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INTEGRAL OPERATORS OF MARCINKIEWICZ TYPE

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ABSTRACT. In this paper we study integral operators of Marcinkiewicz type. We formulate a general method which allows us to obtain the L^p boundedness of several classes of integral operators of Marcinkiewicz type. Our results extend as well as improve previously known results on Marcinkiewicz integral operators.

1. Introduction and statements of results. Let $n \geq 2$ and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. Suppose that Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ that satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(x)\, d\sigma(x) = 0. \tag{1.1}$$

Let $\mathbf{U}(r)$ be the open ball centered at the origin in $\mathbf{R}^n$ with radius 2^r, $r \in \mathbf{R}$. If $r = \infty$, we shall let $\mathbf{U}(r) = \mathbf{R}^n$. For a suitable mapping $\Theta : \mathbf{U}(r) \to \mathbf{R}^d$, $d \in N$ and a measurable function $h : \mathbf{R}^+ \to \mathbf{R}$, let $\{\sigma_{t,\Theta,\Omega,h,r} : t \in \mathbf{R}\}$ be the family of measures defined on $\mathbf{R}^d$ by

$$\int_{\mathbf{R}^d} f\, d\sigma_{t,\Theta,\Omega,h,r} = 2^{-t}\chi_{(-\infty,r)}(t) \int_{|y|\leq 2^t} f(\Theta(y))|y|^{1-n}\Omega(y)h(|y|)\, dy, \tag{1.2}$$

where $\chi_{(-\infty,r)}(t)$ is the characteristic function of the interval $(-\infty, r)$. Define the operator $\mathbf{S}_{\Theta,\Omega,h,r}$ by

$$\mathbf{S}_{\Theta,\Omega,h,r} f(x) = \left(\int_{-\infty}^{\infty} |\sigma_{t,\Theta,\Omega,h,r} * f(x)|^2\, dt \right)^{1/2}. \tag{1.3}$$

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If $r = \infty$, we shall simply denote the measures $\sigma_{t,\Theta,h,r}$ and the operator $\mathbf{S}_{\Theta,\Omega,h,r}$ by $\sigma_{t,\Theta,\Omega,h}$ and $\mathbf{S}_{\Theta,\Omega,h}$, respectively. Obviously, if $r = \infty$, $n = d$ and $\Theta(y) = (y_1, y_2, \dots, y_n)$, then the operator $\mathbf{S}_{\Theta,\Omega,1}$ is the well-known Marcinkiewicz integral operator introduced by Stein which we shall denote by $\mathbf{S}_\Omega$. When $\Omega \in \mathrm{Lip}_\alpha(\mathbf{S}^{n-1})$, $0 < \alpha \leq 1$, Stein proved that $\mathbf{S}_\Omega$ is bounded on L^p for all $1 < p \leq 2$. Subsequently, Benedek, Calderón and Panzone proved the L^p boundedness of $\mathbf{S}_\Omega$ for all $1 < p < \infty$ under the condition $\Omega \in C^1(\mathbf{S}^{n-1})$, [**4**]. Recently, Chen, Fan and Pan [**5**] proved an L^p boundedness result concerning the operator $\mathbf{S}_{\Theta,\Omega,1}$ under the conditions that $n = d$, $\Theta(y) = P(|y|)y'$ where $y' = |y|^{-1}y$ for $y \neq 0$, P is a real polynomial on $\mathbf{R}$ which satisfies $P(0) = 0$, and Ω satisfies Grafakos-Stefanov's condition [**12**]. Very recently, when $n = d$, $\Theta(y) = \mathcal{P}(y) = (P_1(y), \dots, P_d(y))$ is a polynomial mapping, Al-Qassem and Al-Salman [**1**] studied the L^p boundedness of the operator $\mathbf{S}_{\Theta,\Omega,1}$ under the assumption that $\Omega \in \cap_{k=1}^{\infty} F(\alpha, k, n)$ (for the definition of $F(\alpha, k, n)$ see [**1**] or [**3**]).

The main focus of this paper is to prove the L^p boundedness of several classes of Marcinkiewicz integral operators of the form (1.3) with kernels satisfying the natural condition $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$. In fact, we present a systematic method which not only allows us to deal with the operators under consideration, but also has found applications on other problems in this area which will appear in forthcoming papers. The operators which we consider include *Marcinkiewicz integral operators along submanifolds of finite type, Marcinkiewicz integral operators with nonhomogeneous kernels, Marcinkiewicz integral operators along real-analytic manifolds, and Marcinkiewicz integral operators along surfaces of revolutions.* Our results are the following

Theorem 1.1. *Suppose Θ is of finite type at* 0 *and $\Omega \in L(\log^+ L)$ $(\mathbf{S}^{n-1})$. Then $\mathbf{S}_{\Theta,\Omega,1,1}$ is bounded on $L^p(\mathbf{R}^d)$ for $1 < p < \infty$.*

This result was proved by Ding, Fan and Pan in [**7**] under the stronger condition $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$.

Theorem 1.2. *Suppose that*

(i) $\Theta(y) = (P_1(y), \dots, P_d(y))$ *is a polynomial mapping with* $\Theta(-y) = -\Theta(y)$.

(ii) $\sup_{R>0}(1/R\int_0^R |h(t)|^\gamma\, dt)^{1/\gamma} < \infty$ *for some* $\gamma > 1$.

Then $\mathbf{S}_{\Theta,\Omega,h}$ *is bounded on* L^p *for* $1 < p < \infty$ *provided that* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$. *Moreover, the bound for the operator norm* $\|\mathbf{S}_{\Theta,\Omega,h}\|_{p,p}$ *is independent of the coefficients of the polynomials* $\{P_j\}$.

We should point out that, using the technique in [**9**], one can only prove the above result in Theorem 1.2 under the stronger condition $\Omega \in L^q(\mathbf{S}^{n-1})$, $q > 1$. Our next results are motivated by the work on singular integrals in [**6**] and [**11**].

Theorem 1.3. *Suppose that* $|r| < \infty$ *and that* Θ *is real-analytic mapping from* $\mathbf{U}(r)$ *into* $\mathbf{R}^d$. *If* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$, *then* $\mathbf{S}_{\Theta,\Omega,1,r}$ *is bounded on* $L^p(\mathbf{R}^d)$ *for* $1 < p < \infty$.

Theorem 1.4. *Suppose that* $d = n+1$ *and* $\Theta = (y, \Gamma(|y|))$, *where* Γ *is a strictly increasing function on* $[0,\infty)$ *that satisfies*

(a) $\Gamma'(t) \geq C(\Gamma(t)/t)$ *for* $t > 0$;

(b) $\Gamma(2t) \leq c\Gamma(t)$ *for* $t > 0$ *and for some nonnegative constant* c.

Then $\mathbf{S}_{\Theta,\Omega,1}$ *is bounded on* $L^p(\mathbf{R}^{n+1})$ *for* $1 < p < \infty$ *provided that* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$.

Throughout this paper, the letters C and θ_p are positive constants that may vary at each occurrence but they are independent of the essential variables.

2. Main tools. Suppose $0 < \delta < 1$. By an elementary procedure, choose a collection of $\mathcal{C}^\infty$ functions $\{\psi_{\delta,t}\}_{t\in\mathbf{R}}$ on $(0,\infty)$ with the following properties:

$$\operatorname{supp}(\psi_{\delta,t}) \subseteq \left[2^{(-t-1)/\delta}, 2^{(-t+1)/\delta}\right], \quad 0 \leq \psi_{\delta,t} \leq 1,$$

$$\left|\frac{d^s\psi_{\delta,t}}{du^s(u)}\right| \leq \frac{C}{u^s} \quad \text{and} \quad \sum_{j\in\mathbf{Z}} \psi_{\delta,j+t}(u) = 1.$$

For a linear transformation $\mathbf{L} : \mathbf{R}^n \to \mathbf{R}^d$, $d \geq 1$, let $\varphi_{\delta,t,\mathbf{L}}$ be such that $\hat{\varphi}_{\delta,t,\mathbf{L}}(\xi) = \psi_{\delta,t}(|\mathbf{L}(\xi)|)$. For $j \in \mathbf{Z}$, define the operator $\mathbf{g}^{\delta}_{j,\mathbf{L}}$ by

$$\mathbf{g}^{\delta}_{j,\mathbf{L}}(f)(x) = \left(\int_{-\infty}^{\infty} |\varphi_{\delta,j+t,\mathbf{L}} * f(x)|^2 \, dt \right)^{1/2}. \tag{2.1}$$

Then

$$\|\mathbf{g}^{\delta}_{j,\mathbf{L}}(f)\|_p \le C\|f\|_p \tag{2.2}$$

for all $1 < p < \infty$. By using a technique developed in [**9**], one only needs to verify (2.2) in the special case where $d \le n$ and $\mathbf{L}$ is a projection π^n_d. The latter can be obtained by a well-known argument (see [**13**, pp. 26–28]). For a family of measures $\{\sigma_t : t \in \mathbf{R}\}$, define the operator $\mathbf{J}^{\delta}_{j,\mathbf{L}}$ by

$$\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)(x) = \left(\int_{-\infty}^{\infty} |\sigma_{t/\delta} * \varphi_{\delta,j+t,\mathbf{L}} * f(x)|^2 \, dt \right)^{1/2}. \tag{2.3}$$

Let $\boldsymbol{\sigma}^*$ be the maximal function which corresponds to the family $\{\sigma_t : t \in \mathbf{R}\}$, i.e., $\boldsymbol{\sigma}^*(f)(x) = \sup_{t \in \mathbf{R}} ||\sigma_t| * f(x)|$.

Lemma 2.1. *Let A be a positive real number and $\{\sigma_t : t \in \mathbf{R}\}$ be a family of measures on $\mathbf{R}^n$ such that*

(i) $\sup_{t\in\mathbf{R}} \|\sigma_t\| \le 1$;

(ii) $\|\sigma^*(f)\|_q \le A\|f\|_q$ *for some* $q > 1$.

Then, for $|1/2 - 1/p_0| = 1/2q$, there exists a constant C such that

$$\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|_{p_0} \le C\|\mathbf{g}^{\delta}_{j,\mathbf{L}}(f)\|_{p_0}\sqrt{A}\|f\|_{p_0}. \tag{2.4}$$

Proof. We follow a similar argument as in [**8**]. By duality, it suffices to prove (2.4) for $p_0 > 2$. Let $q = (p_0/2)'$ and choose a nonnegative function $v \in L^q_+(\mathbf{R}^n)$ with $\|v\|_q = 1$ such that

$$\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|^2_{p_0} = \int_{\mathbf{R}^n} \int_{-\infty}^{\infty} |\sigma_{t/\delta} * \varphi_{\delta,j+t,\mathbf{L}} * f(x)|^2 v(x) \, dt \, dx.$$

Then it is easy to see that

$$\begin{aligned}\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|^2_{p_0} &\leq \int_{\mathbf{R}^n} [\mathbf{g}^{\delta}_{j,\mathbf{L}}(f)]^2(z)\boldsymbol{\sigma}^*(v)(-z)\,dz \\ &\leq \|\mathbf{g}^{\delta}_{j,\mathbf{L}}(f)\|^2_{p_0}\|\boldsymbol{\sigma}^*(v)\|_q \leq C^2 A\|\mathbf{g}^{\delta}_{j,\mathbf{L}}\|^2_{p_0}\|f\|^2_{p_0},\end{aligned}$$

where the last inequality follows by (2.2) and (ii). This concludes the proof of the lemma.

Theorem 2.2. *Let $0 < \delta < 1$ and $A > 1$. Suppose that $\{\sigma_t : t \in \mathbf{R}\}$ is a family of measures on $\mathbf{R}^n$ such that*

(i) $\sup_{t\in\mathbf{R}} \|\sigma_t\| \leq 1$;

(ii) $\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|_2 \leq 2^{-|j|}\|f\|_2$ *for all* $j \in \mathbf{Z}$;

(iii) $\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|_{p_0} \leq CA\|f\|_{p_0}$ *for all* $j \in \mathbf{Z}$ *and for some* $p_0 > 2$.

Then for $p_0' < p < p_0$, there exists a constant C such that the operator $\mathbf{S}$ *defined by* (1.3) *with $\sigma_{t,\Theta,\Omega,h,r}$ replaced by σ_t satisfies*

$$\|\mathbf{S}f\|_p \leq \frac{AC}{\sqrt{\delta}}\|f\|_p. \tag{2.5}$$

Proof. By interpolation between (ii) and (iii), we get that

$$\|\mathbf{J}^{\delta}_{j,\mathbf{L}}(f)\|_p \leq 2^{-\theta_p|j|}AC\|f\|_p \tag{2.6}$$

for all $p_0' < p < p_0$. By the properties of the collection $\{\psi_{\delta,j}\}_{j\in\mathbf{Z}}$ and a simple change of variable we see that

$$\mathbf{S}f(x) \leq \frac{1}{\sqrt{\delta}}\sum_{j\in\mathbf{Z}} \mathbf{J}^{\delta}_{j,\mathbf{L}}(f)(x). \tag{2.7}$$

Hence, by combining (2.6) and (2.7) we get (2.5) for all $p_0' < p < p_0$. This completes the proof of our theorem.

By minor modifications of the proof of Lemma 2.5 in [**1**], we get

Lemma 2.3. *Let $\{\sigma^l_t : l = 0, 1, \ldots, N, t \in \mathbf{R}\}$ be a family of measures such that $\sigma^0_t = 0$ for all $t \in \mathbf{R}$. Let $\mathbf{D}_l : \mathbf{R}^n \to \mathbf{R}^{m_l}$,*

$l = 0, 1, \ldots, N$ be linear transformations, $m_l \in \mathbf{N}$. Let $d_l \in \mathbf{R}^+$ and $\delta_l \in \mathbf{R}^+$, $l = 1, \ldots, N$. Suppose that for all $t \in \mathbf{R}$ and $l = 0, 1, \ldots, N$, we have

(i) $\|\sigma_t^l\| \le C$;

(ii) $|(\hat{\sigma_t^l})(\xi)| \le C(2^{d_l t}|D_l(\xi)|)^{\delta_l}$;

(iii) $|(\hat{\sigma_t^l})(\xi) - (\hat{\sigma_t^{l-1}})(\xi)| \le C(2^{d_l t}|D_l(\xi)|)^{-\delta_l}$.

Then there exists a family of measures $\{\nu_t^l : l = 1, \ldots, N, t \in \mathbf{R}\}$ such that

(i′) $\|\nu_t^l\| \le C$;

(ii′) $|(\hat{\nu_t^l})(\xi)| \le C \min\{(2^{d_l t}|D_l(\xi)|)^{\delta_l}, (2^{d_l t}|D_l(\xi)|)^{-\delta_l}\}$;

(iii′) $\sigma_t^N = \sum_{l=1}^{N} \nu_t^l$.

Remark 2.4. One can show that, if the families $\{\sigma_t^l : t \in \mathbf{R}\}$, $l = 0, 1, \ldots, N$, have the additional assumption that their corresponding maximal functions are bounded on $L^p(\mathbf{R}^n)$ for some $1 < p < \infty$, then so are the families $\{\nu_t^l : t \in \mathbf{R}\}$, $l = 1, \ldots, N$, with the same L^p bounds.

The following lemma is a key in proving our results and it may be useful in some other situations as well.

Lemma 2.5. *Suppose that for some $\alpha > 0$, $\Omega \in L(\log^+ L)^\alpha(\mathbf{S}^{n-1})$ that satisfies* (1.2). *Then there exists a subset $\mathbf{D}$ of $\mathbf{N}$, a sequence $\{\lambda_m : m \in \mathbf{N}\}$ of nonnegative real numbers, and a sequence of functions $\{\Omega_m : m \in \mathbf{D} \cup \{0\}\}$ in $L^1(\mathbf{S}^{n-1})$ such that*

(i) $\int_{\mathbf{S}^{n-1}} \Omega_m \, d\sigma = 0$ *for* $m \in \mathbf{D} \cup \{0\}$;

(ii) $\|\Omega_m\|_\infty \le 2^{4m}$ *and* $\|\Omega_m\|_1 \le 2$ *for* $m \in \mathbf{D}$;

(iii) $\Omega_0 \in L^2(\mathbf{S}^{n-1})$;

(iv) $\sum_{m \in \mathbf{D}} m^\alpha \lambda_m < \infty$;

(v) $\Omega = \Omega_0 + \sum_{m \in \mathbf{D}} \lambda_m \Omega_m$.

Proof. We argue as in [**2**]. For a natural number $m \in \mathbf{N}$, let $\mathbf{E}_m$ be the set of points $x' \in \mathbf{S}^{n-1}$ which satisfy $2^m \le |\Omega(x')| < 2^{m+1}$. Also we let $\mathbf{E}_0$ be the set of points $x' \in \mathbf{S}^{n-1}$ which satisfy $|\Omega(x')| < 2$. For

$m \in \mathbf{N} \cup \{0\}$, let $b_m = \Omega \chi_{\mathbf{E_m}}$ and $\lambda_m = \|\Omega \chi_{\mathbf{E}_m}\|_1$, where $\chi_{\mathbf{E}_m}$ is the characteristic function of the set $\mathbf{E}_m$. Let $\mathbf{D}$ be the set of all $m \in \mathbf{N}$ with $2^{3m}\lambda_m \geq 1$. Define the sequence of functions $\{\Omega_m\}_{m \in \mathbf{D} \cup \{0\}}$ by

$$\Omega_0(x) = b_0(x) - \int_{\mathbf{S}^{n-1}} b_0(x)\, d\sigma(x) + \sum_{m \notin \mathbf{D}} \Big\{ b_m(x) - \int_{\mathbf{S}^{n-1}} b_m(x)\, d\sigma(x) \Big\}$$

and for $m \in \mathbf{D}$,

$$\Omega_m(x) = (\lambda_m)^{-1}(b_m(x) - \int_{\mathbf{S}^{n-1}} b_m(x)\, d\sigma(x)).$$

Now (i)–(iii) and (v) hold trivially. On the other hand,

$$\sum_{m \in \mathbf{D}} m^\alpha \lambda_m \leq \sum_{m \in \mathbf{D}} \int_{\mathbf{E}_m} (\log^+ \Omega)^\alpha |\Omega|\, d\sigma \leq \|\Omega\|_{L(\log^+ L)^\alpha(\mathbf{S}^{n-1})}.$$

3. Maximal functions. In this section we prove the following general result concerning maximal functions

Theorem 3.1. *Let* $\mathbf{L} : \mathbf{R}^n \to \mathbf{R}^d$ *be a linear transformation and* $0 < \delta < 1$. *Let* $\{\sigma_t : t \in \mathbf{R}\}$ *and* $\{\mu_t : t \in \mathbf{R}\}$ *be two families of measures that satisfy*

(i) $\sup_{t \in \mathbf{R}} \|\sigma_t\| \leq 1$ *and* $\sup_{t \in \mathbf{R}} \|\mu_t\| \leq 1$;

(ii) $|\hat{\sigma}_t(\xi) - \hat{\mu}_t(\xi)| \leq (2^t |\mathbf{L}(\xi)|)^\delta$;

(iii) $|\hat{\sigma}_t(\xi)| \leq (2^t |\mathbf{L}(\xi)|)^{-\delta}$;

(iv) *For any nonnegative function* f, $F(t,x) = |\sigma_t * f(x)|$ *satisfies*

$$F(t,x) \leq 2^{s-t} F(s,x) \quad \textit{for } t \leq s;$$

(v) $\|\mu^*(f)\|_p \leq C/\delta \|f\|_p$ *for all* $1 < p < \infty$.

Then

$$\|\sigma^*(f)\|_p \leq \frac{C}{\delta} \|f\|_p \tag{3.1}$$

for all $1 < p < \infty$.

Proof. By a technique developed in [**9**], we may assume that $d \le n$ and $\mathbf{L} = \pi_d^n$ is a projection. Choose $\mathcal{F} \in \mathcal{S}(\mathbf{R}^d)$ such that $\hat{\mathcal{F}}(\eta) = 1$ for $|\eta| \le 1/2$, and $\hat{\mathcal{F}}(\eta) = 0$ for $|\eta| \ge 1$. Let $\mathcal{F}_r(x) = r^{-d}\mathcal{F}(x/r)$ for $r > 0$. Define the family of measures $\{\tau_t : t \in \mathbf{R}\}$ by

$$\hat{\tau}_t(\xi) = \hat{\sigma}_t(\xi) - \hat{\mathcal{F}}(2^t \pi_d^n(\xi))\hat{\mu}_t(\xi). \tag{3.2}$$

Let $\mathbf{J}^{\delta}_{j,\tau,\pi_d^n}$ be the operator defined by (2.3) with $\sigma_{t/\delta}$ replaced by $\tau_{t/\delta}$. Then by (iv), (3.2) and the properties of the collection $\{\psi_{\delta,j}\}_{j\in\mathbf{Z}}$, we can easily see that the following inequalities hold

$$\sigma^*(f)(x) \le \frac{2}{\sqrt{\delta}} \sum_{j\in\mathbf{Z}} \mathbf{J}^{\delta}_{j,\tau,\pi_d^n}(f)(x) + (M \otimes I_{\mathbf{R}^{n-d}})(\mu^*(f)(x)) \tag{3.3}$$

$$\tau^*(f)(x) \le \frac{2}{\sqrt{\delta}} \sum_{j\in\mathbf{Z}} \mathbf{J}^{\delta}_{j,\tau,\pi_d^n}(f)(x) + 2(M \otimes I_{\mathbf{R}^{n-d}})(\mu^*(f)(x)), \tag{3.4}$$

where M stands for the Hardy-Littlewood maximal function on $\mathbf{R}^d$ and $I_{\mathbf{R}^{n-d}}$ is the identity operator on $\mathbf{R}^{n-d}$.

By the estimates (i)–(iii) and the definition of the measures $\{\tau_t : t \in \mathbf{R}\}$, we have

$$\sup_{t\in\mathbf{R}} \|\tau_{t/\delta}\| \le C; \tag{3.5}$$

$$|\hat{\tau}_{t/\delta}(\xi)| \le C \min\{(2^{t/\delta}|\pi_d^n(\xi)|)^{\delta}, (2^{t/\delta}|\pi_d^n(\xi)|)^{-\delta}\}. \tag{3.6}$$

Now, for $j \in \mathbf{Z}$, by (3.6), it is easy to see that

$$\begin{aligned}\|\mathbf{J}^{\delta}_{j,\tau,\pi_d^n}(f)\|_2^2 &\le 2^{-2|j|+2} \int_{\mathbf{R}^d} |\hat{f}(\xi)|^2 \\ &\qquad \cdot \left(\int_{\frac{\delta}{\log 2}\log(2^{(-j-1)/\delta}|\pi_d^n(\xi)|^{-1})}^{\frac{\delta}{\log 2}\log(2^{(-j+1)/\delta}|\pi_d^n(\xi)|^{-1})} dt \right) d\xi \\ &= 2^{-2|j|+3}\|f\|_2^2.\end{aligned}$$

Thus

$$\|\mathbf{J}^{\delta}_{j,\tau,\pi_d^n}(f)\|_2 \le 2^{-|j|}\sqrt{8}\|f\|_2. \tag{3.7}$$

By (3.4) and (3.7), we have

$$\|\tau^*(f)\|_2 \le \frac{C}{\delta}\|f\|_2. \tag{3.8}$$

Thus, by Lemma 2.1, (2.1) and (3.8) with $q = 2$, we get

$$\|\mathbf{J}^{\delta}_{j,\tau,\pi^n_d}(f)\|_{p_0} \le \frac{C}{\sqrt{\delta}}\|f\|_{p_0} \tag{3.9}$$

for $|1/p_0 - 1/2| = 1/4$. Therefore, by (3.4), (3.7), (3.9) and condition (v) we have

$$\|\tau^*(f)\|_p \le \frac{C}{\delta}\|f\|_p \tag{3.10}$$

for $p \in [(4/3), 4]$. Next, repeating the above argument with $q = (4/3) + \varepsilon(\varepsilon \to 0^+)$, we get that

$$\|\tau^*_\delta\|_p \le \frac{C}{\delta}\|f\|_p \tag{3.11}$$

for $p \in [(7/8), 8]$. By successive applications of the above argument, we get

$$\|\tau^*_\delta\|_p \le \frac{C}{\delta}\|f\|_p \tag{3.12}$$

for all $1 < p < \infty$. Thus by Lemma 2.1, (2.1) and (3.12), we have

$$\|\mathbf{J}^{\delta}_{j,\tau,\pi^n_d}(f)\|_{p} \le \frac{C}{\sqrt{\delta}}\|f\|_{p} \tag{3.13}$$

for all $1 < p < \infty$. Hence (3.1) follows by (3.3), (3.7) and (3.13).

4. Proof of main results.

Definition 4.1. Let U be an open set in $\mathbf{R}^n$ and $\Theta : U \to \mathbf{R}^d$ a smooth mapping. For $x_0 \in U$ we say that Θ is of finite type at x_0 if, for each unit vector η in $\mathbf{R}^d$, there is a multi-index α so that

$$\partial^\alpha_x[\Theta(x) \cdot \eta]_{x=x_0} \ne 0.$$

Proof of Theorem 1.1. Assume that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$. Let $\mathbf{D}$, $\{\lambda_m : m \in \mathbf{N}\}$ and $\{\Omega_m : m \in \mathbf{D} \cup \{0\}\}$ be as in Lemma 2.5 (here $\alpha = 1$). For $m \in \mathbf{D} \cup \{0\}$, let $\{\sigma_{t,\Theta,\Omega_m,1,1} : t \in \mathbf{R}\}$ be the family of measures defined by (1.2) and $\mathbf{S}_{\Theta,\Omega_m,1,1}$ be the integral operator defined by (1.3). Thus

$$\mathbf{S}_{\Theta,\Omega,1,1}(f) \leq \mathbf{S}_{\Theta,\Omega_0,1,1}(f) + \sum_{m\in\mathbf{D}} \lambda_m \mathbf{S}_{\Theta,\Omega_m,1,1}(f).$$

We will show that $\|\mathbf{S}_{\Theta,\Omega_m,1,1}(f)\|_p \leq Cm\|f\|_p$ for $p \in (1,\infty)$ which implies by condition (iv) of Lemma 2.5 that $\|\sum_{m\in\mathbf{D}} \lambda_m \mathbf{S}_{\Theta,\Omega_m,1,1}(f)\|_p \leq C\|f\|_p$ for $p \in (1,\infty)$. On the other hand, a similar but easier argument yields that $\mathbf{S}_{\Theta,\Omega_0,1,1}(f)$ is bounded on L^p for $p \in (1,\infty)$.

By a similar argument as in the proof of Theorem B in [**10**], there exist a natural number N, polynomial mappings $\mathcal{P}_l : \mathbf{R}^n \to \mathbf{R}^d$, linear transformations $\mathbf{L}_l : \mathbf{R}^d \to \mathbf{R}^{\lambda(l)}$, positive integers d_l, $l = 1,\dots,N$, and measures $\{\sigma_{t,\mathcal{P}_l,\Omega_m,1,1} : t \in \mathbf{R}, 0 \leq l \leq N\}$ such that

$$|(\sigma_{t,\mathcal{P}_l,\Omega_m,1,1})\hat{}(\xi) - (\sigma_{t,\mathcal{P}_{l-1},\Omega_m,1,1})\hat{}(\xi)| \leq C\|\Omega_m\|_1 (2^{d_l t}|\mathbf{L}_l(\xi)|)^{\varepsilon_l}; \tag{4.1}$$

$$|\sigma_{t,\mathcal{P}_l,\Omega_m,1,1})\hat{}(\xi)| \leq C\|\Omega_m\|_2 (2^{d_l t}|\mathbf{L}_l(\xi)|)^{-\varepsilon_l}; \tag{4.2}$$

$$\|\sigma_{t,\mathcal{P}_l,\Omega_m,1,1}\| \leq C\|\Omega_m\|_1, \tag{4.3}$$

for $t \leq t_0 < 0$ and $l = 0,1,\dots,N$ with $\sigma_{t,\mathcal{P}_0,\Omega_m,1,1} = 0$ and $\sigma_{t,\mathcal{P}_N,\Omega_m,1,1} = \sigma_{t,\Theta,\Omega_m,1,1}$. Here $\lambda(l)$ denote the cardinality of $\{\beta \in (\mathbf{N} \cup \{0\})^n : |\beta| = l\}$. Thus, by condition (ii) of Lemma 2.5 and an interpolation argument, we get

$$|(\sigma_{t,\mathcal{P}_l,\Omega_m,1,1})\hat{}(\xi) - (\sigma_{t,\mathcal{P}_{l-1},\Omega_m,1,1})\hat{}(\xi)| \leq C(2^{d_l t}|\mathbf{L}_l(\xi)|)^{\varepsilon_l/m}; \tag{4.4}$$

$$\|(\sigma_{t,\mathcal{P}_l,\Omega_m,1,1})\hat{}(\xi)\| \leq C(2^{d_l t}|\mathbf{L}_l(\xi)|)^{-\varepsilon_l/m}; \tag{4.5}$$

$$\|\sigma_{t,\mathcal{P}_l,\Omega_m,1,1}\| \leq C, \tag{4.6}$$

for $l = 0,1,\dots,N$ and $t \leq t_0$.

For $l = 1,\dots,N$, let

$$(\sigma^m_{\mathcal{P}_l})^*(f) = \sup_{t\in\mathbf{R}} ||\sigma_{t,\mathcal{P}_l,\Omega_m,1,1}| * f|$$

and

$$(\sigma_{\mathcal{P}_l}^{m,+})^*(f) = \sup_{t\in\mathbf{R}} \|\sigma_{t,\mathcal{P}_l,|\Omega_m|,1,1}| * f|.$$

By the L^p boundedness result in [**13**, pp. 476–478], and the fact that $\|\Omega_m\|_1 \le 2$, we have

$$\|(\sigma_{\mathcal{P}_l}^{m,+})^*(f)\|_p \le C\|f\|_p \tag{4.7}$$

for all $1 < p < \infty$ and $l = 1, \dots, N-1$.

Since the measures $\sigma_{t,\mathcal{P}_l,|\Omega_m|,1,1}$ satisfy (4.4)–(4.6) for $l = N$, then by Remark 2.4, (4.7) and Theorem 3.1, we get

$$\|(\sigma_{\Theta}^{m,+})^*(f)\|_p \le Cm\|f\|_p \tag{4.8}$$

for all $1 < p < \infty$. Since $(\sigma_{\mathcal{P}_l}^{m})^*(f)(x) \le (\sigma_{\mathcal{P}_l}^{m,+})^*(f)(x)$, then by (4.7) and (4.8) we have

$$\|(\sigma_{\mathcal{P}_l}^{m})^*(f)\|_p \le Cm\|f\|_p \tag{4.9}$$

for all $1 < p < \infty$ and $l = 1, \dots, N$. Therefore, by Lemma 2.3 and Remark 2.4, there exists a family of measures $\{\nu_t^{l,m} : t \in \mathbf{R}, 1 \le l \le N\}$ which satisfies (i′)–(iii′) of Lemma 2.3 with $\delta_l = \varepsilon_l/m$ and

$$\|(\nu^{l,m})^*(f)\|_p \le Cm\|f\|_p \tag{4.10}$$

for all $1 < p < \infty$ and $l = 1, \dots, N$. Thus by (iii′) of Lemma 2.3, we see that

$$\mathbf{S}_{\Theta,\Omega_m,1,1}(f) \le \sum_{l=1}^{N} \mathbf{S}_{l,m}(f), \tag{4.11}$$

where $\mathbf{S}_{l,m}$ has the same definition as $\mathbf{S}_{\Theta,\Omega_m,1,1}$ with $\sigma_{t,\Theta,\Omega_m,1,1}$ replaced by $\nu_t^{l,m}$. Hence by the estimates (i′)–(iii′), (4.10)–(4.11), Lemma 2.1 and Theorem 2.2, we have

$$\|\mathbf{S}_{\Theta,\Omega_m,1,1}(f)\|_p \le Cm\|f\|_p \tag{4.12}$$

for all $1 < p < \infty$. This ends the proof of our theorem.

The proofs of Theorems 1.2–1.4 can be easily obtained by adapting the same arguments employed in the proof of Theorem 1.1 and the techniques in [**6**], [**9**] and [**11**]. We omit the details.

REFERENCES

1. H. Al-Qassem and A. Al-Salman, *Rough Marcinkiewicz integral operators*, Internat. J. Math. Math. Sci. **27** (2001), 495–503.

2. A. Al-Salman and Y. Pan, *Singular integrals with rough kernels in* $L\log^+ L(\mathbf{S}^{n-1})$, J. London Math. Soc. (2) **66** (2002), 153–174.

3. ———, *Singular integrals with rough kernels*, Canad. Math. Bull., to appear.

4. A. Benedek, A. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci. USA **48** (1962), 356–365.

5. J. Chen, D. Fan and Y. Pan, *A note on Marcinkiewicz integral operator*, Math. Nachr. **227** (2001), 33–42.

6. L. Chen and D. Fan, *On singular integrals along surfaces related to block spaces*, Integral Equations Operator Theory **29** (1997), 261–268.

7. Y. Ding, D. Fan and Y. Pan, *On the* L^p *boundedness of Marcinkiewicz integrals*, Michigan Math. J., to appear.

8. J. Duoandikoetxea and J.L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math. **84** (1986), 541–561.

9. D. Fan and Y. Pan, *Singular integrals with rough kernels supported by subvarieties*, Amer. J. Math. **119** (1997), 799–839.

10. D. Fan, K. Guo and Y. Pan, *Singular integral along submanifolds of finite type*, Michigan Math. J. **44** (1997), 135–142.

11. ———, *Singular integrals with rough kernels along real-analytic submanifolds in* $\mathbf{R}^n$, Integral Equations Operator Theory **33** (1999), 8–19.

12. L. Grafakos and A. Stefanov, L^p *bounds for singular integrals and maximal singular integrals with rough kernels*, Indiana Univ. Math. J. **47** (1998), 455–469. MR 99i:42019. zbl913.42014.

13. E.M. Stein, *Harmonic analysis*: *Real-variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

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