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ON NECESSARY AND SUFFICIENT CONDITIONS FOR EXPONENTIAL STABILITY IN LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. Suppose K is a continuous matrix-valued function such that

$$\int_0^\infty t^2|K(t)|\,dt < \infty,$$

and let X be the matrix-valued solution of the resolvent problem

$$X'(t) = AX(t) + \int_0^t K(t-s)X(s)\,ds, \quad t > 0; \quad X(0) = I.$$

If the solution X is in $L^1(0,\infty)$, then the following are equivalent:

(a) There are $\beta > 0$, $c > 0$ such that

$$|K(t)| \le ce^{-\beta t}, \quad t \ge 0.$$

(b) There are $\alpha > 0$, $c_0 > 0$ such that

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad |X''(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.$$

1. Introduction. This paper is a study of the exponential decay to zero of the solution of the resolvent equation

$$X'(t) = AX(t) + \int_0^t K(t-s)X(s)\,ds, \quad t > 0; \tag{1.1a}$$

$$X(0) = I. \tag{1.1b}$$

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Here the solution X is a matrix-valued function on $[0,\infty)$, A is a real matrix, and K is a continuous and integrable matrix-valued function on $[0,\infty)$. The significance of (1.1) is that the vector-valued solution of the Volterra integro-differential equation

$$y'(t) = Ay(t) + \int_0^t K(t-s)y(s)\,ds + f(t), \quad t > 0, \tag{1.2a}$$

$$y(0) = y_0, \tag{1.2b}$$

can be represented in terms of the variation of parameters formula

$$y(t) = X(t)y_0 + \int_0^t X(t-s)f(s)\,ds.$$

For this reason X is referred to as the *resolvent*, or *fundamental solution* of (1.2).

The question arises as to whether the integrability of the solution of (1.1) necessarily implies its exponential decay. This is equivalent to a problem posed in the survey article of Corduneanu and Lakshmikantham [**8**]. The papers of Murakami [**14, 15**], which motivate this work, throw considerable light on this question. Murakami shows that the exponential decay of all solutions of (1.1) is equivalent to an exponential decay property of the kernel K under the restriction that none of the elements K_{ij} change sign on $[0,\infty)$. The exponential decay restriction is

$$\int_0^\infty |K(s)|e^{\gamma s}\,ds < \infty \quad \text{for some } \gamma > 0. \tag{1.3}$$

It should be mentioned that a special case of a result of this type can be proved by slightly extending a result of Burton [**6**, Theorem 1.3.7]. Moreover, the possibility of results like this was anticipated in a remark of MacCamy and Wong [**12**].

In this paper, we remove the sign restriction on the kernel K. We show, whenever the fundamental solution X of (1.1) is integrable, that the following are equivalent:

(a) There is $\beta > 0$, $c > 0$ such that

$$|K(t)| \le ce^{-\beta t}, \quad t \ge 0;$$

(b) There is $\alpha > 0$, $c_0 > 0$ such that

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad |X''(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.$$

In order to prove this equivalence, we require that K be a continuous, integrable function which obeys the integrability condition

$$\int_0^\infty t^2 |K(t)|\, dt < \infty.$$

Murakami's theorem has applications beyond this characterization of exponential stability. His result and methods can be applied in order to determine necessary and sufficient conditions for almost sure and moment exponential asymptotic stability of solutions of stochastically perturbed versions of (1.1a). Results for linear equations are established in Appleby and Freeman [**3**]; nonlinear equations are considered in Appleby [**1**].

An application of his results in the study of exponential stability in the theory of linear viscoelasticity is covered in Appleby, Fabrizio, Lazzari and Reynolds [**2**]. In this setting, physical considerations render a relaxation of the sign condition on the kernel highly desirable.

We also stress that results which identify particular decay properties of the resolvents with the decay properties of the kernel, exist in the literature. Instances of this type of analysis are provided in work of Burton, Huang and Mahfoud [**7**] (wherein the existence of moments is studied), Appleby and Reynolds [**4, 5**] (which are concerned with the existence of subexponential solutions) and Fabrizio and Polidoro [**9**] (in which the polynomial stability in linear models of viscoelasticity is examined). A classical result in this spirit, but one in which converse results are not considered, is given in Shea and Wainger [**16**]. It is proven that whenever the kernel of a linear Volterra integro-differential equation lies in a particular weighted space, and the resolvent is integrable, then the resolvent lies in the same weighted space as the kernel. These results may be seen in Gripenberg, Londen, and Staffans [**10**, Theorems 4.4.13, 4.4.16].

2. Mathematical preliminaries. We introduce some standard notation. We denote by $\mathbf{R}$ the real number set, and $\alpha \wedge \beta$ denotes the minimum of the real numbers α and β. Let $M_n(\mathbf{R})$ be the set of real-valued $n \times n$ matrices, with I the identity matrix and O the zero matrix. $|\cdot|$ denotes the absolute value of real and complex numbers. It also denotes a norm on $M_n(\mathbf{R})$ with the property that $|AB| \le |A||B|$. Thus for example we could take $|A| = \sum_{1\le i,j\le n} |A_{ij}|$. If J is an interval in $\mathbf{R}$ and V a finite-dimensional normed space, we denote by $C(J,V)$ the family of continuous functions $\varphi : J \to V$. Similarly, $C^m(J,V)$ denotes the family of functions which are m times differentiable on J, and have continuous m-th derivative. The space of Lebesgue integrable functions on $\varphi : (0,\infty) \to V$ will be denoted $L^1((0,\infty),V)$. Where the codomain V is clear from the context, we omit it from these notations. The convolution of F and G in $C((0,\infty),M_n(\mathbf{R}))$ is denoted by $F * G$ and defined to be the function given by

$$(F * G)(t) = \int_0^t F(t-s)G(s)\,ds, \quad t \ge 0.$$

We denote by $\mathbf{C}$ the set of complex numbers, the real part of z in $\mathbf{C}$ by $\operatorname{Re} z$, and the imaginary part by $\operatorname{Im} z$. If $F : [0,\infty) \to M_n(\mathbf{R})$, and there is an $\alpha \in \mathbf{R}$ such that $\int_0^\infty |F(t)|e^{-\alpha t}\,dt < \infty$, we can define the Laplace transform of F to be

$$\widehat{F}(z) = \int_0^\infty F(t)e^{-zt}\,dt, \quad \operatorname{Re} z \ge \alpha.$$

In this case, $\widehat{F}(z)$ exists and is continuous in z for $\operatorname{Re} z \ge \alpha$, and analytic on $\operatorname{Re} z > \alpha$.

Next we precisely formulate our problem. Throughout the paper, we assume that $K : [0,\infty) \to M_n(\mathbf{R})$ is a function with the property that

$$(\mathrm{H}_1) \qquad K \in C[0,\infty) \cap L^1(0,\infty).$$

Under the hypothesis (H_1), it is well known that the initial-value problem (1.1) has a unique continuous solution, which is moreover continuously differentiable. Also the solution X of equation (1.1) is in $L^1(0,\infty)$ if and only if

$$\det(zI - A - \widehat{K}(z)) \ne 0, \quad \text{for all } \operatorname{Re} z \ge 0.$$

This result was first established in Grossman and Miller [**11**]. The uniform asymptotic stability of the zero solution of (1.1a) is equivalent to the fundamental solution X of (1.1) being integrable, by a result of Miller [**13**].

3. Discussion of results. In this section, we explain the connection between the results on exponential decay presented in Murakami [**14, 15**] and those in this paper.

In Theorem 1 of [**15**], it is shown that if there is $\gamma > 0$ such that

$$\kappa := \int_0^\infty |K(s)| e^{\gamma s}\, ds < \infty, \tag{3.1}$$

and the solution X of (1.1) is integrable, then there are $\alpha > 0$, $c > 0$ such that

$$|X(t)| \le ce^{-\alpha t}, \quad t \ge 0. \tag{3.2}$$

Proposition 3.1. *Suppose that the solution of* (1.1) *is integrable and* (3.1) *holds. Then there are c, $\beta > 0$ such that*

$$|X(t)| \le ce^{-\beta t}, \quad |X'(t)| \le ce^{-\beta t}, \quad t \ge 0.$$

Proof. If $\gamma \le \alpha$, we have

$$\begin{aligned}\left| \int_0^t K(t-s) X(s)\, ds \right| &\le e^{-\gamma t} \int_0^t |K(t-s)| e^{\gamma(t-s)} \cdot ce^{-(\alpha-\gamma)s}\, ds \\ &\le c\kappa e^{-\gamma t},\end{aligned}$$

while if $\gamma > \alpha$

$$\begin{aligned}\left| \int_0^t K(t-s) X(s)\, ds \right| &\le ce^{-\alpha t} \int_0^t |K(t-s)| e^{\gamma(t-s)} e^{-(\gamma-\alpha)(t-s)}\, ds \\ &\le ce^{-\alpha t} \int_0^t |K(s)| e^{\gamma s}\, ds \le c\kappa e^{-\alpha t}.\end{aligned}$$

Hence from (1.1), (3.1) and (3.2), there is $c' > 0$ such that $|X'(t)| \le c' e^{-(\alpha\wedge\gamma)t}$, $t \ge 0$. $\square$

It is illuminating to observe that Murakami's exponential integrability hypothesis (3.1) is equivalent to a pointwise exponential bound on $t \mapsto \int_t^\infty |K(s)|\, ds$.

Proposition 3.2. *There exist $c > 0$ and $\delta > 0$ such that*

$$\int_t^\infty |K(s)|\, ds \le ce^{-\delta t}, \quad t \ge 0, \tag{3.3}$$

if and only if $\gamma > 0$ satisfies (3.1).

Proof. If γ satisfies (3.1), then

$$\kappa \ge \int_t^\infty e^{\gamma s}|K(s)|\, ds \ge e^{\gamma t} \int_t^\infty |K(s)|\, ds,$$

and (3.3) holds. To see that the converse holds, we note that (3.3) implies that

$$\begin{aligned} \frac{c}{\varepsilon} &\ge \int_0^\infty e^{(\delta-\varepsilon)t} \int_t^\infty |K(s)|\, ds\, dt \\ &= \frac{1}{\delta-\varepsilon} \int_0^\infty \left\{ e^{(\delta-\varepsilon)s} - 1 \right\} |K(s)|\, ds, \end{aligned}$$

for each $0 < \varepsilon < \delta$. This rearranges to give

$$\int_0^\infty |K(s)| e^{\gamma s}\, ds \le \int_0^\infty |K(s)|\, ds + \frac{c}{\varepsilon}(\delta - \varepsilon),$$

with $\gamma := \delta - \varepsilon > 0$. $\square$

These propositions allow us to reformulate Theorem 2 of [**14**].

Theorem 3.3. *Suppose that the solution of* (1.1) *is in $L^1(0,\infty)$ and that K obeys* (H_1). *If*

(C_1) no entry of K changes sign on $[0,\infty)$,

then the following conditions are equivalent:

(a) *There exist* $c > 0$, $\delta > 0$ *such that*

$$\int_t^\infty |K(s)|\,ds \le ce^{-\delta t}, \quad t \ge 0.$$

(b) *There exist* $\alpha > 0$, $c_0 > 0$ *such that*

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.$$

The main result of this paper is the following.

Theorem 3.4. *Suppose that the solution of* (1.1) *is in* $L^1(0,\infty)$ *and that* K *obeys* (H_1). *If*

$$(\mathrm{C}_2) \qquad \int_0^\infty t^2 |K(t)|\,ds < \infty,$$

then the following conditions are equivalent:

(a) *There exist* $\beta > 0$, $c > 0$ *such that*

$$|K(t)| \le ce^{-\beta t}, \quad t \ge 0.$$

(b) *There exist* $\alpha > 0$, $c_0 > 0$ *such that*

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad |X''(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.$$

By restating Murakami's result as Theorem 3.3 above, we can readily see the connection between his result and Theorem 3.4. Apart from the technical conditions (C_1), (C_2), we see that the difference between the results is essentially one of regularity of X and K. Theorem 3.3 identifies the exponential decay of X and X' with the exponential decay of $t \mapsto \int_t^\infty |K(s)|\,ds$, while Theorem 3.4 identifies the exponential decay of X, X', and X'' with the exponential decay of K. Thus, Theorem 3.3 requires weaker conditions on the integrability of K ($\int_0^\infty |K(t)|\,dt < \infty$) and the exponential decay of K ($t \mapsto \int_t^\infty |K(s)|\,ds \to 0$ exponentially fast) in order to prove weaker results on the exponential decay of X

($X, X' \to 0$ exponentially fast). Theorem 3.3 requires stronger conditions on the integrability of K ($\int_0^\infty t^2|K(t)|\,dt < \infty$) and the exponential decay of K ($K \to 0$ exponentially fast) in order to prove stronger results on the exponential decay of X ($X, X', X'' \to 0$ exponentially fast).

We now show that condition (a) of Theorem 3.4 does not hold unless X'' decays exponentially.

Proposition 3.5. *Suppose that K obeys* (H_1) *and* (3.1). *If the solution of* (1.1) *is integrable, then there exist $c_0 > 0$, $\alpha > 0$ such that*

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.$$

Moreover,

$$\limsup_{t\to\infty} |X''(t)|e^{\varepsilon t} = \infty \quad \textit{for all } \varepsilon > 0 \tag{3.4}$$

if and only if

$$\limsup_{t\to\infty} |K(t)|e^{\varepsilon t} = \infty \quad \textit{for all } \varepsilon > 0. \tag{3.5}$$

Proof. We first show that (3.5) implies (3.4). Notice that, as K is continuous, X is in $C^2(0,\infty)$, and indeed

$$X''(t) = AX'(t) + \int_0^t K(s)X'(t-s)\,ds + K(t). \tag{3.6}$$

Then we get

$$\begin{aligned} |X''(t)| &\ge |K(t)| - |A||X'(t)| - \left|\int_0^t K(t-s)X'(s)\,ds\right| \\ &\ge |K(t)| - |A|c_0 e^{-\alpha t} - c_1 e^{-(\alpha\wedge\gamma)t} \\ &\ge |K(t)| - c_2 e^{-(\alpha\wedge\gamma)t}. \end{aligned}$$

Therefore with $0 < \varepsilon < \alpha \wedge \gamma$, we have $\limsup_{t\to\infty} |X''(t)|e^{\varepsilon t} = \infty$, which implies (3.4).

The proof that (3.4) implies (3.5) uses (3.6). Since

$$
\begin{aligned}
|K(t)| &\geq |X''(t)| - |A||X'(t)| - \left| \int_0^t K(t-s)X'(s)\,ds \right| \\
&\geq |X''(t)| - |A|c_0 e^{-\alpha t} - c_1 e^{-(\alpha\wedge\gamma)t},
\end{aligned}
$$

(3.5) follows. $\square$

It is not difficult to find functions K which satisfy (3.1), but also obey

$$
\limsup_{t\to\infty} |K(t)|e^{\varepsilon t} = \infty \quad \text{for all } \varepsilon > 0.
$$

To construct a scalar example, let $\gamma > 0$, and define k by

$$
k(t) = \begin{cases} e^{-\gamma 2^n} + 2^n(e^{-\gamma 2^n} - 1)(t - 2^n), & t \in I_n \text{ for some } n, \\ 1 + 2^n(e^{-\gamma(2^n + 2^{1-n})} - 1)(t - (2^n + 2^{-n})), & t \in J_n \text{ for some } n, \\ e^{-\gamma t}, & \text{otherwise,} \end{cases}
$$

where $I_n = [2^n, 2^n + 2^{-n}]$, $J_n = [2^n + 2^{-n}, 2^n + 2^{1-n}]$ and $n \in \mathbf{N}$. Then for every $\varepsilon \in (0, \gamma)$,

$$
\int_0^\infty e^{(\gamma-\varepsilon)s}|k(s)|\,ds < \infty.
$$

But, because $\limsup_{t\to\infty} k(t) = 1$, $\limsup_{t\to\infty} e^{\varepsilon t}k(t) = \infty$ for all $\varepsilon > 0$.

Proposition 3.5 shows that the exponential stability of the zero solution is not sufficient to imply the *pointwise* exponential decay of the kernel. Furthermore, the exponential decay of the solution and its derivative is *also* insufficient. Thus, the hypotheses of Theorem 3.4, which gives the equivalence of the exponential stability of the solution and its first two derivatives, and the pointwise exponential decay of K, are essential.

We finish our discussion by observing that an infinite family of results similar to Theorem 3.4 can be established, under a stronger differentiability restriction on the kernel K. When $K \in C^m(0, \infty)$, these results connect the exponential decay of the solution X of (1.1) and its first $m+2$ derivatives with the exponential decay of K and its first m derivatives. The general theorem for $K \in C^m(0,\infty)$ (in the case $m \geq 2$) is stated below.

Theorem 3.6. *Suppose that the solution of* (1.1) *is in* $L^1(0,\infty)$. *Suppose there is* $m \geq 2$ *such that* $K \in C^m(0,\infty) \cap L^1(0,\infty)$. *Then the following conditions are equivalent*:

(a) *There exist* $\beta > 0$, $c > 0$ *such that*

$$|K^{(j)}(t)| \leq ce^{-\beta t}, \quad t \geq 0$$

for all $j = 0, 1, \dots, m$.

(b) *There exist* $\alpha > 0$, $c_0 > 0$ *such that*

$$|X^{(i)}(t)| \leq c_0 e^{-\alpha t}, \quad t \geq 0.$$

for all $i = 0, 1, \dots, m+2$.

The methods and argument used to prove this result are similar to those involved in the proof of Theorem 3.4, so we do not supply a proof here.

The reader may note that a moment condition of the form (C_2) is not needed in Theorem 3.6. However, in order to prove a comparable result when $K \in C^1(0,\infty)$, a moment condition on K of the form (C_2) must be reimposed.

Theorem 3.7. *Suppose that the solution of* (1.1) *is in* $L^1(0,\infty)$. *Suppose that* $K \in C^1(0,\infty) \cap L^1(0,\infty)$. *If*

$$\int_0^\infty t|K(t)|\,ds < \infty,$$

then the following conditions are equivalent:

(a) *There exist* $\beta > 0$, $c > 0$ *such that*

$$|K(t)| \leq ce^{-\beta t}, \quad |K'(t)| \leq ce^{-\beta t}, \quad t \geq 0.$$

(b) *There exist* $\alpha > 0$, $c_0 > 0$ *such that*

$$
\begin{aligned}
|X(t)| &\le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t},\\
|X''(t)| &\le c_0 e^{-\alpha t}, \quad |X'''(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0.
\end{aligned}
$$

Once again, the interested reader is invited to establish this assertion.

4. Preparatory results. To prove Theorem 3.4, it is first necessary to establish a number of supporting results, and introduce some auxiliary functions.

We observe that if K obeys

$$
\text{(H}_2\text{)} \qquad \int_0^\infty t^2 |K(t)|\, ds < \infty,
$$

then under (H_1), (H_2), the functions

$$
K_1(t) = \int_t^\infty K(s)\, ds, \quad K_2(t) = \int_t^\infty K_1(s)\, ds \tag{4.1}
$$

are well defined.

We first show that the exponential decay of K_2 defined in (4.1), and the exponential decay of X, X' and X'' imply the pointwise exponential decay of K.

Lemma 4.1. *Suppose that* K *satisfies* (H_1), (H_2), *and that there is* $c_2 > 0$, $\gamma_2 > 0$ *such that* K_2 *defined by* (4.1) *obeys* $|K_2(t)| \le c_2 e^{-\gamma_2 t}$. *If there exist* $c_0 > 0$, $\alpha > 0$ *such that the solution of* (4.1) *obeys*

$$
|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad |X''(t)| \le c_0 e^{-\alpha t}, \quad t \ge 0,
$$

then there is $c > 0$ *such that* $|K(t)| \le c e^{-(\alpha \wedge \gamma_2) t}$, *for* $t \ge 0$.

Proof. We first prove that there is $c_1 > 0$ and $\gamma_1 > 0$ such that $|K_1(t)| \le c_1 e^{-\gamma_1 t}$.

Integrating the convolution $K * X$ by parts, we have

$$\begin{aligned} \int_0^t K(s)X(t-s)\,ds = -K_1(t) + \int_0^\infty K(u)\,du\,X(t) \\ - \int_0^t K_1(s)X'(t-s)\,ds, \end{aligned} \tag{4.2}$$

and integrating the last integral on the righthand side by parts gives

$$\begin{aligned} \int_0^t K_1(s)X'(t-s)\,ds = -K_2(t)X'(0) + \int_0^\infty K_1(s)\,ds\,X'(t) \\ - \int_0^t K_2(s)X''(t-s)\,ds. \end{aligned}$$

Thus there is $c'' > 0$ such that

$$\left| \int_0^t K_1(s)X'(t-s)\,ds \right| \le c'' e^{-(\alpha\wedge\gamma_2)t}.$$

Substituting (4.2) into (1.1) yields

$$K_1(t) = AX(t) - X'(t) + \int_0^\infty K(u)\,du\,X(t) - \int_0^t K_1(s)X'(t-s)\,ds,$$

so $|K_1(t)| \le c_1 e^{-(\alpha\wedge\gamma_2)t}$ for some $c_1 > 0$.

We now show that K is exponentially bounded. Let $\gamma_1 = \alpha\wedge\gamma_2 > 0$. We note that

$$\begin{aligned} \int_0^t K(s)X'(t-s)\,ds = -K_1(t) + \int_0^\infty K(s)\,ds\,X'(t) \\ - \int_0^t K_1(s)X''(t-s)\,ds. \end{aligned}$$

Hence there exists a $c' > 0$ such that

$$\left| \int_0^t K(s)X'(t-s)\,ds \right| \le c' e^{-(\alpha\wedge\gamma_1)t}.$$

By (3.6), $|K(t)| \le c e^{-(\alpha\wedge\gamma_1)t}$ for some $c > 0$. $\square$

We now appeal to Theorem 1 of [**15**] to prove the following result.

Theorem 4.2. *Suppose that K obeys* (H_1) *and that there exist $c > 0$, $\beta > 0$ such that $|K(t)| \le ce^{-\beta t}$. Suppose that the solution of* (1.1) *is in $L^1(0,\infty)$. Then, there exist $c_0 > 0$, $\alpha > 0$ such that for $t \ge 0$,*

$$|X(t)| \le c_0 e^{-\alpha t}, \quad |X'(t)| \le c_0 e^{-\alpha t}, \quad |X''(t)| \le c_0 e^{-\alpha t}. \tag{4.3}$$

Proof. Since $X \in L^1(0,\infty)$ and $|K(t)| \le ce^{-\beta t}$, we have automatically by Theorem 1 of [**15**] that $|X(t)| \le c_1 e^{-\alpha_1 t}$ for some $c_1 > 0$, $\alpha_1 > 0$. Using this estimate and $|K(t)| \le ce^{-\beta t}$ in (1.1) gives $|X'(t)| \le c_2 e^{-(\alpha_1 \wedge \beta)t}$, and in turn using this in (3.6) yields

$$|X''(t)| \le c_3 e^{-(\alpha_1 \wedge \beta)t}.$$

Hence the result is true with $\alpha = \alpha_1 \wedge \beta > 0$. $\square$

We now give an additional condition on K under which (4.3) implies $|K_2(t)| \le ce^{-\gamma t}$ for some c, $\gamma > 0$.

Theorem 4.3. *Suppose that K obeys* (H_1) *and* (H_2)*, and that there exist $c > 0$, $\gamma > 0$ such that* (4.3) *holds. Then there exist $c_2 > 0$, $\gamma_2 > 0$ such that K_2 defined by* (4.1) *obeys $|K_2(t)| \le c_2 e^{-\gamma_2 t}$, $t \ge 0$.*

We now see that this result enables us to prove Theorem 3.4: the proof that (b) implies (a) follows from Theorem 4.3 and Lemma 4.1. The fact that (a) implies (b) is nothing but the subject of Theorem 4.2.

5. Proof of Theorem 4.3. In this proof, we follow Murakami's presentation of Theorem 1 in [**15**] and give a self-contained demonstration. It relies upon obtaining an integral equation for K_2 which is then analyzed using transform techniques. In this section, X and its derivatives are viewed as data.

By (H_2), K_1 and K_2 are in $L^1(0,\infty)$. Since $K_1'(t) = -K(t)$, and

$$\begin{aligned}(K_1 * X)'(t) &= \int_0^t K_1'(t-s)X(s)\,ds + \int_0^\infty K(s)\,ds X(t) \\ &= -X'(t) + AX(t) + \int_0^\infty K(s)\,ds X(t),\end{aligned}$$

(1.1) can be integrated to give

$$\int_0^t K_1(t-s)X(s)\,ds = -X(t) - \left(A + \int_0^\infty K(s)\,ds\right)\int_t^\infty X(s)\,ds.$$

Similarly, because $K_2'(t) = -K_1(t)$,

$$\begin{aligned}(K_2 * X)'(t) &= -(K_1 * X)(t) + K_2(0)X(t)\\ &= (I + K_2(0))X(t) + \left(A + \int_0^\infty K(s)\,ds\right)\int_t^\infty X(s)\,ds,\end{aligned}$$

we see that

$$\int_0^t K_2(t-s)X(s)\,ds = \Phi(t). \tag{5.1}$$

where

$$\begin{aligned}\Phi(t) = L - \int_t^\infty \Bigg\{(I + K_2(0))X(s) + \left(A + \int_0^\infty K(v)\,dv\right)&\\ \times \int_s^\infty X(u)\,du\Bigg\}\,ds.&\end{aligned}$$

The constant of integration L is chosen so that $\Phi(0) = 0$.

Since $|X(t)| \le c_0 e^{-\alpha t}$ and $K_2(0) = \int_0^\infty sK(s)\,ds$,

$$\begin{aligned}\Phi(0) = L - \Bigg\{ \left(I + \int_0^\infty sK(s)\,ds\right)\int_0^\infty X(s)\,ds&\\ + \left(A + \int_0^\infty K(s)\,ds\right)\int_0^\infty sX(s)\,ds\Bigg\}.&\end{aligned}$$

If it can be shown that

$$\begin{aligned}\left(I + \int_0^\infty sK(s)\,ds\right)\int_0^\infty X(s)\,ds&\\ + \left(A + \int_0^\infty K(s)\,ds\right)\int_0^\infty sX(s)\,ds = 0&\end{aligned} \tag{5.2}$$

then $\Phi(0) = L$, and so $L = 0$. The proof of (5.2) follows by taking Laplace transforms of (1.1). Since X and K are in $L^1(0,\infty)$, we have

$$(zI - A - \widehat{K}(z))\widehat{X}(z) = I, \quad \operatorname{Re} z \ge 0. \tag{5.3}$$

Since $z \mapsto \widehat{X}(z)$, $z \mapsto \widehat{K}(z)$ are analytic in $\operatorname{Re} z > 0$, we can differentiate both sides of (5.3) to give

$$(zI - A - \widehat{K}(z))\widehat{X}'(z) + (I - \widehat{K}'(z))\widehat{X}(z) = 0, \quad \operatorname{Re} z > 0.$$

Now, because

$$\lim_{z \to 0,\, \operatorname{Re} z > 0} \widehat{K}'(z) = -\int_0^\infty sK(s)\,ds,$$

and $z \mapsto \widehat{X}(z)$ is analytic at $z = 0$, we get

$$-\left(A + \int_0^\infty K(s)\,ds\right)\widehat{X}'(0) + \left(I + \int_0^\infty sK(s)\,ds\right)\widehat{X}(0) = 0,$$

which is nothing other than (5.2). Therefore

$$\Phi(t) = -\int_t^\infty \left\{(I + K_2(0))X(s) + \left(A + \int_0^\infty K(v)\,dv\right) \times \int_s^\infty X(u)\,du\right\} ds. \tag{5.4}$$

The asymptotic behavior of K_2 is now investigated using transform methods. An exponential bound of the form $|\Phi(t)| \le c_1 e^{-\alpha t}$ follows from $|X(t)| \le c_0 e^{-\alpha t}$ and (5.4). In particular, $\Phi \in L^1(0,\infty)$. Also, $X \in L^1(0,\infty)$, and $K_2 \in L^1(0,\infty)$ by (H_2). Therefore, taking Laplace transforms of (5.1) gives

$$\widehat{K_2}(z)\widehat{X}(z) = \widehat{\Phi}(z), \quad \operatorname{Re} z \ge 0. \tag{5.5}$$

We now show that $\widehat{X}(z)$ is invertible for all $\operatorname{Re} z \ge 0$. Suppose to the contrary that there is $z_0 \in \mathbf{C}$ with $\operatorname{Re} z_0 \ge 0$ such that $\det(\widehat{X}(z_0)) = 0$. By (5.3) we get

$$1 = \det(I) = \det([z_0 I - A - \widehat{K}(z_0)]\widehat{X}(z_0)) = 0,$$

a contradiction. Therefore, we may define

$$F(z) = \widehat{\Phi}(z)\widehat{X}(z)^{-1}, \quad \operatorname{Re} z \ge 0,$$

and so, by (5.5)

$$\widehat{K}_2(z) = F(z), \quad \operatorname{Re} z \geq 0.$$

Let us now show that F can also be defined on a strip in the negative real half-plane.

Observe that $z \mapsto \widehat{\Phi}(z)$, $z \mapsto \widehat{X}(z)$, $z \mapsto \widehat{X'}(z)$ are defined for $\operatorname{Re} z > -\alpha$, and therefore for $\operatorname{Re} z \geq -(2\alpha)/3$. By the Riemann-Lebesgue Lemma, and the fact that $|X'(t)| \leq c_0 e^{-\alpha t}$, there exists $T_0 > 0$ such that $|\widehat{X'}(z)| < 1/2$ for all $z \in \mathbf{C}$ with $-(2\alpha)/3 \leq \operatorname{Re} z < 0$ and $|\operatorname{Im} z| > T_0$. Therefore $z\widehat{X}(z) = I + \widehat{X'}(z)$ is invertible in the region $-(2\alpha)/3 \leq \operatorname{Re} z < 0$, $|\operatorname{Im} z| > T_0$, and so $z \mapsto \widehat{X}(z)$ is invertible in that region. Now consider

$$D = \left\{ z \in \mathbf{C} : -\frac{2\alpha}{3} \leq \operatorname{Re} z \leq 0,\, |\operatorname{Im} z| \leq T_0 \right\}.$$

Since D is compact and $z \mapsto \det(\widehat{X}(z))$ is analytic on D, $\det(\widehat{X})$ has only finitely many zeros in D. Since we have already established that $\det(\widehat{X}(z)) \neq 0$ for $\operatorname{Re} z \geq 0$, it follows that if there is $z \in D$ such that $\det(\widehat{X}(z)) = 0$, we can define $\gamma_0 < 0$ such that

$$\gamma_0 = \max\{\operatorname{Re} z : \, z \in D,\, \det(\widehat{X}(z)) = 0\}.$$

In the case that $\det(\widehat{X}(z)) \neq 0$ for all $z \in D$, we can define $\gamma_0 = -\alpha/2$. Thus in either case, there exists $\gamma_0 < 0$ such that $\det(\widehat{X}(z)) \neq 0$ for $\operatorname{Re} z > \gamma_0$, or that $z \mapsto \widehat{X}(z)$ is invertible for $\operatorname{Re} z > \gamma_0$. Thus we can extend F by

$$F(z) = \widehat{\Phi}(z)\widehat{X}(z)^{-1}, \quad \gamma_0 < \operatorname{Re} z < 0,$$

and F is analytic for all $\operatorname{Re} z > \gamma_0$.

Let $\varepsilon \in (0, -\gamma_0)$. For $T > 0$ the analyticity of F in the box $\{\xi + i\eta \, : \, -\varepsilon \leq \xi \leq \varepsilon,\, -T \leq \eta \leq T\}$ gives

$$\left(\int_{\varepsilon - iT}^{\varepsilon + iT} + \int_{\varepsilon + iT}^{-\varepsilon + iT} + \int_{-\varepsilon + iT}^{-\varepsilon - iT} + \int_{-\varepsilon - iT}^{\varepsilon - iT} \right) F(z) e^{zt}\, dz = 0.$$

If we can show that

$$\text{(5.6)} \qquad \lim_{T \to \infty} \int_{-\varepsilon - iT}^{\varepsilon - iT} F(z) e^{zt}\, dz = 0, \quad \lim_{T \to \infty} \int_{-\varepsilon + iT}^{\varepsilon + iT} F(z) e^{zt}\, dz = 0,$$

the inversion formula for Laplace transforms gives

$$\begin{aligned} K_2(t) &= \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} F(z)e^{zt}\,dz \\ &= \frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} z\widehat{\Phi}(z)\left(z\widehat{X}(z)\right)^{-1} e^{zt}\,dz, \quad t>0. \end{aligned} \tag{5.7}$$

The exponential bounds on $X^{(j)}(t)$, $j=0,1$ and (5.4) imply the existence of positive constants c_1, c_2 and c_3 such that $|\Phi^{(j)}(t)| \le c_{j+1}e^{-\alpha t}$, $j=0,1,2$. Therefore, the expansion

$$\widehat{\Phi}(z) = \frac{\Phi(0)}{z} + \frac{\Phi'(0)}{z^2} + \frac{\widehat{\Phi''}(z)}{z^2}, \quad \operatorname{Re} z \ge -\varepsilon,$$

is valid. Because $\Phi(0)=0$, there is a constant $M>0$ such that $|z|^2|\widehat{\Phi}(z)| \le M$, $\operatorname{Re} z \ge -\varepsilon$. Due to the expansion

$$\widehat{X}(z) = \frac{I}{z} + \frac{\widehat{X'}(z)}{z}, \quad \operatorname{Re} z \ge -\varepsilon,$$

there exists $T>0$ such that

$$\left|\left(z\widehat{X}(z)\right)^{-1}\right| \le 2, \quad \operatorname{Re} z \ge -\varepsilon,\ |\operatorname{Im} z| > T.$$

Hence

$$\begin{aligned} &\left|\int_{-\varepsilon+iT}^{\varepsilon+iT} z\widehat{\Phi}(z)\left(z\widehat{X}(z)\right)^{-1} e^{zt}\,dz\right| \\ &\le \int_{-\varepsilon}^{\varepsilon} \frac{1}{|\xi+iT|}|\xi+iT|^2|\widehat{\Phi}(\xi+iT)|\left|\left((\xi+iT)\widehat{X}(\xi+iT)\right)^{-1}\right| e^{\varepsilon t}\,d\xi \\ &\le e^{\varepsilon t}\int_{-\varepsilon}^{\varepsilon}\frac{2M}{T}\,d\xi = \frac{4\varepsilon M}{T}\,e^{\varepsilon t}. \end{aligned}$$

Therefore, this term has zero limit as $T\to\infty$. This establishes the first formula in (5.6). The second formula can be obtained in a similar fashion, and so (5.7) follows.

Next, define

$$G(z) = F(z) - \frac{\Phi'(0)}{z-\gamma_0} = \widehat{\Phi}(z)\widehat{X}(z)^{-1} - \Phi'(0)\,\frac{1}{z-\gamma_0}.$$

Now we show that

$$\lim_{|z|\to\infty} z^2 G(z) \text{ exists for } \operatorname{Re} z \geq -\varepsilon. \tag{5.8}$$

Since $|X''(t)| \leq c_0 e^{-\alpha t}$, (5.4) implies that $|\Phi'''(t)| \leq c_4 e^{-\alpha t}$. Therefore

$$\widehat{\Phi}(z) = \frac{\Phi'(0)}{z^2} + \frac{\Phi''(0)}{z^3} + \frac{\widehat{\Phi'''}(z)}{z^3}, \quad z \neq 0.$$

Since $(I+C)^{-1} = I - C + C^2 + \cdots$ if $|C| < 1$ and $|\widehat{X'}(z)| \to 0$ as $|z| \to \infty$, we see that for large enough $|z|$ with $\operatorname{Re} z \geq -\varepsilon$,

$$(I + \widehat{X'}(z))^{-1} = I + Y_1(z) = I - \widehat{X'}(z) + Y_2(z),$$

where $|Y_1(z)| \leq \mu |\widehat{X'}(z)|$ and $|Y_2(z)| \leq \mu |\widehat{X'}(z)|^2$ for some constant $\mu > 0$. Then,

$$\begin{aligned} z^2 G(z) = \Phi'(0) \left[\frac{\gamma_0 z}{\gamma_0 - z} I - X'(0) - \widehat{X''}(z) + z Y_2(z) \right] \\ + \left(\Phi''(0) + \widehat{\Phi'''}(z) \right) (I + Y_1(z)). \end{aligned}$$

By the Riemann-Lebesgue lemma, $Y_1(z) \to 0$ and $zY_2(z) \to 0$ as $|z| \to \infty$; also $\widehat{X''}(z) \to 0$ and $\widehat{\Phi'''}(z) \to 0$ as $|z| \to \infty$. These combine to establish (5.8).

Therefore, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(-\varepsilon + i\eta)|\, d\eta =: \kappa_1 < \infty.$$

By (5.7) and the definition of G, we have

$$\begin{aligned} |K_2(t)| &\leq \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} G(-\varepsilon + i\eta) e^{(-\varepsilon + i\eta)t}\, d\eta \right| \\ &\quad + \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Phi'(0)}{-\varepsilon + i\eta - \gamma_0} e^{(-\varepsilon + i\eta)t}\, d\eta \right| \\ &\leq \kappa_1 e^{-\varepsilon t} + |\Phi'(0)| \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{(-\varepsilon + i\eta)t}}{-\varepsilon + i\eta - \gamma_0}\, d\eta \right|. \end{aligned}$$

Since

$$\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}\frac{e^{zt}}{z-\gamma_0}\,dz=e^{\gamma_0 t},$$

we have $|K_2(t)| \le (\kappa_1 + |\Phi'(0)|)e^{-\varepsilon t}$ as $\gamma_0 < -\varepsilon$, which proves Theorem 4.3.

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