_n - A(\phi_n)^{-1} F(\phi_n), \quad n \geq 0, \tag{5}$$

for $\phi_0 \in C[a,b]$ given, where $A(\phi)$ is an approximation of $F'(\phi)$ when the kernel $K(x,t)$ is approximated by a degenerated kernel $\widetilde{K}(x,t)$. These types of iterative processes have been studied, for other nonlinear problems, by other authors [**3, 5**], but in our study we establish a semi-local convergence result under weak conditions for the operator H.

We finish with several numerical examples where algorithm (5) is applied.

2. A semi-local convergence result. To study the semi-local convergence of iterative process (5), we make sure that the sequence $\{\phi_n\}$ is well-defined and converges to a solution of equation (2). Firstly, we consider a starting point $\phi_0 \in C[a,b]$ and we want to know if $A_{\phi_0}^{-1}$ exists. To do this, we use Banach's lemma about the inversion of operators, to have the following result:

Lemma 2.1. *Let $\phi_0 \in C[a,b]$, and we assume that*

$$|\lambda| < \frac{1}{L\varepsilon}. \tag{I}$$

Then, $A_{\phi_0}^{-1}$ *exists and*

$$\|A_{\phi_0}^{-1}\| \le \frac{1}{1-|\lambda|L\varepsilon} \equiv \delta,$$

where $L = \max_{x\in[a,b]} \int_a^b |\widetilde{K}(x,t)|\,dt$ *and* $\varepsilon \equiv \|H'(\phi_0)\|$.

Proof. Taking into account that

$$(I - A_{\phi_0})[\psi](x) = \lambda \int_a^b \widetilde{K}(x,t)H'(\phi_0(t))\psi(t)\,dt,$$

we have that $\|I - A_{\phi_0}\| \le |\lambda|L\varepsilon < 1$, and the result holds by Banach's lemma. □

Now, in the following lemma, we give conditions for the existence of A_ϕ^{-1}, for each $\phi \in \overline{B(\phi_0,R)} = \{\phi \in X; \|\phi - \phi_0\| \le R\}$, where R is a fixed positive real number.

Lemma 2.2. *Let* $\phi_0 \in C[a,b]$ *be such that* $A_{\phi_0}^{-1}$ *exists and* $\|A_{\phi_0}^{-1}\| \le \delta$. *Assume*:

(II) $\|H'(\phi) - H'(\psi)\| \le \omega(\|\phi-\psi\|)$, *for* $\phi,\psi \in \overline{B(\phi_0,R)}$, *where* ω *is a nondecreasing positive real function,* $\omega : \mathbf{R}_+ \to \mathbf{R}_+$, *and*

(III) $h(R) \equiv \delta|\lambda|L\omega(R) < 1$.

Then, for each $\phi \in \overline{B(\phi_0,R)}$, A_ϕ^{-1} *exists and*

$$\|A_\phi^{-1}\| \le \frac{1}{1-h(R)}.$$

Proof. By using Banach's Lemma, we obtain that $(A_\phi)^{-1}$ exists for each $\phi \in \overline{B(\phi_0,R)}$. From the hypotheses, we have

$$\|(I - A_{\phi_0}^{-1}A_\phi)\psi\| \le \|A_{\phi_0}^{-1}\|\,\|(A_{\phi_0} - A_\phi)\psi\| \le \delta|\lambda|L\|H'(\phi) - H'(\phi_0)\|\,\|\psi\|.$$

Then,

$$\|I - A_{\phi_0}^{-1}A_\phi\| \le \delta|\lambda|L\omega(R) = h(R) < 1$$

and the result holds. □

Besides, if we prove that $\phi_1 \in \overline{B(\phi_0, R)}$, we deduce that ϕ_2 is well defined by Lemma 2.2. By means of a recursive procedure, if we prove that $\phi_n \in \overline{B(\phi_0, R)}$, we deduce that the sequence $\{\phi_n\}$, given by (5), is well defined.

Next, we give a semi-local convergence result for this sequence. For that, we take into account the following expression

$$\begin{aligned} F(\phi_n)(x) &= F(\phi_n)(x) - A_{\phi_{n-1}}(\phi_n - \phi_{n-1})(x) - F(\phi_{n-1})(x) \\ &= \int_{\phi_{n-1}}^{\phi_n} [F'_\phi - A_{\phi_{n-1}}](x)\, d\phi \\ &= \int_0^1 [F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - A_{\phi_{n-1}}](x)\, ds. \end{aligned} \tag{6}$$

In the following lemma we give a bound for $\|\phi_{n+1} - \phi_n\|$.

Lemma 2.3. *Let $\phi_0 \in C[a,b]$ be such that $A_{\phi_0}^{-1}$ exists and $\|A_{\phi_0}^{-1}\| \leq \delta$. If $\phi_n \in \overline{B(\phi_0, R)}$, for each $n \in \mathbf{N}$, assume the conditions of Lemma 2.2 and*

(IV) $\|H'(\phi)\| \leq \tilde{\omega}(\|\phi\|)$, $\quad \phi \in \overline{B(\phi_0, R)}$, *where $\tilde{\omega}$ is a nondecreasing positive real function, $\tilde{\omega} : \mathbf{R}_+ \to \mathbf{R}_+$. Then,*

$$\|\phi_{n+1} - \phi_n\| \leq \frac{1}{1 - h(R)}\, |\lambda| g(R) \|\phi_n - \phi_{n-1}\|, \quad n \geq 1,$$

where $g(R) = N\omega(2R) + M\tilde{\omega}(\|\phi_0\| + R)$, $N = \max_{x \in [a,b]} \int_a^b |K(x,t)|\, dt$ and $M = \max_{x \in [a,b]} \int_a^b |R(x,t)|\, dt$.

Proof. Firstly, we need a bound for $\|F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - A_{\phi_{n-1}}\|$. From the expression

$$\begin{aligned} &[F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - A_{\phi_{n-1}}]\psi(x) \\ &\quad = -\lambda \int_a^b \big[K(x,t)H'(\phi_{n-1}+s(\phi_n-\phi_{n-1})) - \widetilde{K}(x,t)H'(\phi_{n-1})\big]\psi(t)\, dt \\ &\quad = -\lambda \int_a^b \big[K(x,t)\left(H'(\phi_{n-1} + s(\phi_n - \phi_{n-1})) - H'(\phi_{n-1})\right) \\ &\qquad\qquad - R(x,t)H'(\phi_{n-1})\big]\psi(t)\, dt, \end{aligned}$$

it follows that

$$\begin{aligned}
&\|F'_{\phi_{n-1}+s(\phi_n-\phi_{n-1})} - A_{\phi_{n-1}}\| \\
&\quad \le |\lambda|\left[N\|H'(\phi_{n-1}+s(\phi_n-\phi_{n-1})) - H'(\phi_{n-1})\| + M\|H'(\phi_{n-1})\|\right] \\
&\quad \le |\lambda|\left[N\omega(s\|\phi_n-\phi_{n-1}\|) + M\tilde{\omega}(\|\phi_{n-1}\|)\right].
\end{aligned}$$

Thus, we deduce, by (6), that

$$\begin{aligned}
\|F(\phi_n)\| &\le |\lambda|\left[N\omega(2R) + M\tilde{\omega}(\|\phi_0\|+R)\right]\|\phi_n-\phi_{n-1}\| \\
&= |\lambda| g(R)\|\phi_n-\phi_{n-1})\|.
\end{aligned}$$

So we have

$$\|F(\phi_n)\| \le |\lambda| g(R)\|\phi_n-\phi_{n-1}\|, \tag{7}$$

and the result holds from Lemma 2.2. □

Finally, we give the main result of semi-local convergence for the sequence $\{\phi_n\}$. For that, we denote

$$\Delta(R) = \frac{1}{1-h(R)}\,|\lambda| g(R).$$

Then, if we consider $|\lambda| < 1/L\varepsilon$, by Lemma 2.1, $A_{\phi_0}^{-1}$ exists and $\|A_{\phi_0}^{-1}\| \le \delta$. Now, we denote $\|A_{\phi_0}^{-1}F(\phi_0)\| \le \eta$, and we consider the following auxiliary scalar equation

$$t(1-\Delta(t)) - \eta = 0. \tag{8}$$

Theorem 2.4. *Assume the previous conditions* (I), (II), (III) *and* (IV)*. Let R be the minimum positive solution of equation* (8) *and we suppose that $\Delta \equiv \Delta(R) < 1$. Then, the sequence $\{\phi_n\}$, given by* (5)*, converges to a solution ϕ^* of equation* (2) *and $\phi_n, \phi^* \in \overline{B(\phi_0,R)}$. Moreover, we have the following error bounds:*

$$\|\phi^* - \phi_n\| \le \frac{\Delta^n}{1-\Delta}\,\eta.$$

Proof. Notice that $\|\phi_1 - \phi_0\| \leq \|A_{\phi_0}^{-1} F(\phi_0)\| \leq \eta$, and therefore it follows that $\phi_1 \in \overline{B(\phi_0, R)}$, since $\eta < R$. Then, by applying Lemma 2.2, we obtain that $A_{\phi_1}^{-1}$ exists and ϕ_2 is well defined. Now, from Lemma 2.3, we have that

$$\|\phi_2 - \phi_1\| \leq \frac{1}{1 - h(R)} |\lambda| g(R) \|\phi_1 - \phi_0\| \leq \Delta\eta.$$

Besides,

$$\|\phi_2 - \phi_0\| \leq \|\phi_2 - \phi_1\| + \|\phi_1 - \phi_0\| \leq (1 + \Delta)\eta < R$$

and $\phi_2 \in \overline{B(\phi_0, R)}$.

Thus, by an inductive procedure and taking into account Lemmas 2.2 and 2.3, we obtain that, for all $n \in \mathbf{N}$, ϕ_n is well defined, $\phi_n \in \overline{B(\phi_0, R)}$ and $\|\phi_n - \phi_{n-1}\| \leq \Delta^{n-1}\eta$, for $n \geq 0$.

We only need to prove that $\{\phi_n\}$ is a Cauchy sequence. Taking into account the previous lemmas, we have

$$\begin{aligned}
\|\phi_{n+m} - \phi_n\| &\leq \|\phi_{n+m} - \phi_{n+m-1}\| + \|\phi_{n+m-1} - \phi_{n+m-2}\| + \cdots \\
&\quad + \|\phi_n - \phi_{n-1}\| \\
&\leq \left[\Delta^{n+m-1} + \Delta^{n+m-2} + \cdots + \Delta^n\right] \|\phi_1 - \phi_0\| \\
&\leq \Delta^n \sum_{k=0}^{m-1} \Delta^k \|\phi_1 - \phi_0\| \leq \frac{\Delta^n}{1 - \Delta}\, \eta.
\end{aligned}$$

But this quantity goes to zero when $n \to \infty$. If $\phi^* = \lim_{n\to\infty} \phi_n$, then, by letting $m \to \infty$ and $n = 0$ in the previous inequality, we have

$$\|\phi^* - \phi_0\| \leq \frac{1}{1 - \Delta}\, \eta = R.$$

We only need to prove that $F(\phi^*) = 0$. From (7), we have

$$\|F(\phi_n)\| \leq |\lambda| g(R) \|\phi_n - \phi_{n-1}\| \leq (1 - h(R))\eta\Delta^n.$$

Then, as $\Delta < 1$, $F(\phi^*) = 0$ by the continuity of the operator F. $\square$

Remark. Notice that condition (IV) can be obtained from (II) by taking into account that, if $\phi \in B(\phi_0, R)$,

$$\|H'(\phi)\| \leq \|H'(\phi_0)\| + \omega(\|\phi - \phi_0\|) \leq \varepsilon + \omega(R).$$

We have demanded both conditions because, in practice, from condition (IV), we can improve the error bounds, as we can see in Example 1.

3. A practical construction of A_ϕ. In this section we consider a particular case of the construction of the operator A_ϕ. Then, by assuming that $K(x,t)$ has $(m+1)$ partial derivatives in the second variable t, we consider the Taylor formula of $K(x,t)$ in the variable t, we obtain

$$K(x,t) = \sum_{i=1}^{m} \frac{1}{i!} \frac{\partial^i K}{\partial t^i}(x,0)t^n + R_m(x,t) = \sum_{i=1}^{m} \alpha_i(x)\beta_i(t) + R_m(x,t),$$

then we take

$$\widetilde{K}(x,t) = \sum_{i=1}^{m} \alpha_i(x)\beta_i(t) = \sum_{i=1}^{m} \frac{1}{i!} \frac{\partial^i K}{\partial t^i}(x,0)t^n.$$

Our next aim is to obtain the inverse operator $[A_\phi]^{-1}$ for the last $\widetilde{K}(x,t)$. For this, if we denote $I_j = \int_a^b \beta_j(t)H'(\phi(t))\psi(t)\,dt$, we have

$$A_\phi[\psi](x) = \theta(x) = \psi(x) - \lambda \sum_{j=1}^{n} \alpha_j(x) I_j \tag{9}$$

and

$$[A_\phi]^{-1}[\theta](x) = \psi(x) = \theta(x) + \lambda \sum_{j=1}^{m} \alpha_j(x) I_j,$$

when the integrals I_j can be calculated independently of ψ. This can be done in the following form: we multiply equality (9) by $\beta_i(x)H'(\phi(x))$, then we integrate in the x variable the equality obtained, and we have

$$I_i - \lambda \sum_{j=1}^{n} \left(\int_a^b \beta_i(x)H'(\phi(x))\alpha_j(x)\,dx \right) I_j = \int_a^b \beta_i(x)H'(\phi(x))\theta(x)\,dx.$$

Now, if we denote

$$a_{ij}(\phi) = \int_a^b \beta_i(x) H'(\phi(x)) \alpha_j(x)\, dx$$

and

$$b_i(\phi) = \int_a^b \beta_i(x) H'(\phi(x)) \theta(x)\, dx,$$

we have the following linear system of equations

$$I_i - \lambda \sum_{j=1}^{m} a_{ij}(\phi) I_j = b_i(\phi), \quad i = 1, \ldots, m. \tag{10}$$

This system has a unique solution if

$$(-\lambda)^m \begin{pmatrix} a_{11}(\phi)-(1/\lambda) & a_{12}(\phi) & a_{13}(\phi) & \ldots & a_{1m}(\phi) \\ a_{21}(\phi) & a_{22}(\phi)-(1/\lambda) & a_{23}(\phi) & \ldots & a_{2m}(\phi) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1}(\phi) & a_{m2}(\phi) & a_{m3}(\phi) & \ldots & a_{mm}(\phi)-(1/\lambda) \end{pmatrix} \neq 0.$$

Then, we assume $1/\lambda$ is not an eigenvalue of the matrix $(a_{ij}(\phi))$. Thus, if $I_1, I_2, \ldots, I_m$ is the solution of system (10), we can define

$$[A_\phi]^{-1}[\theta](x) = \theta(x) + \lambda \sum_{j=1}^{m} \alpha_j(x) I_j,$$

and we obtain iteration (5), whose convergence was established in Theorem 2.4. Notice that the last condition required for λ has to be satisfied in each iteration ϕ_n.

Notice that, if the differentiability conditions are satisfied by the kernel in the first argument, we can make a similar procedure.

To illustrate the above theoretical results, we provide some examples.

Example 1. We consider the following nonlinear integral equation

$$\phi(x) = \sin(\pi x) + \frac{1}{20} \int_0^1 e^{xt} \sin(\phi(t))\, dt, \quad x \in [0,1]. \tag{11}$$

Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, where the max-norm is used and let $F : X \to X$ be the nonlinear operator given by

$$F(\phi)(x) = \phi(x) - \sin(\pi x) - \frac{1}{20}\int_0^1 e^{xt}\sin(\phi(t))\,dt, \quad x \in [0,1]. \tag{12}$$

By differentiating (12), we have:

$$F'_\phi[\psi](x) = \psi(x) - \frac{1}{20}\int_0^1 e^{xt}\cos(\phi(t))\psi(t)\,dt, \quad x \in [0,1], \tag{13}$$

and we define

$$A_\phi[\psi](x) = \psi(x) - \frac{1}{20}\int_0^1 \widetilde{K}(x,t)\cos(\phi(t))\psi(t)\,dt, \quad x \in [0,1], \tag{14}$$

where we take as $\widetilde{K}(x,t)$ the Taylor's formula for the second variable when $t = 0$, and $R(x,t)$ is Taylor's rest. Namely,

$$\widetilde{K}(x,t) = \sum_{i=1}^{3}\alpha_i(x)\beta_i(t) = 1 + xt + x^2\,\frac{t^2}{2} \tag{15}$$

and

$$R(x,t) = e^{xt} - \widetilde{K}(x,t) = x^3\,\frac{e^{x\theta}}{6}\,t^3, \quad \theta \in (0,t).$$

Firstly we check the convergence conditions of Theorem 2.4. For this, we calculate the constants of Section 2,

$$\lambda = \frac{1}{20}, \quad N = e - 1, \quad L = \frac{5}{3}, \quad M = \frac{e}{24}.$$

We take the starting-point $\phi_0(x) = \sin(\pi x)$. Then, we obtain, from (14),

$$A_{\phi_0}[\psi](x) = \psi(x) - \frac{1}{20}\int_0^1 \left(1 + xt + x^2\,\frac{t^2}{2}\right)\cos(\sin(\pi t))\psi(t)\,dt,$$

and $H'(\phi_0) = \cos(\sin(\pi x))$. Therefore $\varepsilon = \|H'(\phi_0)\| = 1$ and the hypotheses of Lemma 2.1 are satisfied, since $|\lambda| = 1/20 < 1/L\varepsilon = 3/5$.

Then, the eigenvalues of the matrix (a_{ij}) are: $\rho_1 = 0.0928853$, $\rho_2 = 8.32667 \times 10^{-17}$ and $\rho_3 = 2.30925 \times 10^{-18}$, and $\lambda \neq 1/\rho_i$, for $i = 1, 2, 3$. So, $[A_{\phi_0}]^{-1}$ exists and $\delta = 60/55$. The function ω is the identity and $\tilde{\omega}$ is the constant function 1. The functions $h(R)$ and $g(R)$ are

$$h(R) = \frac{1}{11}\, R \quad \text{and} \quad g(R) = (e-1)2x + \frac{e}{24}\,(x+1).$$

Hence

$$\Delta(R) = \frac{(e-1)2x + (e/24)(x+1)}{1 - (1/11)R}.$$

Next, we have to solve auxiliary equation (8). To do this, we need a bound for $\|F(\phi_0)\|$. By using a Gauss-Legendre numerical integration formula with two nodes, we obtain $\|F(\phi_0)\| \leq 0.0496412$. Then $\eta = 0.0541541$ and the solutions of equation (8) are $r_1 = 0.111454$ and $r_2 = 0.133458$. From Theorem 2.4, $R = r_1 = 0.111454$. For this value, $h(R) = 0.0101322 < 1$ and $\Delta = 0.514115 < 1$, and the hypotheses of Theorem 2.4 hold. So, we can deduce that the sequence given by (5) converges to a solution of the equation $F(\phi) = 0$, and the iterates and the solution are in $\overline{B(\phi_0, 0.111454)}$.

To finish this example, we are going to deal with the computational aspect of the method (5) to solve (12). To calculate the iterations $\phi_n(x)$, with starting-point $\phi_0(x)$, we proceed in the following way:

1. We compute $A_{\phi_0}^{-1}$. For this, $\alpha_1(x) = 1$, $\alpha_2(x) = x$, $\alpha_3(x) = x^2$ and $\beta_1(t) = 1$, $\beta_2(t) = t$, $\beta_3 = t^2/2$, by (15). Linear system (10) is then given by:

$$\left\{ \begin{array}{rcrcrcr} 0.96174 I_1 & - & 0.0191299 I_2 & - & 0.0133386 I_3 & = & -0.0292223 \\ -0.0191299 I_1 & + & 0.986661 I_2 & - & 0.0104429 I_3 & = & -0.0161642 \\ -0.00666928 I_1 & - & 0.00522143 I_2 & + & 0.995677 I_3 & = & -0.00588616. \end{array} \right.$$

whose solutions are $I_1 = -0.03081, I_2 = -0.0170458$ and $I_3 = -0.00620747$.

2. We define

$$\begin{aligned} \phi_1(x) &= \phi_0(x) - F(\phi_0)(x) - \lambda \sum_{j=1}^{3} \alpha_j(x) I_j \\ &= \sin(\pi x) + 0.0144483(e^{0.211325x} + e^{0.788675x}) \\ &\quad + 0.05(0.03081 + 0.0170458x + 0.00620747x^2) \end{aligned}$$

By using the Mathematica program, we obtain the following approximations:

$$\begin{aligned}
\phi_0(x) &= \sin(\pi x),\\
\phi_1(x) &= \sin(\pi x) + 0.0144483(e^{0.211325x} + e^{0.788675x})\\
&\quad + 0.0015405 + 0.00085229x + 0.000310374x^2,\\
\phi_2(x) &= \sin(\pi x) + 0.0151317e^{0.211325x} + 0.0153788e^{0.788675x}\\
&\quad + 4.28099 \times 10^{-6} + 2.53791 \times 10^{-6}x + 9.63343 \times 10^{-7}x^2,\\
\phi_3(x) &= \sin(\pi x) + 0.0151334e^{0.211325x} + 0.0153816e^{0.788675x}\\
&\quad + 9.96116 \times 10^{-9} + 5.79993 \times 10^{-9}x + 2.19471 \times 10^{-9}x^2,\\
\phi_4(x) &= \sin(\pi x) + 0.0151334e^{0.211325x} + 0.0153816e^{0.788675x}\\
&\quad + 2.07491 \times 10^{-11} + 1.20498 \times 10^{-11}x + 4.56067 \times 10^{-12}x^2,\\
\phi_5(x) &= \sin(\pi x) + 0.0151334e^{0.211325x} + 0.0153816e^{0.788675x}\\
&\quad + 4.11672 \times 10^{-14} + 2.38836 \times 10^{-14}x + 9.04246 \times 10^{-15}x^2.
\end{aligned}$$

We can consider the last iteration $\phi_5(x)$ as a good approximation of the solution taking into account the difference

$$\begin{aligned}
\phi_5(x) - \phi_4(x) = {} & 8.55373 \times 10^{-12}e^{0.211325x} + 1.30048 \times 10^{-12}e^{0.788675x}\\
& - 2.0708 \times 10^{-11} - 1.2026 \times 10^{-11}x\\
& + 4.55163 \times 10^{-12}x^2.
\end{aligned}$$

Finally, a graphic of this error and the approximate solution are given in Figures 1 and 2, respectively.

If we take more terms in Taylor's formula to construct $\widetilde{K}(x,t)$, the obtained accuracy does not compensate the operational cost.

Example 2. The aim of this example is to show a situation where the error made can be exactly computed. Moreover, the Picard iteration does not converge and method (5) converges to a solution of $F(\phi) = 0$.

We consider the following nonlinear integral equation

$$\phi(x) = x^3 + \frac{1}{7}\int_0^1 e^{-x-t}\phi(t)^3\,dt, \quad x \in [0,1]. \tag{16}$$

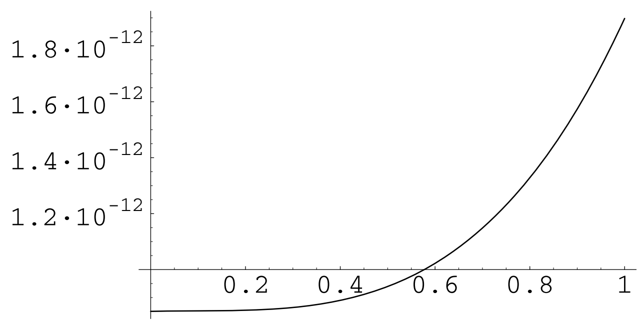

FIGURE 1. $\phi_5(x) - \phi_4(x)$.

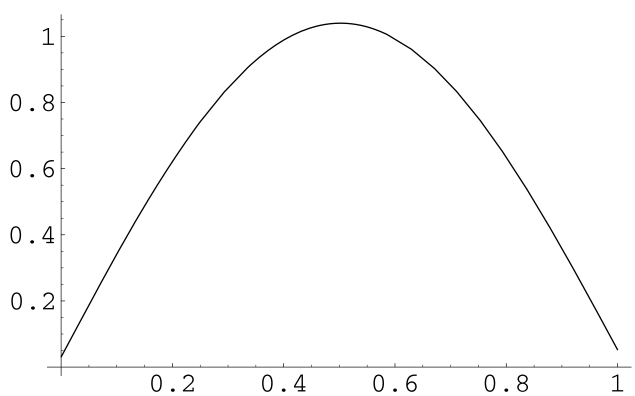

FIGURE 2. $\phi_5(x)$.

Let $X = C[0,1]$ be the space of continuous functions defined on the interval $[0,1]$, where the max-norm is used and $F : X \to X$ is the nonlinear operator given by

$$F(\phi)(x) = \phi(x) - x^3 - \frac{1}{7}\, e^{-x} \int_0^1 e^{-t} \phi(t)^3 \, dt, \quad x \in [0,1]. \tag{17}$$

As the kernel of integral equation (17) is degenerated, we can find the exact solutions of this equation:

$$\begin{aligned} &u_1(x) = 5.15105 e^{-x} + x^3, \quad u_2(x) = 0.00584052 e^{-x} + x^3, \\ &u_3(x) = -5.47631 e^{-x} + x^3. \end{aligned} \tag{18}$$

We take the starting-point $\phi_0(x) = x^3$ and we prove that the hypotheses of Theorem 2.4 hold. So, we can deduce that the sequence given by (5) converges to a solution of the equation $F(\phi) = 0$. Moreover, the iterates and the solution are in $\overline{B(\phi_0, 0.0102037)}$.

In a similar way, we compute the iterations of method (5) starting in different points. Then, by using the Mathematica program, we obtain the following approximations:

- For $\phi_0 = x^3$, with a precision of 10^{-16}:

$$\begin{aligned} \phi_0(x) &= x^3, \\ \phi_1(x) &= x^3 + 0.00585398 e^{-x}, \\ \phi_2(x) &= x^3 + 0.00584049 e^{-x}, \\ \phi_3(x) &= x^3 + 0.00584052 e^{-x}, \\ \phi_4(x) &= x^3 + 0.00584052 e^{-x}. \end{aligned}$$

We observe the sequence $\{\phi_n\}$ converges to the solution $u_2(x)$. In Table 1, we show the error $\|u_2(x) - \phi_n(x)\|$ of these approximations.

If we use Picard's iteration starting at the same initial point, we obtain the following sequences:

$$\begin{aligned} \phi_0(x) &= x^3, \\ \phi_1(x) &= x^3 + 0.0057763 e^{-x}, \\ \phi_2(x) &= x^3 + 0.00583981 e^{-x}, \\ \phi_3(x) &= x^3 + 0.00584051 e^{-x}, \\ \phi_4(x) &= x^3 + 0.00584052 e^{-x}. \end{aligned}$$

whose errors can be seen in Table 2.

TABLE 1. The error of sequence (5).

$\Vert u_2(x) - \phi_0(x)\Vert$	$=$	0.00584052
$\Vert u_2(x) - \phi_1(x)\Vert$	$=$	0.0000134541
$\Vert u_2(x) - \phi_2(x)\Vert$	$=$	3.21471×10^{-8}
$\Vert u_2(x) - \phi_3(x)\Vert$	$=$	7.68056×10^{-11}
$\Vert u_2(x) - \phi_4(x)\Vert$	$=$	1.83503×10^{-13}

TABLE 2. The error for the Picard iteration.

$\Vert u_2(x) - \phi_0(x)\Vert$	$=$	0.00584052
$\Vert u_2(x) - \phi_1(x)\Vert$	$=$	0.0000642242
$\Vert u_2(x) - \phi_2(x)\Vert$	$=$	7.10535×10^{-7}
$\Vert u_2(x) - \phi_3(x)\Vert$	$=$	7.86144×10^{-9}
$\Vert u_2(x) - \phi_4(x)\Vert$	$=$	8.69798×10^{-11}

• If we start at $\phi_0 = x - 6$, we obtain the following sequence that converges to $u_3(x)$

$$\begin{aligned}
\phi_0(x) &= x - 6, \\
\phi_1(x) &= x^3 - 6.88453e^{-x}, \\
\phi_2(x) &= x^3 - 5.83435e^{-x}, \\
\phi_3(x) &= x^3 - 5.51213e^{-x}, \\
\phi_4(x) &= x^3 - 5.47723e^{-x}, \\
\phi_5(x) &= x^3 - 5.47633e^{-x}, \\
\phi_6(x) &= x^3 - 5.47631e^{-x}.
\end{aligned}$$

The error of the last iteration is $\Vert 2.4277 \times 10^{-7} e^{-x}\Vert = 2.4277 \times 10^{-7}$.

The Picard iteration does not converge in this case.

REFERENCES

1. J. Appell, E. DePascale and PP. Zabrejko, *On the application of Newton-Kantorovich method to nonlinear integral equations of Uryson type*, Numer. Funct. Anal. Optim. **12** (1991), 271–283.

2. I. K. Argyros, *Newton-like methods under mild differentiability conditions with error analysis*, Bull. Austral. Math. Soc. **37** (1998), 131–147.

3. I. K. Argyros and F. Szidarovszky, *Theory and applications of iteration methods*, CRC-Press Inc., Boca Raton, Florida, USA, 1993.

4. K.E. Atkinson, *A survey of numerical methods for solving nonlinear integral equations*, J. Integral Equations Appl. **4** (1992), 15–46.

5. M.A. Hernández and M. A. Salanova, *A family of Newton type iterative processes*, Internat. J. Computer Math. **51** (1994), 205–214.

6. Abdul J. Jerri, *Introduction to integral equations with applications*, Wiley-Interscience, New York, 1999.

7. L.V. Kantorovich and G.P. Akilov, *Functional analysis*, Pergamon Press, New York, 1982.

8. G. Miel, *Numerical solution on parallel processors of two-point boundary problems of astrodynamics*, Math. Concepts Methods Sci. Engrg. **42**, (1990), 131–182.

9. D. Porter and D.S.G. Stirling, *Integral equations, a practical treatment, from spectral theory to applications*, Cambridge Univ. Press, Cambridge, 1990.

10. A.M. Wazwaz, *A first course in integral equations*, World Scientific, Singapore, 1997.

University of La Rioja, Department of Mathematics and Computation, C/ Luis de Ulloa s/n. 26006 Logroño, Spain
E-mail address: `mahernan@dmc.unirioja.es`

University of La Rioja, Department of Mathematics and Computation, C/ Luis de Ulloa s/n. 26006 Logroño, Spain
E-mail address: `masalano@dmc.unirioja.es`