JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 15, Number 4, Winter 2003

EXISTENCE RESULTS FOR BV-SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

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ABSTRACT. In this paper we deal with the existence of global bounded variation (BV) solutions as well as continuous BV-solutions of nonlinear Hammerstein and Volterra-Hammerstein integral equations formulated in terms of the Lebesgue and the Denjoy-Perron integral. The method of proof is based on an application of the Leray-Schauder alternative for contractions.

1. Introduction. Functions of bounded variation appear frequently as solutions to many integral equations which describe concrete physical phenomena. This fact motivated us to investigate bounded variation solutions as well as continuous bounded variation solutions of the Hammerstein type integral equation

$$x(t) = g(t) + \int_0^T K(t,s) f(x(s))\, ds, \quad \text{for } t \in I = [0,T], \tag{1}$$

and the Volterra-Hammerstein integral equation

$$x(t) = g(t) + \int_0^t K(t,s) f(x(s))\, ds, \quad \text{for } t \in I, \tag{2}$$

where I is a compact interval in $\mathbf{R}$. First we will investigate equations (1) and (2) with the Lebesgue integral and then later with the Denjoy-Perron integral.

The theory of the Denjoy-Perron integral gives a pure theoretical motivation for the need to investigate BV-solutions of equations (1) and (2). More precisely, it is well-known that if $h : I \to \mathbf{R}$ is any function integrable in the Denjoy-Perron sense and ϕ is a function of bounded variation, then $h\phi$ is also integrable in this sense, see [**5**] for details.

Received by the editors on December 31, 2002, and in revised form on August 10, 2003.

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A variety of existence theorems for BV-solutions and continuous BV-solutions of equation (2) and the Hammerstein integral equation of the form

$$x(t) = g(t) + \lambda \int_0^T K(t,s) f(x(s))\, ds, \quad \text{for } t \in I, \text{ and } \lambda \in \mathbf{R}, \tag{3}$$

with the Lebesgue integral were proved in [**4**]. In [**3**] similar results were obtained for equations (2) and (3) with the Denjoy-Perron integral. Existence theorems for equation (2) in [**3**] and [**4**] have a local character. Their proofs were based on the classical Banach contraction principle.

This paper develops existence theory for global BV-solutions and continuous BV-solutions of equations (1) and (2). The proofs of the results in Section 3 and Section 4 are based on the following Leray-Schauder alternative, see [**6**].

Let U be an open subset of a Banach space $(X, \|\cdot\|)$ with $0 \in U$. Suppose $F : \bar{U} \to X$ and assume there exists a continuous nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that for $x, y \in \bar{U}$ we have $\|F(x) - F(y)\| \le \phi(\|x - y\|)$; here $\bar{U}$ denotes the closure of U in X. In addition assume $F(\bar{U})$ is bounded and $x \ne \lambda F(x)$ for $x \in \partial U$ and $\lambda \in (0,1]$; here ∂U denotes the boundary of U in X. Then F has a fixed point in U.

We refer the reader to [**2**] for basic results concerning the superposition operator in the space of functions of bounded variation and to [**8, 9**] for an introduction to integral operators in the space of functions of bounded variation.

2. Preliminaries. In this section we collect some definitions and results which will be needed in the sequel. Consider a function $x : [a,b] \to \mathbf{R}$, where $[a,b]$ is a compact interval in $\mathbf{R}$. Recall that the number

$$\bigvee_a^b (x) = \sup \sum_{i=1}^n |x(s_i) - x(s_{i-1})|,$$

where the supremum is taken over all (finite) partitions $\{s_0, s_1, \dots, s_n\}$ of $[a,b]$, is called the variation of x over $[a,b]$. By $BV = BV([a,b])$ we will denote the space of all functions x defined on $[a,b]$ such that

$\bigvee_a^b(x) < +\infty$, with the norm

$$\|x\|_{BV} = |x(a)| + \bigvee_a^b(x).$$

It is well known that BV considered with the above norm is a (real or complex) Banach space. Functions of bounded variation will be called BV-functions in this paper.

Recall that the superposition operator generated by a function $f = f(u)$ acts in the space BV if and only if f satisfies a local Lipschitz condition, cf. [**2**, p. 174].

Next we present the concept of the Denjoy-Perron integral. First we define ACG^* functions.

A function $f : [a,b] \to \mathbf{R}$ is said to be generalized absolutely continuous on $[a,b]$, written ACG^* on $[a,b]$, if it is continuous on $[a,b]$ and if this interval can be expressed as the sum of a finite or countable sequence of sets on each of which the function f is absolutely continuous in the restricted sense.

We say that a function $f : [a,b] \to \mathbf{R}$ is integrable in the Denjoy-Perron sense on $[a,b]$ if there exists an ACG^* function $F : [a,b] \to \mathbf{R}$ with $F' = f$ almost everywhere on $[a,b]$. The increment $F(b) - F(a)$ over the interval $[a,b]$ is termed the definite D-P integral of f over $[a,b]$ and is denoted by $(D-P)\int_a^b f(s)\,ds$.

In this paper, instead of "$(D-P)\int_a^b f(s)\,ds$," we use "$\int_a^b f(s)\,ds$". More information about the Denjoy-Perron integral and its properties can be found in the books [**5**] and [**7**]. One of the properties of the Denjoy-Perron integral frequently used in Section 4 is mentioned next.

If $x \in BV([a,b])$ and $\phi : [a,b] \to \mathbf{R}$ is integrable on $[a,b]$ in the Denjoy-Perron sense, then the following inequality holds:

$$\left|\int_a^b x(s)\phi(s)\,ds\right| \le |x(a)| \left|\int_a^b \phi(s)\,ds\right| + \bigvee_a^b(x) O(\Phi; [a,b]),$$

where $\Phi(s) = \int_a^s \phi(t)dt$ and $O(\Phi; [a,b])$ denotes the oscillation of Φ on $[a,b]$.

3. Equations with the Lebesgue integral. Consider equation (1), where "$\int$" stands for the Lebesgue integral. Assume that:

1^0 $g : I \to \mathbf{R}$ is a BV-function;

2^0 $f : \mathbf{R} \to \mathbf{R}$;

3^0 $K : I \times I \to \mathbf{R}$ is a function such that $K(t, \cdot)$ is a Lebesgue integrable for any $t \in I$ and

$$\bigvee_0^T (K(\cdot, s)) \leq M(s) \quad \text{for a.e. } s \in I,$$

where $M : I \to \mathbf{R}_+$ is integrable in the Lebesgue sense;

4^0 there exists $\Psi : [0, +\infty) \to [0, +\infty)$ with $\Psi(u) > 0$ for $u > 0$ and $\sup_{s\in[0,T]} |f(x(s))| \leq \Psi(\|x\|_{BV})$ for any $x \in BV(I)$;

5^0 there exists $M_0 > 0$ with $M_0/(\|g\|_{BV} + \Psi(M_0) \cdot T_0) > 1$, where $T_0 = \int_0^T (|K(0, s)| + M(s))\, ds$;

6^0 there exists $\phi_{M_0} : [0, +\infty) \to [0, +\infty)$ continuous and nondecreasing with $T_0\phi_{M_0}(z) < z$ for $z > 0$ and with $|f(x) - f(y)| \leq \phi_{M_0}(|x - y|)$ for $|x|$, $|y| \leq M_0$.

Remark 1. Notice in 4^0 it is enough to assume $\sup_{s\in[0,T]} |f(x(s))| \leq \Psi(\|x\|_{BV})$ for $x \in BV(I)$ with $\|x\|_{BV} = M_0$.

Now we prove the following existence result.

Theorem 1. *Under the above assumptions equation* (1) *has a BV-solution, defined on I.*

Proof. The proof is based on an idea from [**1**, pp. 681–682]. First, let us observe that by 3^0, for $t \in I$, we have

$$|K(t, s)| \leq |K(0, s)| + \bigvee_0^T (K(\cdot, s)) \leq |K(0, s)| + M(s) \quad \text{for a.e. } s \in I.$$

Denote by $\overline{B}_{M_0}$ the closed ball of center zero and radius M_0 in the space $BV(I)$. Define $G(x)(t) = g(t) + \int_0^T K(t, s) f(x(s))\, ds$ for $x \in \bar{B}_{M_0}$ and

$t \in I$. For any $x, y \in \bar{B}_{M_0}$ we have

$$\begin{aligned}
&\|G(x) - G(y)\|_{BV} \\
&\le \int_0^T |K(0,s)||f(x(s)) - f(y(s))|\, ds \\
&\quad + \sup_{0=t_0<\ldots<t_n=T} \int_0^T \sum_{i=1}^n |K(t_i,s) - K(t_{i-1},s)||f(x(s)) - f(y(s))|\, ds \\
&\le \sup_{s\in I} |f(x(s)) - f(y(s))| \int_0^T (|K(0,s)| + M(s))\, ds \\
&\le T_0 \sup_{s\in I} \phi_{M_0}(|x(s) - y(s)|) \\
&\le T_0 \phi_{M_0}(\|x - y\|_{BV}).
\end{aligned}$$

Thus, in particular, $G(\bar{B}_{M_0})$ is bounded. Now suppose $x \in BV(I)$ with $\|x\|_{BV} = M_0$ is a solution of

$$x(t) = \lambda(g(t) + \int_0^T K(t,s) f(x(s))\, ds) \quad \text{for } t \in I, \tag{4}$$

for $\lambda \in (0,1]$. Then, by 5^0, we have

$$\begin{aligned}
||x\|_{BV} &\le \|g\|_{BV} + \| \int_0^T K(t,s) f(x(s))\, ds \|_{BV} \\
&\le \|g\|_{BV} + \int_0^T |K(0,s)||f(x(s))|\, ds + \bigvee_0^T \left(\int_0^T K(t,s) f(x(s))\, ds \right) \\
&\le \|g\|_{BV} + \sup_{s\in[0,T]} |f(x(s))| \left(\int_0^T (|K(0,s)| + M(s))\, ds \right) \\
&\le \|g\|_{BV} + \Psi(\|x\|_{BV}) \cdot T_0.
\end{aligned}$$

Thus

$$\frac{\|x\|_{BV}}{\|g\|_{BV} + \Psi(\|x\|_{BV}) T_0} \le 1 \tag{5}$$

(note that without loss of generality we may assume $\|g\|_{BV} + \Psi(\|x\|_{BV}) T_0 > 0$).

Now $\|x\|_{BV} = M_0$, so (5) implies that

$$\frac{M_0}{\|g\|_{BV} + \Psi(M_0)T_0} \leq 1,$$

which contradicts 5^0. Apply the nonlinear alternative of Leray-Schauder type in Section 1 to deduce that G has a fixed point in $B_{M_0} = \{x \in BV(I) : \|x\|_{BV} < M_0\}$. It is clear that this fixed point is a BV-solution of (1).

Remark 2. It is clear that 4^0 is equivalent to

$4^{0\prime}$ there exists a nondecreasing function $\Psi : [0, +\infty) \to [0, +\infty)$ with $\Psi(u) > 0$ for $u > 0$ and $|f(x)| \leq \Psi(|x|)$ for $x \in \mathbf{R}$.

Remark 3. Instead of equation (1) one can consider a more general one, namely

$$x(t) = g(t) + \sum_{i=1}^{n} \int_0^T K_i(t,s) f_i(x(s))\, ds \quad \text{for } t \in I, \tag{6}$$

where g satisfies 1^0, f_i satisfies 2^0 and 6^0 with $\phi^i_{M_0}$ for $1 \leq i \leq n$, K_i satisfies 3^0 with M_i integrable in the Lebesgue sense for $1 \leq i \leq n$. Moreover, we assume that

7^0 there exists $\Psi_i : [0, +\infty) \to [0, +\infty)$ such that $\Psi_i(u) > 0$ for $u > 0$, and $\sup_{s \in [0,T]} |f_i(x(s))| \leq \Psi_i(\|x\|_{BV})$ for any $x \in BV(I)$ with $\|x\|_{BV} = M_0$, $i = 1, \dots, n$;

8^0 there exists $M_0 > 0$ such that

$$\frac{M_0}{\|g\|_{BV} + \sum_{i=1}^{n} \Psi_i(M_0) T_i} > 1,$$

where $T_i = \int_0^T (|K_i(0,s)| + M_i(s))\, ds$;

Under the above assumptions one can prove that (6) has a BV-solution, defined on I.

Now we shall consider continuous BV-solutions of (1). Assume that 2^0–6^0 are satisfied. Moreover, suppose that

9^0 $g : I \to \mathbf{R}$ is a continuous BV-function;

10^0 for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t, \tau, s \in I$:

$$|\tau - t| < \delta \Longrightarrow \int_0^T |K(\tau, s) - K(t, s)|\, ds < \varepsilon.$$

Theorem 2. *Under the above assumptions equation* (1) *has a continuous BV-solution defined on* I*.*

Proof. Consider the Banach space $BV_C(I) = BV(I) \cap C(I)$ with the norm $\|\cdot\|_{BV}$. Additional assumptions 9^0 and 10^0 guarantee that the mapping G defined in the proof of Theorem 1 maps continuous functions into continuous ones. Indeed, for $x \in \bar{B}_{M_0}$ and $t, \tau \in I$, we have

$$|G(x)(t) - G(x)(\tau)| \leq |g(t) - g(\tau)| + \sup_{s \in I} |f(x(s))| \int_0^T |K(t, s) - K(\tau, s)|\, ds.$$

Hence G maps $\bar{B}_{M_0} \subset BV_C(I)$ into $BV_C(I)$. Essentially the same reasoning as in the proof of Theorem 1 establishes the result.

Equation (2) is a special case of equation (1). Putting

$$\tilde{K}(t, s) = \begin{cases} K(t, s) & 0 \leq s \leq t, \\ 0 & t < s \leq T, \end{cases}$$

we can write (2) in the following equivalent form

$$x(t) = g(t) + \int_0^T \tilde{K}(t, s) f(x(s))\, ds \quad \text{for } t \in I. \tag{7}$$

Suppose that 1^0, 2^0, 4^0 and 6^0 are satisfied. Moreover, assume that

11^0 $\tilde{T} = \{(t, s) : 0 \leq t \leq T,\ 0 \leq s \leq t\}$ and $K : \tilde{T} \to \mathbf{R}$ is a function such that $|K(s, s)| + \bigvee_s^T (K(\cdot, s)) \leq m(s)$ for almost every $s \in I$, where

$m : I \to \mathbf{R}_+$ is integrable in the Lebesgue sense and $K(t,\cdot)$ is integrable in the Lebesgue sense on $[0,t]$ for every $t \in I$;

12^0 there exists $M_0 > 0$ with

$$\frac{M_0}{\|g\|_{BV} + \Psi(M_0)\cdot T_1} > 1,$$

where $T_1 = \int_0^T m(s)\,ds$.

Theorem 3. *Under the above assumptions equation* (7) *has a* BV-*solution, defined on* I.

Proof. The result follows immediately from Theorem 1. Indeed, we have $\bigvee_0^T(\tilde{K}(\cdot,s)) = |K(s,s)| + \bigvee_s^T(K(\cdot,s)) \le m(s)$ for almost every $s \in I$, so $\tilde{K}$ satisfies 3^0. Moreover $\tilde{K}(0,s) = 0$ for $0 < s \le T$ and thus one can take $T_0 = T_1 = \int_0^T m(s)\,ds$. □

To deal with continuous BV-solutions of (2) we need the following assumption

13^0 for each $t \in I$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\tau \in I$ and $s \in [0,t] \cap [0,\tau]$:

$$|t-\tau| < \delta \Longrightarrow \int_0^{\min(t,\tau)} |K(\tau,s) - K(t,s)|ds < \varepsilon.$$

Theorem 4. *Assume that* 9^0, 2^0, 4^0, 6^0, 11^0, 12^0 *and* 13^0 *hold. Then the equation* (2) *has a continuous* BV-*solution, defined on* I.

Proof. Define $G(x)(t) = g(t) + \int_0^t K(t,s)f(x(s))\,ds$ for $x \in \overline{B}_{M_0} \subset BV_C(I)$ and $t \in I$. Assumptions 9^0 and 13^0 guarantee that G maps continuous functions into continuous ones. For example, for fixed $t_0 \in I$ and $\tau < t_0$,

$$\begin{aligned}&|G(x)(t_0) - G(x)(\tau)|\\ &\le |g(t_0) - g(\tau)|\\ &\quad + \sup_{s\in I}|f(x(s))|\left(\int_0^{\tau} |K(t_0,s) - K(\tau,s)|\,ds + \int_{\tau}^{t_0} |K(t_0,s)|\,ds\right),\end{aligned}$$

for $x \in \overline{B}_{M_0}$. Hence the mapping G maps $\bar{B}_{M_0} \subset BV_C(I)$ into $BV_C(I)$. Essentially the same reasoning as in Theorem 1 establishes the result. $\square$

4. Equations with the Denjoy-Perron integral. Consider again equation (1), where "$\int$" now stands for the Denjoy-Perron integral. Suppose that 1^0 and 2^0 hold. Moreover, assume that

14^0 $K : I \times I \to \mathbf{R}$ is function such that $K(t, \cdot)$ is integrable in the Denjoy-Perron sense for every $t \in I$ and

$$\bigvee_0^T \left(\int_I K(t,s)\, ds \right) < +\infty;$$

15^0 there exists a number $0 < c < +\infty$ such that

$$\sup_{0=t_0<\ldots<t_n=T} \sum_{i=1}^n O\left(\int_0^s (K(t_i,z) - K(t_{i-1},z))\, dz; I \right) < c;$$

16^0 there exists $\Psi : [0,+\infty) \to [0,+\infty)$ such that $\Psi(u) > 0$ for $u > 0$, and $\|f(x)\|_{BV} \le \Psi(\|x\|_{BV})$ for any $x \in BV(I)$;

17^0 there exists $M_0 > 0$ with

$$\frac{M_0}{\|g\|_{BV} + \Psi(M_0) \cdot T'} > 1,$$

where

$$T' = \max\left(\Big\| \int_I K(t,s)\, ds \Big\|_{BV}; O(P; I) + c \right), \quad P(s) = \int_0^s K(0,z)\, dz$$

for $s \in I$;

18^0 there exists $\phi_{M_0} : [0,+\infty) \to [0,+\infty)$ continuous and nondecreasing with $T'\phi_{M_0}(z) < z$ for $z > 0$ and with $\|f(x) - f(y)\|_{BV} \le \phi_{M_0}(\|x - y\|_{BV})$ for $x, y \in BV(I)$ with $\|x\|_{BV}$, $\|y\|_{BV} \le M_0$.

Theorem 5. *Under the above assumptions equation* (1) *has a BV-solution, defined on I.*

Proof. Let G and $\overline{B}_{M_0}$ be as in the proof of Theorem 1. For any $x, y \in \overline{B}_{M_0}$ we obtain

$$\begin{aligned}
&\|G(x) - G(y)\|_{BV} \\
&\quad = \Big| \int_0^T K(0,s)[f(x(s)) - f(y(s))]\,ds \Big| \\
&\qquad + \sup_{0=t_0<\ldots<t_n=T} \sum_{i=1}^n \Big| \int_0^T (K(t_i,s) - K(t_{i-1},s))[f(x(s)) - f(y(s))]ds \Big| \\
&\quad \le |f(x(0)) - f(y(0))| \Big| \int_0^T K(0,s)\,ds \Big| + \bigvee_0^T (f(x) - f(y)) \cdot O(P;I) \\
&\qquad + |f(x(0)) - f(y(0))| \bigvee_0^T \Big(\int_0^T K(t,s)\,ds \Big) + \bigvee_0^T (f(x) - f(y)) c \\
&\quad \le \max \Big(\| \int_0^T K(t,s)\,ds \|_{BV}, O(P;I) + c \Big) \|f(x) - f(y)\|_{BV} \\
&\quad \le \max \Big(\| \int_0^T K(t,s)\,ds \|_{BV}, O(P;I) + c \Big) \phi_{M_0}(\|x - y\|_{BV}) \\
&\quad \le T' \phi_{M_0}(\|x - y\|_{BV}).
\end{aligned}$$

In particular, $G(\bar{B}_{M_0})$ is bounded. Let $x \in BV(I)$ with $\|x\|_{BV} = M_0$ be a solution to (6) for $\lambda \in (0,1]$ (the integral in (6) is the Denjoy-Perron integral). Then

$$\begin{aligned}
\|x\|_{BV} &\le \|g\|_{BV} + \Big\| \int_0^T K(t,s) f(x(s))\,ds \Big\|_{BV} \\
&\le \|g\|_{BV} + \Big| \int_0^T K(0,s) f(x(s))\,ds \Big| + \bigvee_0^T \Big(\int_0^T K(t,s) f(x(s))\,ds \Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \|g\|_{BV} + |f(x(0))| \left| \int_0^T K(0,s)\,ds \right| + \bigvee_0^T (f(x)) O(P;I) \\
&\quad + |f(x(0))| \bigvee_0^T \left(\int_0^T K(t,s)\,ds \right) + c \bigvee_0^T (f(x)) \\
&\le \|g\|_{BV} + |f(x(0))| \left\| \int_I K(t,s)\,ds \right\|_{BV} + \bigvee_0^T (f(x))[O(P;I) + c] \\
&\le \|g\|_{BV} + \|f(x)\|_{BV} T' \\
&\le \|g\|_{BV} + \Psi(\|x\|_{BV}) T'.
\end{aligned}
$$

Thus

$$
\frac{\|x\|_{BV}}{\|g\|_{BV} + \Psi(\|x\|_{BV})T'} \le 1. \tag{8}
$$

Now $\|x\|_{BV} = M_0$, so (8) implies

$$
\frac{M_0}{\|g\|_{BV} + \Psi(M_0)T'} \le 1,
$$

which contradicts 17^0. Apply the nonlinear alternative of Leray-Schauder type for contraction in Section 1 to deduce that G has a fixed point in B_{M_0}. This fixed point is a BV-solution of (1).

For equation (1) with the Denjoy-Perron integral we can also consider continuous BV-solutions. For this purpose we need the additional assumption:

19^0 for each $t \in I$ for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\tau \in I$:

$$
|t - \tau| < \delta \Longrightarrow \sup_{s \in I} \left| \int_0^s [K(t,z) - K(\tau,z)]\,dz \right| < \varepsilon.
$$

Assumptions 9^0 and 19^0 guarantee that the mapping G defined in the proof of Theorem 1 maps $\bar{B}_{M_0} \subset BV_C(I)$ into $BV_C(I)$. Indeed, for

$t, \tau \in I$ and $x \in \bar{B}_{M_0}$ we have

$$
\begin{aligned}
&|G(x)(t) - G(x)(\tau)| \\
&\leq |g(t) - g(\tau)| + \left| \int_0^T [K(t,s) - K(\tau,s)] f(x(s))\, ds \right| \\
&\leq |g(t)-g(\tau)| + |f(x(0))| \left| \int_0^T [K(t,s)-K(\tau,s)]\, ds \right| + \bigvee_0^T (f(x)) \cdot O(H_\tau^t; I) \\
&\leq |g(t)-g(\tau)| + \|f(x)\|_{BV} \max\left(\left| \int_0^T [K(t,s)-K(\tau,s)]\, ds \right|, O(H_\tau^t; I) \right),
\end{aligned}
$$

where $H_\tau^t(s) = \int_0^s [K(t,z) - K(\tau,z)]\, dz$ for $s \in I$. Hence the same reasoning as in Theorem 5 yields

Theorem 6. *Suppose* 2^0, 9^0, $14^0 - 18^0$ *are satisfied. Then equation* (1) *has a continuous* BV*-solution defined on* I.

Now, consider equation (2), where "$\int_0^t$" now stands for the integral in the Denjoy-Perron sense. As in Section 3 it is easily seen that, instead of (2) one can consider an equivalent form (7), where "$\int_0^t$" now stands for the Denjoy-Perron integral. Suppose 1^0, 2^0 and 16^0 are satisfied. Moreover assume that

20^0 $\tilde{T} = \{(t,s) : 0 \leq t \leq T,\, 0 \leq s \leq t\}$ and $K : \tilde{T} \to \mathbf{R}$ is a function such that $K(t,\cdot)$ is integrable in the Denjoy-Perron sense for every $t \in I$ and

$$
\bigvee_0^T \left(\int_0^t K(t,s)\, ds \right) < +\infty;
$$

21^0 there exists a number $0 < c < +\infty$ such that

$$
\sup_{0=t_0 \ldots < t_n = T} \sum_{i=1}^n O\left(\int_0^s K(t_i, z)\, dz - \int_0^{\min(t_{i-1}, s)} K(t_{i-1}, z)\, dz; [0, t_i] \right) < c;
$$

22^0 there exists $M_0 > 0$ with $M_0/(\|g\|_{BV} + \Psi(M_0) \cdot T'') > 1$, where $T'' = \max(\bigvee_0^T (\int_0^t K(t,s)\,ds, c)$.

Finally, we assume that 18^0 with M_0 defined above, is satisfied.

Theorem 7. *Under the above assumptions equation* (2) *has a BV-solution, defined on I.*

Proof. The result follows immediately from Theorem 5. Indeed, we have $\bigvee_0^T (\int_I \tilde{K}(t,s)\,ds = \bigvee_0^T (\int_0^t K(t,s)\,ds) < +\infty$, so $\tilde{K}$ satisfies 14^0. Furthermore, we have

$$\sup_{0=t_0\ldots<t_n=T} \sum_{i=1}^n O\Bigg(\int_0^s (\tilde{K}(t_i,z) - \tilde{K}(t_{i-1},z)\,dz; I \Bigg)$$

$$= \sup_{0=t_0\ldots<t_n=T} \sum_{i=1}^n O\Bigg(\int_0^s K(t_i,z)\,dz - \int_0^{\min(t_{i-1},s)} K(t_{i-1},z)\,dz; [0,t_i] \Bigg) < c,$$

so $\tilde{K}$ satisfies 15^0. Moreover, $\|\int_0^t K(t,s) ds\| = \bigvee_0^T (\int_0^t K(t,s)\,ds)$ and $P(s) = 0$ for $s \in I$, and thus one can take $T' = T''$. $\square$

Finally let's consider continuous BV-solutions of equation (2) with the Denjoy-Perron integral. We need two additional assumptions:

23^0 for each $t \in I$ and for each $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\tau \in I$:

$$|t-\tau| < \delta \Longrightarrow \sup_{s\in[0,\min(t,\tau)]} \left| \int_0^s [K(t,z) - K(\tau,z)]\,dz \right| < \varepsilon;$$

24^0 $\lim_{\tau\to t^+} \sup_{t<s\le\tau} \int_t^s K(\tau,z)\,dz = 0$ for fixed $t \in I$.

We can prove the following

Theorem 8. *Assume that* 9^0, 2^0, 16^0, 18^0, 20^0–24^0 *are satisfied. Then equation* (2) *has a continuous BV-solution, defined on I.*

Proof. Fix $t \in I$ and let $\tau \in (t, T)$. Consider the mapping G defined in the proof of Theorem 1. Then for $x \in \bar{B}_{M_0} \subset BV_C(I)$ we have

$$\begin{aligned}
&|G(x)(t) - G(x)(\tau)| \\
&\le |g(t) - g(\tau)| + \left| \int_0^t [K(t,s) - K(\tau,s)] f(x(s))\, ds \right| + \left| \int_t^\tau K(\tau,s) f(x(s))\, ds \right| \\
&\le |g(t) - g(\tau)| + |f(x(0))| \left| \int_0^t [K(t,s) - K(\tau,s)]\, ds \right| \\
&\quad + \bigvee_0^t (f(x)) \cdot O(H_\tau^t; [0,t]) + |f(x(t))| \left| \int_t^\tau K(\tau,s)\, ds \right| \\
&\quad + \bigvee_t^\tau (f(x)) \cdot O\biggl(\int_t^s K(\tau,z)\, dz; [t,\tau] \biggr) \\
&\le |g(t) - g(\tau)| + \|f(x)\|_{BV} \max \left\{ \left| \int_0^t [K(t,s) - K(\tau,s)]\, ds \right| ; O(H_\tau^t; [0,t]) \right\} \\
&\quad + \sup_{t \in [-M_0, M_0]} |f(t)| \left| \int_t^\tau K(\tau,s)\, ds \right| + \bigvee_0^T (f(x)) \cdot O\biggl(\int_t^s K(\tau,z)\, dz; [t,\tau] \biggr).
\end{aligned}$$

By 9^0, 23^0 and 24^0 we infer that in this case $|G(x)(t) - G(x)(\tau)|$ can be made sufficiently small. Similar reasoning is used in the case when $\tau \in (0, t)$. Hence the mapping G maps $\bar{B}_{M_0}$ into $BV_C(I)$. Essentially the same reasoning as in Theorem 5 establishes the result.

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