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ON THE LIMIT MEASURE TO STOCHASTIC VOLTERRA EQUATIONS

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ABSTRACT. The paper is concerned with a limit measure of stochastic Volterra equation driven by a spatially homogeneous Wiener process with values in the space of real tempered distributions. Necessary and sufficient conditions for the existence of the limit measure are provided and a form of any limit measure is given as well.

1. Introduction. In the paper we investigate the limit measure to the stochastic Volterra equation of the form

$$X(t,\theta) = \int_0^t v(t-\tau) AX(\tau,\theta)\, d\tau + X_0(\theta) + W(t,\theta), \tag{1}$$

where $t \in \mathbf{R}_+$, $\theta \in \mathbf{R}^d$, $v \in L^1_{\text{loc}}(\mathbf{R}_+)$, $X_0 \in S'(\mathbf{R}^d)$ and W is a spatially homogeneous Wiener process which takes values in the space of real, tempered distributions $S'(\mathbf{R}^d)$. The class of operators A contains, in particular, the Laplace operator Δ and its fractional powers $-(-\Delta)^{\alpha/2}$, $\alpha \in (0,2]$.

Description of asymptotic properties of solutions to stochastic evolution equations in finite dimensional spaces and Hilbert spaces is well known and has been collected in the monograph [**7**]. Recently this problem has been studied for generalized Langevin equations in conuclear spaces by Bojdecki and Jakubowski [**1**]. The question of existence of invariant and limit measures in the space of distributions seems to be particularly interesting, especially for stochastic Volterra equations, because this class of equations is not well investigated.

In the paper we give a necessary and sufficient condition for the existence of a limit measure and describe all limit measures to the equation (1). Our results are in a sense analogous to those formulated for the

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finite-dimensional and Hilbert space cases obtained for stochastic evolution equations, see [**7**, Chapter 6].

The paper is organized as follows. We start by recalling some concepts and results from the paper [**12**], where the regularity of stochastic convolutions connected with stochastic Volterra equations driven by a spatially homogeneous Wiener process was studied. In [**12**], we considered existence of solutions to (1) in the space of tempered distributions $S'(\mathbf{R}^d)$ and next derived conditions under which the solutions to (1) were function-valued or continuous with respect to the space variable. In Section 2 we recall some facts concerning generalized homogeneous Gaussian random fields. Section 3 contains the precise meaning and properties of the stochastic integral with values in the space of tempered distributions $S'(\mathbf{R}^d)$. Section 4 consists of the auxiliary results for stochastic Volterra equations recalled from [**12**]. In Section 5 we formulate and prove the main results of the paper providing the existence and the form of the limit measure for the stochastic Volterra equation (1). Section 6 presents some special case of limit measure.

2. Generalized homogeneous Gaussian random fields. By $S(\mathbf{R}^d)$, $S_c(\mathbf{R}^d)$, we denote, respectively, the spaces of all infinitely differentiable rapidly decreasing real and complex functions on $\mathbf{R}^d$ and by $S'(\mathbf{R}^d)$, $S'_c(\mathbf{R}^d)$ the spaces of real and complex tempered distributions. The value of a distribution $\xi \in S'_c(\mathbf{R}^d)$ on a test function ψ will be written as $\langle \xi, \psi \rangle$. For $\psi \in S(r^d)$ we set $\psi_{(s)}(\theta) = \overline{\psi(-\theta)}$, where $\theta \in \mathbf{R}^d$.

We denote by $\mathcal{F}$ the Fourier transform on $S_c(\mathbf{R}^d)$ and on $S'_c(\mathbf{R}^d)$. Let us note that, if $\xi \in S'_c(\mathbf{R}^d)$, then $\langle \mathcal{F}\xi, \psi \rangle = \langle \xi, \mathcal{F}^{-1}\psi \rangle$ for all $\psi \in S_c(\mathbf{R}^d)$ and that $\mathcal{F}$ transforms the space of tempered distributions $S'(\mathbf{R}^d)$ into $S'(\mathbf{R}^d)$.

For any $h \in \mathbf{R}^d$, $\psi \in S(\mathbf{R}^d)$ and $\xi \in S'(\mathbf{R}^d)$, the *translations* $\tau_h\psi, \tau'_h\xi$ are defined as follows: $\tau_h\psi(x) = \psi(x-h)$ and $\langle \tau'_h\xi, \psi \rangle = \langle \xi, \tau_h\psi \rangle$ for $x \in \mathbf{R}^d$.

By $\mathcal{B}(S'(\mathbf{R}^d))$ and $\mathcal{B}(S'_c(\mathbf{R}^d))$, we denote the smallest σ-algebras of subsets of $S'(\mathbf{R}^d)$ and $S'_c(\mathbf{R}^d)$, respectively, such that for any test function φ the mapping $\xi \to \langle \xi, \varphi \rangle$ is measurable.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Any measurable mapping $Y : \Omega \to S'(\mathbf{R}^d)$ will be called a *generalized random field*. A generalized random field Y is called *Gaussian* if $\langle Y, \varphi\rangle$ is a Gaussian random variable for any $\varphi \in S(\mathbf{R}^d)$. One says that a generalized random field Y is *homogeneous* or *stationary* if, for all $h \in \mathbf{R}^d$, the translation $\tau'_h(Y)$ of Y has the same distribution as Y.

If Y is a homogeneous, Gaussian random field, then for each $\psi \in S(\mathbf{R}^d)$, $\langle Y, \psi\rangle$ is a Gaussian random variable and the bilinear functional $p : S(\mathbf{R}^d) \times S(\mathbf{R}^d) \to \mathbf{R}$ defined by the formula $p(\varphi, \psi) = \mathcal{E}(\langle Y, \varphi\rangle\langle Y, \psi\rangle)$ for $\varphi, \psi \in S(\mathbf{R}^d)$ is continuous and positive-definite. Since $p(\varphi, \psi) = p(\tau_h\varphi, \tau_h\psi)$ for all $\varphi, \psi \in S(\mathbf{R}^d)$, $h \in \mathbf{R}^d$, there exists (see e.g. [**8**]) a unique positive-definite distribution $\Gamma \in S'(\mathbf{R}^d)$ such that for all $\varphi, \psi \in S(\mathbf{R}^d)$, one has $p(\varphi, \psi) = \langle \Gamma, \varphi * \psi_{(s)}\rangle$.

The distribution Γ is called the *space correlation* of the field Y. By Bochner-Schwartz theorem the positive-definite distribution Γ is the inverse Fourier transform of a unique positive, symmetric, slowly increasing measure μ on $\mathbf{R}^d : \Gamma = \mathcal{F}^{-1}(\mu)$. The measure μ is called the *spectral measure* of Γ and of the field Y. Let us recall from [**8**], [**14**] or [**15**] that the symmetric, non-negative measure μ on $\mathbf{R}^d$ is called *slowly increasing* (or *tempered*) if $\int_{\mathbf{R}^d}(1+|x|^2)^{-k}\, d\mu(x) < +\infty$ for some $k > 0$.

Any random field Y may be identified with the family of random variables $\{Y_0\}$ parametrized by $\theta \in \mathbf{R}^d$. In particular, a homogeneous Gaussian random field is a family of Gaussian random variables $Y(\theta)$, $\theta \in \mathbf{R}^d$, with Gaussian laws invariant with respect to all translations. That is, for any $\theta_1, \dots, \theta_n \in \mathbf{R}^d$ and any $h \in \mathbf{R}^d$, the law of $(Y(\theta_1 + h), \dots, (Y(\theta_n + h))$ does not depend on h.

In the paper we assume that W is a spatially homogeneous Wiener process with values in the space of real tempered distributions $S'(\mathbf{R}^d)$. This means that W is a continuous process with independent increments taking values in $S'(\mathbf{R}^d)$. Moreover, the process W is space homogeneous in the sense that, for each $t \geq 0$, random variables $W(t)$ are stationary, Gaussian, generalized random fields.

Let us recall two examples of spatially homogeneous Wiener processes.

Example 1. Symmetric α-stable distributions $\Gamma(x) = e^{-|x|^\alpha}$, with $\alpha \in (0, 2]$ provide examples of random fields. For $\alpha = 1$ and $\alpha = 2$, the densities of the spectra measures are given by the formulas

$c_1(1+|x|^2)^{(-d+1)/2}$ and $c_2 e^{-|x|^2}$, where c_1 and c_2 are appropriate constants.

Example 2. Let $p(\varphi,\psi)=\langle\varphi,\psi\rangle$, $\varphi,\psi\in S(\mathbf{R}^d)$. Then Γ is equal to the Dirac δ_0-function, its spectral density $d\mu/dx$ is the constant function $(2\pi)^{-d/2}$ and $\partial W/\partial t$ is a white noise on $L^2([0,\infty)\times\mathbf{R}^d)$. If $B(t,x)$, $t\geq 0$ and $x\in\mathbf{R}^d$ is a Brownian sheet on $[0,\infty)\times\mathbf{R}^d$, then W can be defined by the formula

$$W(t,x)=\frac{\partial^d B(t,x)}{\partial x_1\ldots\partial x_d},\quad t\geq 0.$$

3. Stochastic integration. In the paper we integrate operator-valued functions $\mathcal{R}(t)$, $t\geq 0$, with respect to a spatially homogeneous Wiener process W. The operators $\mathcal{R}(t)$, $t\geq 0$, are non-random and act from some linear subspaces of $S'(\mathbf{R}^d)$ into $S(\mathbf{R}^d)$. By Γ we denote the covariance of $W(1)$ and the associated spectral measure by μ. To underline the fact that the distributions of W are determined by Γ we will write W_Γ. We denote by q a scalar product on $S(\mathbf{R}^d)$ given by the formula $q(\varphi,\psi)=\langle\Gamma,\varphi*\psi_{(s)}\rangle$, where $\varphi,\psi\in S(\mathbf{R}^d)$. In other words (see e.g. [**1**]) such a process W may be called *associated* with q.

The crucial role in the theory of stochastic integration with respect to W_Γ is played by the Hilbert space $S'_q\subset S'(\mathbf{R}^d)$ called the *kernel* of W_Γ. Namely the space S'_q consists of all distributions $\xi\in S'(\mathbf{R}^d)$ for which there exists a constant C such that

$$|\langle\xi,\psi\rangle|\leq C\sqrt{q(\psi,\psi)},\quad \psi\in S(\mathbf{R}^d).$$

The norm in S'_q is given by the formula

$$|\xi|_{S'_q}=\sup_{\psi\in S}\frac{|\langle\xi,\psi\rangle|}{\sqrt{q(\psi,\psi)}}.$$

Let us assume that we require that the stochastic integral should take values in a Hilbert space H continuously imbedded into $S'(\mathbf{R}^d)$. Let $L_{HS}(S'_q,H)$ be the space of Hilbert-Schmidt operators from S'_q into H.

Assume that $\mathcal{R}(t)$, $t \geq 0$, is measurable $L_{HS}(S'_q, H)$-valued function such that

$$\int_0^t \|\mathcal{R}(\sigma)\|^2_{L_{HS}(S'_q,H)}\, d\sigma < +\infty, \quad \text{for all } t \geq 0.$$

Then the stochastic integral

$$\int_0^t \mathcal{R}(\sigma)\, dW_\Gamma(\sigma), \quad t \geq 0$$

can be defined in a standard way, see [**10**] or [**6**].

Of special interest are operators $\mathcal{R}(t)$, $t \geq 0$, of convolution type

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi, \quad t \geq 0, \quad \xi \in S'(\mathbf{R}^d),$$

with $\mathbf{r}(t) \in S'(\mathbf{R}^d)$. The convolution operator is not, in general, defined for all $\xi \in S'(\mathbf{R}^d)$. For many cases the Fourier transform $\mathcal{F}\mathbf{r}(t)(\lambda)$, $t \geq 0$, $\lambda \in \mathbf{R}^d$, is continuous in both variables and, for any $T \geq 0$,

$$\sup_{t\in[0,T]} \sup_{\lambda\in\mathbf{R}^d} |\mathcal{F}\mathbf{r}(t)(\lambda)| = M_T < +\infty. \tag{2}$$

Then the above convolution is well-defined and the operators $\mathcal{R}(t)$ may be defined as the Fourier inverse transform

$$\mathcal{R}(t)\xi = \mathcal{F}^{-1}(\mathcal{F}\mathbf{r}(t)\mathcal{F}(\xi),$$

for all ξ such that $\mathcal{F}\xi$ has a representation as a function.

Proposition 1 [**12**, Theorem 1]. *Assume that the function $\mathcal{F}\mathbf{r}$ is continuous in both variables and satisfies* (2). *Then the stochastic convolution*

$$\mathcal{R} * W_\Gamma(t) = \int_0^t \mathcal{R}(t-\sigma)\, dW_\Gamma(\sigma), \quad t \geq 0,$$

is a well-defined $S'(\mathbf{R}^d)$-valued stochastic process. For each $t \geq 0$, $\mathcal{R} * W_\Gamma(t)$ is a Gaussian, stationary, generalized random field with the spectral measure

$$\mu_t(d\lambda) = \left(\int_0^t |\mathcal{F}\mathbf{r}(\sigma)(\lambda)|^2\, d\sigma\right)\mu(d\lambda),$$

and with the covariance Γ_t

$$\Gamma_t = \int_0^t \mathbf{r}(\sigma) * \Gamma * \mathbf{r}_{(s)}(\sigma)\, d\sigma.$$

4. Auxiliary results. In this section we set the considered problem more precisely and recall from [**12**] auxiliary theorem which will be useful for formulating the limit results in the next section.

Let us rewrite the stochastic Volterra equation (1) in the simpler form

$$X(t) = \int_0^t v(t-\tau) A X(\tau)\, d\tau + X_0 + W_\Gamma(t). \tag{3}$$

We study the equation (3) in the space $S'(\mathbf{R}^d)$ where $X_0 \in S'(\mathbf{R}^d)$, A an operator given in the Fourier transform form

$$\mathcal{F}(A\xi)(\lambda) = -a(\lambda)\mathcal{F}(\xi)(\lambda), \tag{4}$$

v is a locally integrable function and W_Γ is an $S'(\mathbf{R}^d)$-valued space homogeneous Wiener process.

Note that if $a(\lambda) = |\lambda|^2$, then $A = \Delta$ and if $a(\lambda) = |\lambda|^\alpha$, $\alpha \in (0,2)$, then $A = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian. Some other cases of equation (3) with the operator A in the form (4) are provided in the paper [**12**].

The deterministic version of the equation has been investigated by many authors, see the monographs: Gripenberg, Londen and Staffans [**9**] and Prüss [**13**]. Stochastic Volterra equation (3) has been considered, among others, by Clément, Da Prato and Prüss [**2–5**].

We shall assume the following

Hypothesis (H). *For any* $\gamma \geq 0$, *the unique solution* $s(\cdot,\gamma)$ *to the equation*

$$\mathbf{s}(t) + \gamma \int_0^t v(t-\tau)\mathbf{s}(\tau)\, d\tau = 1, \quad t \geq 0 \tag{5}$$

fulfills the following condition: for any $T \geq 0$, $\sup_{t\in[0,T]} \sup_{\gamma\geq 0} |\mathbf{s}(t,\gamma)| < +\infty$.

Comment 1. Let us note that by assumption the function v is a locally integrable function. The solution $s(\cdot,\gamma)$ of the equation (5) is a locally integrable function and measurable with respect to both variables $\gamma \geq 0$ and $t \geq 0$.

For some special cases the function $\mathbf{s}(t,\gamma)$ may be found explicitly

$$\begin{aligned}
&\text{for } v(t)=1, && \mathbf{s}(t,\gamma)=e^{-\gamma t}, \quad t\geq 0, \quad \gamma \geq 0;\\
&\text{for } v(t)=1, && \mathbf{s}(t,\gamma)=\cos(\sqrt{\gamma}t), \quad t\geq 0, \quad \gamma \geq 0;\\
&\text{for } v(t)=e^{-t}, && \mathbf{s}(t,\gamma)=(1+\gamma)^{-1}[1+\gamma e^{-(1+\gamma)t}], \quad t\geq 0, \quad \gamma\geq 0.
\end{aligned}$$

The linear stochastic Volterra equation (3) with the operator A given in the form (4) is determined by three objects: the spatial correlation Γ of the process W_Γ, the operator A and the function v. Equivalently, the equation (3) with (4) is determined by the spectral measure μ, the functions a and $\mathbf{s}$, respectively.

Let us introduce the so-called *resolvent* family $\mathcal{R}(\cdot)$ determined by the operator A and the function v:

$$\mathcal{R}(t)\xi = \mathbf{r}(t) * \xi, \quad \xi \in S'(\mathbf{R}^d),$$

where

$$\mathbf{r}(t) = \mathcal{F}^{-1}\mathbf{s}(t, a(\cdot)), \quad t \geq 0.$$

As in the deterministic case, the solution to the stochastic Volterra equation (3) is of the form

$$X(t) = \mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau)\, dW_\Gamma(\tau), \quad t \geq 0. \tag{6}$$

We have the following corollary of the previous results on stochastic integration.

Proposition 2 [**12**, Theorem 2]. *Let W_Γ be a spatially homogeneous Wiener process and $\mathcal{R}(t)$, $t \geq 0$, the resolvent for the equation* (3). *If Hypothesis* (H) *holds, then the stochastic convolution*

$$Z(t) = \int_0^t \mathcal{R}(t-\sigma)\, dW_\Gamma(\sigma), \quad t \geq 0, \tag{7}$$

is a well-defined $S'(\mathbf{R}^d)$-valued process. For each $t \geq 0$, the random variable $Z(t)$ is a Gaussian, generalized, stationary random field on $\mathbf{R}^d$ with the spectral measure

$$\mu_t(d\lambda) = \left[\int_0^t (\mathbf{s}(\sigma, a(\lambda)))^2 \, d\sigma\right] \mu(d\lambda). \tag{8}$$

5. The main results. In this section we formulate results providing the existence of a limit measure and the form of any limit measure for the stochastic Volterra equation (3) with the operator A given by (4). In our considerations we assume that the hypothesis H holds and then we use the auxiliary results recalled in Section 4. Particularly, we use the fact (see Proposition 2) that the stochastic convolution defined by (7) is a Gaussian field with the spectral measure μ_t given by (8).

First let us introduce the following notation. By ν_t we denote the law $\mathcal{L}(Z(t)) = \mathcal{N}(0, \Gamma_t)$ of the process $Z(t)$, $t \geq 0$, the stochastic convolution defined by the formula (7).

Let us define

$$\mu_\infty(d\lambda) = \left[\int_0^\infty (\mathbf{s}(\sigma, a(\lambda)))^2 \, d\sigma\right] \mu(d\lambda). \tag{9}$$

From now on we assume that the hypothesis H *is satisfied.*

We have the following results.

Lemma 1. *Let μ_t and μ_∞ be measures defined by* (8) *and* (9), *respectively. If μ_∞ is a slowly increasing measure, then the measures $\mu_t \to \mu_\infty$ as $t \to +\infty$ in the distribution sense.*

Proof. First of all, let us notice that, by Proposition 2, the measures μ_t, $t \geq 0$, are spectral measures of stationary generalized Gaussian random fields. Moreover, the measures μ_t, $t \geq 0$, are slowly increasing. Since the function $\mathbf{s}(\tau, a(\lambda))$, $\tau \geq 0$, $\lambda \in \mathbf{R}^d$ is bounded, then the integral $g_t(\lambda) = \int_0^t (\mathbf{s}(\tau, a(\lambda)))^2 \, d\tau$, for $t < +\infty$, is bounded as well. In the proof we shall use the specific form of the measures μ_t, $t \geq 0$, defined by (8).

We assume that the measure μ_∞ is slowly increasing, that is, there exists $k > 0$:

$$\int_{\mathbf{R}^d} (1+|\lambda|^2)^{-k}\, d\mu_\infty(\lambda) = \int_{\mathbf{R}^d} (1+|\lambda|^2)^{-k} \left[\int_0^\infty (\mathbf{s}(t, a(\lambda)))^2\, d\tau \right] d\mu(\lambda) < +\infty.$$

Hence, the function $g_\infty(\lambda) = \int_0^\infty (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau < +\infty$ for almost every λ.

Convergence of measures in the distribution sense is a special kind of weak convergence of measures. This means that

$$\int_{\mathbf{R}^d} \varphi(\lambda)\, d\mu_t(\lambda) \overset{t\to+\infty}{\longrightarrow} \int_{\mathbf{R}^d} \varphi(\lambda)\, d\mu_\infty(\lambda) \tag{10}$$

for any test function $\varphi \in S(\mathbf{R}^d)$.

In our case, because of formulae (8) and (9), we have to prove the following convergence

$$\lim_{t\to+\infty} \int_{\mathbf{R}^d} \varphi(\lambda) g_t(\lambda)\, d\mu(\lambda) = \int_{\mathbf{R}^d} \varphi(\lambda) g_\infty(\lambda)\, d\mu(\lambda), \tag{11}$$

where $\varphi \in S(\mathbf{R}^d)$ and g_t and g_∞ are as above.

In other words, the convergence (10) of the measures μ_t, $t \geq 0$, to the measure μ_∞ in the distribution sense, in our case is equivalent to the weak convergence (11) of functions g_t, $t \geq 0$, to the function g_∞.

Let us recall that the function $\mathbf{s}$ determining the measures μ_t, $t \geq 0$, and μ_∞, satisfies the Volterra equation (5) (see Hypothesis (H)):

$$\mathbf{s}(t) + \gamma \int_0^t v(t-\tau)\mathbf{s}(\tau)\, d\tau = 1.$$

Additionally, by Lemma 2.1 from [**3**], $\lim_{t\to+\infty} \mathbf{s}(t) = 0$.

We can estimate as follows, for any $\varphi \in S(\mathbf{R}^d)$,

(12)

$$\begin{aligned}\Big| \int_{\mathbf{R}^d} \varphi(\lambda) g_t(\lambda)\, d\mu(\lambda) &= \int_{\mathbf{R}^d} \varphi(\lambda) g_\infty(\lambda)\, d\mu(\lambda) \Big| \\ &\le \int_{\mathbf{R}^d} |\varphi(\lambda)|\, |g_t(\lambda) - g_\infty(\lambda)|\, d\mu(\lambda) \\ &= \int_{\mathbf{R}^d} |\varphi(\lambda)| \, \Big| \int_0^t (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau \\ &\quad - \int_0^{+\infty} (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau \Big|\, d\mu(\lambda) \\ &\le \int_{\mathbf{R}^d} |\varphi(\lambda)| \Big(\int_t^{+\infty} (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau \Big)\, d\mu(\lambda).\end{aligned}$$

The right-hand side of (12) tends to zero because $h_\infty(\lambda) = \int_t^{+\infty} (\mathbf{s}(\tau, a(\lambda)))^2\, d\tau$ tends to zero when $t \to +\infty$.

Hence, we have proved the convergence (11) which is equivalent to the convergence (10) of the measures μ_t, as $t \to +\infty$, to the measure μ_∞ in the distribution sense. □

Lemma 2. *Let Γ_t, Γ_∞ be covariance kernels of the stochastic convolution* (7) *for $t < +\infty$ and $t = +\infty$, respectively, and μ_t, μ_∞ are defined by* (8) *and* (9). *Assume that μ_∞ is a slowly increasing measure on $\mathbf{R}^d$. Then $\Gamma_1 \to \Gamma_\infty$, as $t \to +\infty$, in the distribution sense if and only if the measures $\mu_t \to \mu_\infty$, for $t \to +\infty$, in the distribution sense.*

Proof. The sufficiency comes from the convergence of measures in the distribution sense which, in fact, is a type of weak convergence of measures. Actually, the convergence of μ_t, $t \ge 0$, to the measure μ_∞ in the distribution sense means that $\langle \mu_t, \varphi \rangle \overset{t\to+\infty}{\longrightarrow} \langle \mu_\infty, \varphi \rangle$ for any $\varphi \in S(\mathbf{R}^d)$. Particularly, because the Fourier transform acts from $S(\mathbf{R}^d)$ into $S(\mathbf{R}^d)$, we have $\langle \mu_t, \mathcal{F}(\varphi) \rangle \overset{t\to+\infty}{\to} \langle \mu_\infty, \mathcal{F}(\varphi) \rangle$ for any $\varphi \in S(\mathbf{R}^d)$. This is equivalent to the convergence $\langle \mathcal{F}^{-1}(\mu_t), \varphi \rangle \overset{t\to+\infty}{\to} \langle \mathcal{F}^{-1}(\mu_\infty), \varphi) \rangle$, $\varphi \in S(\mathbf{R}^d)$.

This means the convergence of the Fourier inverse transforms of considered measures μ_t, as $t \to +\infty$, to the inverse transform of the measure μ_∞ in the distribution sense. Because the measures μ_t, $t \geq 0$ and μ_∞ are positive, symmetric and slowly increasing on $\mathbf{R}^d$, then their Fourier inverse transforms define, by Bochner-Schwartz theorem, covariance kernels $\Gamma_t = \mathcal{F}^{-1}(\mu_t)$, $t \geq 0$, and $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$, respectively. Hence, $\Gamma_t \to \Gamma_\infty$ as $t \to +\infty$, in the distribution sense.

The necessity is the version of Lévy-Cramér's theorem generalized for a sequence of slowly increasing measures $\{\mu_t\}$, $t \geq 0$, and their Fourier inverse transforms which are their characteristic functionals. □

Now we are able to formulate the main results of the paper.

First let us recall the definition of weak convergence of probability measures defined on the space $S'(\mathbf{R}^d)$ of tempered distributions.

Definition. A sequence $\{\gamma_t\}$ of probability measures on $S'(\mathbf{R}^d)$ converges weakly to probability measure γ on $S'(\mathbf{R}^d)$ if, for any function $f \in C_b(S')$,

$$\lim_{t\to+\infty} \int_{S'(\mathbf{R}^d)} f(x)\gamma_t(dx) = \int_{S'(\mathbf{R}^d)} f(x)\gamma(dx). \tag{13}$$

A more general definition on weak convergence of probability measures defined on topological spaces may be found, e.g., in [**11**].

Theorem 1. *There exists the limit measure $\nu_\infty = \mathcal{N}(0,\Gamma_\infty)$, the weak limit of the measures $\nu_t = \mathcal{N}(0,\Gamma_t)$, as $t \to +\infty$, if and only if the measure μ_∞ defined by* (9) *is slowly increasing.*

Theorem 2. *Assume that the measure μ_∞ defined by* (9) *is slowly increasing. Then any limit measure of the stochastic Volterra equation* (3) *is of the form*

$$m_\infty * \mathcal{N}(0,\Gamma_\infty), \tag{14}$$

where m_∞ is the limit measure for the deterministic version of the equation (3) *with condition* (4) *and $\mathcal{N}(0,\Gamma_\infty)$ is the limit measure of the measures ν_t as $t \to +\infty$.*

We would like to emphasize that Theorems 1 and 2 have been formulated in the spirit analogous to well-known theorems giving invariant measures for linear evolution equations, see, e.g., [**7**] or [**1**]. Such results first give conditions for the existence of invariant measure and next describe all invariant measures provided they exist. Our theorems extend, in some sense, Theorem 6.2.1 from [**7**]. Because we consider stochastic Volterra equations we cannot study invariant measures but limit measures.

5.1 Proofs of theorems.

Proof of Theorem 1. (Necessity.) Let us notice that, by Proposition 2, the laws $\nu_t = \mathcal{N}(0, \Gamma_t)$, $t \geq 0$, are laws of Gaussian, stationary, generalized random fields with the spectral measures μ_t and the covariances Γ_t. The weak convergence (13) is equivalent to the convergence of the characteristic functionals corresponding to the measures ν_t, $t \geq 0$ and ν_∞, respectively. Particularly,

$$\hat{\nu}_t(\varphi) \overset{t\to+\infty}{\longrightarrow} \hat{\nu}_\infty(\varphi) \quad \text{for any } \varphi \in S(\mathbf{R}^d).$$

We may use the specific form of the characteristic functionals of Gaussian fields. Namely, we have

$$\hat{\nu}_t(\varphi) = \mathbf{E} e^{i\langle Z(t), \varphi\rangle} = \exp\left(-\frac{1}{2}\, q_t(\varphi,\varphi)\right) = \exp\left(-\frac{1}{2}\, \langle \Gamma_t, \varphi * \varphi_{(s)}\rangle\right),$$

where $t \geq 0$, $\varphi \in S(\mathbf{R}^d)$ and $Z(t)$ is the stochastic convolution given by (7).

Analogously,

$$\hat{\nu}_\infty(\varphi) = \exp\left(-\frac{1}{2}\, \langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\right), \quad \varphi \in S(\mathbf{R}^d).$$

Hence, we have the following convergence

$$\exp\left(-\frac{1}{2}\, \langle \Gamma_t, \varphi * \varphi_{(s)}\rangle\right) \overset{t\to+\infty}{\longrightarrow} \exp\left(-\frac{1}{2}\, \langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\right)$$

for any $\varphi \in S(\mathbf{R}^d)$.

Because Γ_t, $t \geq 0$, are positive-definite generalized functions, then Γ_∞ is a positive-definite generalized function, too. So by the Bochner-Schwartz theorem, there exists a slowly increasing measure μ_∞ such that $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$.

(Sufficiency.) Assume that the measure μ_∞, defined by the formula (9) is slowly increasing. Then, by the Bochner-Schwartz theorem, there exists a positive-definite distribution Γ_∞ on S such that $\Gamma_\infty = \mathcal{F}^{-1}(\mu_\infty)$ and

$$\langle \Gamma_\infty, \varphi \rangle = \int_{\mathbf{R}^d} \varphi(x)\, d\mu_\infty(x).$$

Now we have to show that Γ_∞ is the limit, in the distribution sense, of the functionals Γ_t, $t \geq 0$. In order to do this, by Lemma 2, we have to prove the convergence of the spectral measures $\mu_t \to \mu_\infty$ as $t \to +\infty$ in the distribution sense. But, by Lemma 1, the measures μ_t, $t \geq 0$, defined by (8), converge to the measure μ_∞ in the distribution sense. This fact implies, by Lemma 2, that $\Gamma_t \to \Gamma_\infty$, as $t \to +\infty$, in the distribution sense.

Then, the following convergence

$$\exp\left(-\frac{1}{2}\langle \Gamma_t, \varphi * \varphi_{(s)} \rangle\right) \overset{t\to+\infty}{\longrightarrow} \exp\left(-\frac{1}{2}\langle \Gamma_\infty \varphi * \varphi_{(s)} \rangle\right)$$

holds for any $\varphi \in S(\mathbf{R}^d)$. This means the convergence of characteristic functionals of the measures $\nu_t = \mathcal{N}(0, \Gamma_t)$, $t \geq 0$, to the characteristic functional of the measure $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$. Hence, there exists the weak limit ν_∞ of the sequence ν_t, $t \geq 0$, and $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$. □

Proof of Theorem 2. Now we consider a limit measure for the stochastic Volterra equation (3) with the condition (4). This means that we study a limit distribution of the solution given by (6) to the considered equation (3).

Let us introduce the following notation for distributions, when $0 \leq t < \infty$:

$\eta_t = \mathcal{L}(X(t))$ means the distribution of the solution $X(t)$;

$m_t = \mathcal{L}(\mathcal{R}(t)X_0)$ denotes the distribution of the part $\mathcal{R}(t)X_0$ of the solution $X(t)$;

$\nu_t = \mathcal{L}(Z(t)) = \mathcal{L}(\int_0^t \mathcal{R}(t-\tau)\, dW_\Gamma(\tau))$ is, as earlier, the distribution of the stochastic convolution $Z(t)$ given by (7), that is, $\nu_t = \mathcal{N}(0, \Gamma_t)$.

We assume that η_∞ is any limit measure of the stochastic Volterra equation (3) with the condition (4). This means that distributions η_t of the solution $X(t)$, as $t \to +\infty$, converge weakly to η_∞.

We have to show the formula (14), that is, the distribution η_∞ has the form $\eta_\infty = m_\infty * \mathcal{N}(0, \Gamma_\infty)$.

The distribution of the solution (6) can be written

$$\mathcal{L}(X(t)) = \mathcal{L}\left(\mathcal{R}(t)X_0 + \int_0^t \mathcal{R}(t-\tau)\, dW_\Gamma(\tau)\right)$$

for any $0 \le t < +\infty$.

Because the initial value X_0 is independent of the process $W_\Gamma(t)$, we have

$$\mathcal{L}(X(t)) = \mathcal{L}(\mathcal{R}(t)X_0) * \mathcal{L}(Z(t))$$

or, using the above notation,

$$\eta_t = m_t * \nu_t \quad \text{for any } 0 \le t < +\infty.$$

This formula can be rewritten in terms of characteristic functionals of the above distributions:

$$\hat{\eta}_t(\varphi) = \hat{m}_t(\varphi)\hat{\nu}_t(\varphi), \tag{15}$$

where $\varphi \in S(\mathbf{R}^d)$ and $0 \le t < +\infty$.

Then, letting t tend to $+\infty$ in (15), we have

$$\hat{\eta}_\infty(\varphi) = \mathcal{C}(\varphi)\hat{\nu}_\infty(\varphi), \quad \varphi \in S(\mathbf{R}^d),$$

where $\hat{\eta}_\infty(\varphi)$ is the characteristic functional of the limit distribution η_∞, $\mathcal{C}(\varphi) = \lim_{t\to\infty} \hat{m}_t(\varphi)$ and $\hat{\nu}_\infty(\varphi)$ is the characteristic functional of the limit measure $\nu_\infty = \mathcal{N}(0, \Gamma_\infty)$; moreover,

$$\hat{\nu}_\infty(\varphi) = \exp\left(-\frac{1}{2}\langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\right).$$

Now we have to prove that $\mathcal{C}(\varphi)$ is the characteristic functional of the weak limit measure m_∞ of the distributions $m_t = \mathcal{L}(\mathcal{R}(t)X_0)$.

In fact,

$$\mathcal{C}(\varphi) = \hat{\eta}_\infty(\varphi) \exp\left(\frac{1}{2}\langle \Gamma_\infty, \varphi * \varphi_{(s)}\rangle\right),$$

where the right-hand side of this formula, as the product of characteristic functionals, satisfies conditions of the generalized Bochner's theorem, see, e.g., [**10**].

So, using the generalized Bochner's theorem once again, there exists a measure m_∞ in $S'(\mathbf{R}^d)$ such that $\mathcal{C}(\varphi) = \hat{m}_\infty$, as required. Hence, we have obtained $\eta_\infty = m_\infty * \mathcal{N}(0, \Gamma_\infty)$. □

6. Some special case. Stochastic Volterra equations have been considered by several authors, see, e.g., [**2–5**] and [**12**], and are studied in connection with problems arising in viscoelasticity. Particularly, in [**3**] the heat equation in materials with memory is treated. In that paper the authors consider an auxiliary equation of the form

$$z(t) + \int_0^t [\mu c(t-\tau) + \beta(t-\tau)] z(\tau)\, d\tau = 1, \tag{16}$$

$t \geq 0$, where μ is a positive constant and c, β are some functions specified below.

Let us notice that, if in the Volterra equation (5), we take $v(\tau) = [\mu c(\tau) + \beta(\tau)]/\gamma$, we arrive at the equation (16). On the contrary, if we assume in the equation (16) that $\beta(\tau) = 0$, $\mu = \gamma$ and $v(\tau) = c(\tau)$, we obtain the equation (5).

Assume, like in [**3**], the following

Hypothesis (H1). 1. *Function β is nonnegative nonincreasing and integrable on $\mathbf{R}^+$.*

2. *The constants μ, c_0 are positive.*

3. *There exists a function $\delta \in L^1(\mathbf{R}^+)$ such that*

$$c(t) := c_0 - \int_0^t \delta(\sigma)\, d\sigma \quad \text{and} \quad c_\infty := c_0 - \int_0^{+\infty} |\delta(\sigma)|\, d\sigma > 0.$$

Proposition 3 [**3**, Lemma 2.1]. *Let functions β, δ and c be as in Hypothesis* (H1). *Then the solution to* (16) *satisfies*

1. $0 \leq |z(t)| \leq 1$, $t \geq 0$;
2. $\int_0^{+\infty} |z(t)|\, dt \leq (\mu c_\infty)^{-1} < +\infty$.

In the next result we will use the above assumption and proposition of Clément and DaPrato and follow the spirit of their argumentation.

Proposition 4. *Assume that the stochastic Volterra equation* (3) *has the kernel function v given in the form*

$$v(t) = c_0 - \int_0^t |\delta(\sigma)|\, d\sigma > 0, \quad c_0 > 0 \quad \textit{where } \delta \in L^1(\mathbf{R}^+),$$

and the operator A is given by (4). *In this case the limit measure μ_∞ given by the formula* (9) *is a slowly increasing measure.*

Proof. This proposition is the direct consequence of the definition (9) of the measure μ_∞ and Proposition 3. In fact, from Proposition 3 we have

$$\int_0^{+\infty} |\mathbf{s}(\tau,\gamma)|\, d\tau \leq (\gamma c_\infty)^{-1}, \tag{17}$$

where γ satisfies Hypothesis (H1), so c_∞ is finite. Hence, the right-hand side of (17) is finite for any finite γ. In our case, because A satisfies (4), $\gamma = a(\lambda)$.

From the definition (9) of the measure μ_∞ we have

(18)

$$\int_{\mathbf{R}^d} (1+|\lambda|^2)^{-k}\, d\mu_\infty(\lambda) = \int_{\mathbf{R}^d} (1+|\lambda|^2)^{-k} \left[\int_0^\infty (\mathbf{s}(t,a(\lambda)))^2\, d\tau \right] d\mu(\lambda)$$

for $k > 0$. Let us notice that, by Proposition 3, $0 \leq |\mathbf{s}(t,a(\lambda))| \leq 1$ for $t \geq 0$. So, $(\mathbf{s}(t,a(\lambda)))^2 \leq |\mathbf{s}(t,a(\lambda))|$. Therefore, because (17) holds and the measure μ is slowly increasing, the right-hand side of (18) is finite. Hence, the measure μ_∞ is slowly increasing, too. □

Comment 2. Let us emphasize that in the special case considered in this section, when the Hypothesis (H1) holds, the main result of the paper, that is, Theorems 1 and 2, may be obtained more easily than in Section 5.

6.1 Concluding remarks. The main result of the paper, Theorem 1, gives an equivalence between the existence of a limit law ν_∞ of the stochastic convolution defined by (7) and the fact that the measure μ_∞, defined by (9), is slowly increasing (tempered). The result is expressed in terms of an auxiliary function $\mathbf{s}$ satisfying the equation (5). It would be natural and useful to link, as an application of Theorem 1, this property of the measure μ_∞ to the properties of the parameters of the Volterra equation under consideration. In particular, one would like to know sufficient and necessary conditions to guarantee that the measure μ_∞ is slowly increasing.

Let us recall that the Volterra equation (1) is characterized by three elements: the operator A, the function v and the spatial correlation Γ of the process W_Γ. Equivalently, when the operator A is given in the form (4), the Volterra equation may be characterized by the triple: the function a, the function $\mathbf{s}$ satisfying (5) and the spectral measure μ of the process W_Γ.

The relation between Γ $(\Gamma_t, t \geq 0)$, and μ $(\mu_t, t \geq 0)$ is, by Bochner-Schwartz theorem, unique and invertible: $\Gamma = \mathcal{F}^{-1}(\mu)(\Gamma_t = \mathcal{F}^{-1}(\mu_t))$. The relation between the operator A and the function a is given by the formula (4). This means that we consider the class of such operators A which fulfill (4). The class of operators under consideration covers, in particular the Laplace operator and its fractional powers.

The most interesting is the relation between the functions v and $\mathbf{s}$. These functions are connected by the equation (5), in which v is the kernel and $\mathbf{s}$ is the solution. Therefore, for given v we can find $\mathbf{s}$, but in general, not vice versa. Only in particular cases, having the function $\mathbf{s}$, we are able to obtain the function v. Hence, it is very difficult to reformulate the results obtained in the paper in terms of the function v.

The problem of finding the kernel function of Volterra equation for given solution, called in the literature *the inverse problem* is, in general, extremely difficult. Now there are not many papers dealing with this problem. Although not much has been done, one observes growing interest in it. The progress in the field of the inverse problem in Volterra equation is needed in order to make the required reformulation.

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