JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 2, Summer 1999

ELECTROMAGNETIC SCATTERING FROM AN ORTHOTROPIC MEDIUM

ROLAND POTTHAST

ABSTRACT. We investigate electromagnetic wave propagation in an inhomogeneous anisotropic medium. For the case of an orthotropic medium we derive the Lippmann-Schwinger equation, which is equivalent to a system of strongly singular integral equations. Uniqueness and existence of a solution is shown and we examine the regularity of the solution by means of integral equations. We prove the infinite Fréchet differentiability of the scattered field in its dependence on the refractive index of the anisotropic medium and we derive a characterization of the Fréchet derivatives as a solution of an anisotropic scattering problem.

1. Introduction. Integral equation methods play a central role in the study of electromagnetic scattering problems. This is primarily due to the fact that the mathematical formulation of scattering problems leads to equations defined over unbounded domains, and hence by the reformulation in terms of integral equations one can replace the problem over an unbounded domain by one over a bounded domain. They also form a powerful tool to study the various features of the problem and to treat the corresponding inverse scattering problems, cf., [**3**].

Although integral equation methods for electromagnetic scattering from obstacles and *isotropic* inhomogeneous media have been quite far developed, the corresponding theory for *anisotropic* media is yet in its infancy. In many cases of practical importance, however, the assumption of an isotropic medium is unwarranted. There is a wide range of materials with an anisotropic behavior in the presence of electromagnetic waves. For example, in medical imaging the nerves and organs such as the brain, the heart and the liver are strongly anisotropic.

With this paper we want to start the investigation of electromagnetic scattering from *anisotropic* inhomogeneous bounded media by means of integral equations. For the sake of simplicity we will restrict our

Accepted for publication on April 27, 1997.

Copyright ©1999 Rocky Mountain Mathematics Consortium

attention to the case of orthotropic media since in this case we can reduce our problem to a problem in two dimensions. We will derive an integral equation of *Lippmann-Schwinger* type for the anisotropic scattering problem. Due to the anisotropic structure this equation no longer has the form 'identity operator plus a compact operator' (as in the isotropic case), but it is an integro-differential equation. We transform the equation into a *system of strongly singular integral equations.*

We investigate the *uniqueness* and the *existence* of solutions to our system of integral equations in $L^2(\Re^2)$ in Section 3. In Section 4 we then derive *regularity properties* of the solution by means of integral equations, i.e., we develop the existence theory in $H^l(\Re^2)$, $l \geq 1$. Here we give a proof which is independent of the existence and regularity theorems which are derived within the general theory of partial differential equations.

In recent years the development of the theory of *inverse* problems has led to many further questions on the corresponding direct scattering problems. Of central importance is the fact that the inverse scattering problem is both nonlinear and improperly posed. A lot of work is done to develop Newton-type methods for the solution of inverse problems based on recent results for the Fréchet differentiability of boundary value problems with respect to the domain, cf., [**6, 9, 10**]. In Section 5 we derive the *infinite Fréchet differentiability* of the solution to the anisotropic scattering problem with respect to the refractive index N which, due to anisotropy, is now a matrix. Further we give a *characterization* of the nth derivative by means of a corresponding anisotropic scattering problem. By this, we lay a basis for the application of Newton-type methods for the solution of the inverse anisotropic scattering problem.

2. Maxwell's equations in an orthotropic medium. We consider time-harmonic electromagnetic wave propagation in an inhomogeneous anisotropic medium in $\Re^3$. The electric field E and the magnetic field H satisfy the *reduced Maxwell equations*

$$\begin{aligned} \operatorname{curl} E - i\kappa H &= 0 \\ \operatorname{curl} H + i\kappa \mathcal{N}(x) E &= 0, \end{aligned} \tag{2.1}$$

where the wave number $\kappa > 0$ is defined by $\kappa^2 = \varepsilon_0 \mu_0 \omega^2$ and where we assume the *refractive index* $\mathcal{N} = \mathcal{N}(x)$ to be a matrix which is given by

$$\mathcal{N}(x) = \frac{1}{\varepsilon_0}\left(\varepsilon(x) + i\frac{\sigma(x)}{w}\right). \tag{2.2}$$

Here μ_0 denotes the magnetic permeability which is assumed to be constant, $\varepsilon = \varepsilon(x)$ denotes the tensor of the electric permittivity, $\sigma = \sigma(x)$ denotes the tensor of the electric conductivity and ω is the frequency of the electromagnetic wave. For physical reasons we can assume that ε is positive definite and σ is positive semi-definite. In the case of a diagonal tensor this can be reduced to the usual positivity-conditions which are imposed on ε and σ in the scalar case. Further we assume that $\mathcal{N}(x) - I$ has compact support $D \subset \Re^2$, i.e., the inhomogeneity is bounded.

By $H^{(l)}(D)$ we denote the Sobolev space of all functions $f \in L^2(D)$ which possess generalized derivatives $f^{(k)} \in L^2(D)$, $k = 1, 2, \dots, l$. The space $H^{(l)}_{\mathrm{loc}}(\Re^2)$ is the space of functions f for which $f\phi \in H^{(l)}(D)$ for all $\phi \in C_0^\infty(\Re^2)$ where $D = \mathrm{supp}\,(\phi)$.

Let E^i, H^i be a solution of Maxwell's equations for a homogeneous medium $\mathcal{N}(x) \equiv I$ representing an incident field. We want to find a sufficiently smooth solution E, H of (2.1) in $\Re^3$ such that the scattered field E^s, H^s defined by

$$\begin{aligned} E &= E^i + E^s, \\ H &= H^i + H^S, \end{aligned} \tag{2.3}$$

satisfies the *Silver-Müller radiation condition*

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0 \tag{2.4}$$

uniformly for all directions $\hat{x} = x/|x|$ where $r = |x|$. In this paper we consider continuously differentiable matrix functions $\mathcal{N}$ and an orthotropic medium, i.e., we assume for $\mathcal{N}$ the form

$$\mathcal{N}(x) = \begin{pmatrix} n_{11}(x) & n_{12}(x) & 0 \\ n_{21}(x) & n_{22}(x) & 0 \\ 0 & 0 & n_{33}(x) \end{pmatrix} \tag{2.5}$$

where the matrix is independent of the z-coordinate. We consider electromagnetic fields which are also independent of the z-coordinate. Then we get the following two groups of scalar equations which are called TM-mode and TE-mode, respectively:

$$\begin{aligned} \frac{\partial E_3}{\partial y} &= i\kappa H_1 \\ \frac{\partial E_3}{\partial x} &= -i\kappa H_2 \\ \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} &= -i\kappa n_{33} E_3 \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \frac{\partial H_3}{\partial y} &= -i\kappa(n_{11}E_1 + n_{12}E_2) \\ \frac{\partial H_3}{\partial x} &= i\kappa(n_{21}E_1 + n_{22}E_2) \\ \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} &= i\kappa H_3, \end{aligned} \tag{2.7}$$

where the first group involves only H_1, H_2 and E_3 and the second group involves only E_1, E_2 and H_3. The first group of equations describe the scattering problem for an electromagnetic wave polarized perpendicular to the z-axis. The solution to this problem is well known, cf. [**3**]. From the second group of equations we obtain for $u = H_3$ the equation

$$(\nabla \cdot N(x)\nabla + \kappa^2)u = 0, \tag{2.8}$$

or in a weak sense

$$\int_D \nabla v \cdot N\nabla u \, dx - \int_{\partial D} \nu \cdot (vN\nabla u) \, ds = \kappa^2 \int_D vu \, dx \tag{2.9}$$

for all domains D with C^1-boundary having unit outward normal vector ν and all $v \in H^1(D)$, where the matrix $N(x)$ is given by

$$N(x) = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix}. \tag{2.10}$$

In the case of the TM or TE-mode we replace the Silver-Müller radiation condition for E^s, H^s by the *Sommerfeld radiation condition* for the field $u^s := H_3^s$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{2.11}$$

uniformly in all directions where $r = |x|$, $x \in \Re^2$.

We will work with matrices $N(x)$ which can be pointwise diagonalized with a unitary complex matrix $U(x)$, i.e., we have $N(x) = U^*(x)N_D(x)U(x)$ with a diagonal matrix $N_D(x)$ and $U^*(x)U(x) = I$ for every $x \in \Re^2$. Further we will assume that N_D has a positive definite real part and a negative semi-definite imaginary part (for diagonal matrices this can be obtained from the assumptions on the matrix $\mathcal{N}$). We call this set of matrices $\mathcal{S}$. For a wide range of practical applications these conditions are fulfilled.

3. Uniqueness and existence. In this section we will establish the uniqueness and the existence of solutions to the scattering problem (2.8)–(2.11). For the uniqueness proof we use the unique continuation principle. The existence will be shown by means of integral equations. A complex matrix $N(x)$ is called *coercive*, (*semi-coercive*), if

$$\operatorname{Im}(a \cdot \overline{Na}) \geq \gamma(x)|a|^2 \tag{3.1}$$

for every $a \in \mathbf{C}^2$ where $\gamma(x) > 0$, $\gamma \geq 0$, compare [**1**]. Matrices $N \in \mathcal{S}$ are semi-coercive. A complex matrix $N(x)$ is called *elliptic*, if $a \cdot Na \neq 0$ for all $a \neq 0$, $a \in \Re^2$. Matrices $N \in \mathcal{S}$ are elliptic.

Theorem 1. *Assume that N is semi-coercive and elliptic. Then the scattering problem* (2.8) *has at most one solution.*

Proof. Let u be a solution of (2.8) with $u^i = 0$. Apply equation (2.9) to $v = \bar{u}$ for $D \supset \operatorname{supp}(M)$ and use the coercivity of N to obtain

$$\operatorname{Im}\left(\int_{\partial D} u\overline{\frac{\partial u}{\partial \nu}}\, ds\right) = \int_D \operatorname{Im}\left(\nabla u \cdot (\overline{N\nabla u})\right) dy \geq 0. \tag{3.2}$$

From Theorem 2.12 of [**3**] we obtain $u \equiv 0$ in the exterior of D (note that $u \in H^1_{\mathrm{loc}}(\Re^2)$ solves in a weak sense the Helmholtz equation in the exterior of $\operatorname{supp}(M)$ and is therefore analytic) and from the unique continuation principle, see [**5**, Theorem 17.2.1], we get $u = 0$, which ends the proof (in Hörmander's theorem it is shown that the set of generalized normal vectors to the subset $\operatorname{supp}(u) \subset \Re^2$ is empty and therefore $u = 0$ in all of $\Re^2$). $\square$

We now transform problem (2.8)–(2.11) into an integral equation. Let $D \subset \Re^2$ be a bounded domain which contains the support of M. The fundamental solution to the Helmholtz equation in two dimensions is given by

$$\Phi(x,y) := \frac{i}{4} H_0^{(1)}(\kappa|x-y|), \quad x \neq y$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero. Let us denote the ball with center x and radius ε by $B(\varepsilon, x)$. From (2.9) for $v(y) = \Phi(x,y)$ we obtain for $D(\varepsilon, x) := D \setminus \overline{B(\varepsilon, x)}$ the equation

$$(3.3) \quad \begin{aligned} &\int_{D(\varepsilon,x)} \nabla_y \Phi(x,y) \cdot \nabla u(y)\, dy \\ &\qquad - \int_{D(\varepsilon,x)} \nabla_y \Phi(x,y) \cdot M(y) \nabla u(y)\, dy \\ &\qquad - \int_{\partial D(\varepsilon,x)} \nu(y) \cdot \Phi(x,y) \nabla u(y)\, ds(y) \\ &\qquad + \int_{\partial B(\varepsilon,x)} \nu(y) \cdot \Phi(x,y) M(y) \nabla u(y)\, ds(y) \\ &\quad = \kappa^2 \int_{D(\varepsilon,x)} \Phi(x,y) u(y)\, dy. \end{aligned}$$

The function $\Phi(x, .)$ solves (2.9) in $D(\varepsilon, x)$ for $N \equiv I$. Therefore with $v = u$ we obtain

$$(3.4) \quad \begin{aligned} &\int_{D(\varepsilon,x)} \nabla u(y) \cdot \nabla \Phi(x,y)\, dy - \int_{\partial D(\varepsilon,x)} \nu(y) \cdot u(y) \nabla \Phi(x,y)\, ds(y) \\ &\qquad = \kappa^2 \int_{D(\varepsilon,x)} u(y) \Phi(x,y)\, dy. \end{aligned}$$

We subtract (3.4) from (3.3) and get

$$
\begin{aligned}
(3.5)\qquad & -\int_{D(\varepsilon,x)} \nabla\Phi(x,y)\cdot M(y)\nabla u(y)\,dy \\
& \quad - \int_{\partial D(\varepsilon,x)} \left\{ \frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y) \\
& \qquad = -\int_{\partial B(\varepsilon,x)} \nu(y)\cdot \Phi(x,y) M(y)\nabla u(y)\,ds(y).
\end{aligned}
$$

Using Green's formula [**3**, (2.5)] applied to u^i and Green's theorem [**3**, (2.3)] and the radiation condition (2.11) applied to u^s we derive

$$
(3.6)\qquad
\begin{aligned}
&\int_{\partial D} \left\{ \Phi(x,y)\frac{\partial u^i(y)}{\partial \nu(y)} - u^i(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y) = u^i(x), \\
&\int_{\partial D} \left\{ \Phi(x,y)\frac{\partial u^s(y)}{\partial \nu(y)} - u^s(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y) = 0.
\end{aligned}
$$

Finally we estimate the integrals over $\partial B(\varepsilon,x)$ in (3.5). We estimate (note that ν is directed into the interior of $B(\varepsilon,x)$)

$$
(3.7)\qquad \int_{\partial B(\varepsilon,x)} u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)}\,ds(y) = \frac{1}{2\pi}\int_{\partial B(\varepsilon,x)} u(y)\frac{ds(y)}{\varepsilon} + O(\varepsilon)
$$

and

$$
\int_{\partial B(\varepsilon,x)} \frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y)\,ds(y) = O(\varepsilon\ln(\varepsilon)).
$$

uniformly for x in compact subsets of $\Re^2$. Using the convergence

$$
\lim_{\varepsilon\to 0}\left\| \int_{\partial B(\varepsilon,x)} (u(y)-u(x))\frac{ds(y)}{\varepsilon} \right\|_{L^2(D)} = 0,
$$

which is valid for functions in $H^1(D)$, we get the L^2-convergence

$$
(3.8)\qquad -\int_{\partial B(\varepsilon,x)} \left\{ \frac{\partial u(y)}{\partial \nu(y)}\Phi(x,y) - u(y)\frac{\partial \Phi(x,y)}{\partial \nu(y)} \right\} ds(y) \longrightarrow u(x).
$$

Now collecting the equations (3.3)–(3.8) and passing to the limit $\varepsilon \to 0$ we obtain the integral equation

$$
(3.9)\qquad u(x) - \int_D \nabla\Phi(x,y)\cdot M\nabla u(y)\,dy = u^i(x), \quad x\in D,
$$

which is an equation of *Lippmann-Schwinger* type. Note that the integral exists in the sense of an improper integral.

We will prove existence of a solution of (3.9) in $H^1(D)$. For a domain $D \subset \Re^2$ define the space $X(D) := L^2(D) \times L^2(D)$ equipped with the norm

$$||v||_X := (||v_1||^2_{L^2(D)} + ||v_2||^2_{L^2(D)})^{1/2}$$

for $v = (v_1, v_2) \in X(D)$. We consider matrices $M(x)$, $x \in \Re^2$, to be multiplication operators on $X(D)$ and we abbreviate $X := X(\Re^2)$. Take the gradient of equation (3.9), i.e.,

$$\nabla u(x) + \nabla\left(\nabla \cdot \int_D \Phi(x,y)(M\nabla u)(y)\,dy\right) = \nabla u^{in}(x). \tag{3.10}$$

Here we work with the weak derivative of a function in $H^1(D)$, the integrals exist in the sense of Cauchy's principal value. We use the formula

$$\begin{aligned}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\int_D \Phi(x,y)\varphi(y)\,dy \\ = \int_D \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}\Phi(x,y)\varphi(y)dy - \frac{1}{2}\delta_{ki}\varphi(x),\end{aligned} \tag{3.11}$$

see [**8**, p. 310] for L^2-densities to obtain the equation

$$\nabla u(x) - \frac{1}{2}(M\nabla u)(x) + \int_D \nabla_x(\nabla_x \cdot \{\Phi(x,y)(M\nabla u)(y)\}\,dy = \nabla u^{in}(x). \tag{3.12}$$

Using the diagonal matrix

$$I - N_D =: M_D = \begin{pmatrix} d_1(x) & 0 \\ 0 & d_2(x)\end{pmatrix},$$

we have

$$\left(I - \frac{1}{2}M\right)(x) = U^*(x)\begin{pmatrix} 1-(1/2)d_1(x) & 0 \\ 0 & 1-(1/2)d_2(x)\end{pmatrix} U(x) \tag{3.13}$$

where from the conditions on $\mathcal{N}$ we obtain $1 - \mathrm{Re}(d_1) > 0$ and $1 - \mathrm{Re}(d_2) > 0$. We use this to derive that the matrix $I - (1/2)M$ is invertible. We multiply the vector equation (3.12) by $M(I-(1/2)M)^{-1}$. Using the new function $v := M\nabla u$ we get the equation

$$v(x) + M\left(I - \frac{1}{2}M\right)^{-1}(x)\int_D \nabla_x(\nabla_x \cdot \{\Phi(x,y)v(y)\}\,dy = M\left(I - \frac{1}{2}M\right)^{-1}\nabla u^{in}(x), \quad x \in D \tag{3.14}$$

which is a *system of strongly singular integral equations* for the components v_i, $i = 1, 2$ of v.

We first show the equivalence of (3.14) to (3.9). Assume that the L^2-vector field v solves (3.14). We note that the integral in (3.14) maps $X(D)$ into $X(D)$, see [**8**, p. 257]. We define

$$\hat{v}(x) := -\left(I - \frac{1}{2}M\right)^{-1}(x) \cdot \int_D \nabla_x(\nabla_x \cdot \{\Phi(x,y)v(y)\}\,dy + \left(I - \frac{1}{2}M\right)^{-1}\nabla u^{in}(x)$$

and obtain $M\hat{v} = v$ from (3.14). From the definition of $\hat{v}$ using (3.11) we derive that $\hat{v}$ satisfies

$$\hat{v}(x) + \nabla\left(\nabla \cdot \int_D \Phi(x,y)(M\hat{v})(y)\,dy\right) = \nabla u^{in}(x). \tag{3.15}$$

From (3.15) we see that $\hat{v}$ is the gradient of a vector field

$$u(x) := -\nabla \cdot \int_D \Phi(x,y)(M\hat{v})(y)\,dy + u^{in}(x) \tag{3.16}$$

and that u satisfies

$$u(x) + \nabla \cdot \int_D \Phi(x,y)(M\nabla u)(y)\,dy = u^{in}(x),$$

which is equation (3.9).

Note that if $u \in H^2(D)$, we also easily obtain (2.8) from (3.9) by partial integration, application of $\triangle + \kappa^2$ and (3.11). Let us consider the integral

$$(Tv)(x) := \int_{\Re^2} \nabla_x(\nabla_x \cdot \Phi(x,y)v(y))\,dy, \quad x \in \Re^2. \tag{3.17}$$

T defines a strongly singular bounded integral operator in the space $X(\Re^2)$, cf., [**8**, p. 257) and therefore also on the space $X(D) \subset X(\Re^2)$. Let T_0 denote the operator T in the case $\kappa = 0$, i.e., where Φ is replaced by the fundamental solution of Laplace's equation. Using the operators T_0 and $T - T_0$ we can rewrite equation (3.14) in the form

$$v + M\left(I - \frac{1}{2}M\right)^{-1} T_0 v + M\left(I - \frac{1}{2}M\right)^{-1}(T - T_0)v = M\left(I - \frac{1}{2}M\right)^{-1} \nabla u^{in}. \tag{3.18}$$

We estimate the norm of the operator T_0 in the following

Lemma 2. *On the space $X = L^2(\Re^2) \times L^2(\Re^2)$ equipped with the norm*

$$||v||_X = (||v_1||_{L^2}^2 + ||v_2||_{L^2}^2)^{1/2}, \quad v = (v_1, v_2) \in X$$

the operator norm of T_0 is given by

$$||T_0||_{BL(X,X)} = 1/2$$

where $BL(X, X)$ denotes the space of bounded linear operators on X equipped with the canonical operator norm.

Proof. For the kernel k_0 of the operator T_0 we compute

$$k_0(x, y) = \frac{-1}{2\pi} \frac{1}{|x - y|^2} \begin{pmatrix} 1 - 2\dfrac{(x_1 - y_1)^2}{|x - y|^2} & \dfrac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} \\ \dfrac{(x_1 - y_1)(x_2 - y_2)}{|x - y|^2} & 1 - 2\dfrac{(x_2 - y_2)^2}{|x - y|^2} \end{pmatrix}. \tag{3.19}$$

We use $r = |x - y|$, $\phi = \arg[(y - x)/r]$ and the trigonometric formulas $1 - 2\cos^2\phi = -\cos(2\phi)$ and $2\sin\phi\cos\phi = \sin(2\phi)$ to obtain

$$k_0(x, y) = \frac{-1}{2\pi} \frac{1}{r^2} \begin{pmatrix} -\cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & \cos(2\phi) \end{pmatrix} \tag{3.20}$$

Since $\sin(2\phi)$ and $\cos(2\phi)$ are the spherical harmonics of order 2 in two dimensions we can use [**8**, Chapter 10] to compute the Fourier transform of k_0. We obtain

$$\hat{k}_0(\xi) = \frac{1}{2}\begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \tag{3.21}$$

with $\xi = |\xi|(\cos\theta, \sin\theta)$. In the Fourier-space the operator T_0 becomes a multiplication operator $\hat{k}_0(\xi)$. For every point $\xi \in \Re^2$ the matrix

$$\begin{pmatrix} -\cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \tag{3.22}$$

simply rotates and reflects the vector $(\mathcal{F}v_1(\xi), \mathcal{F}v_2(\xi))$ which does not affect its L^2-norm $||.||_X$. Since v and $\mathcal{F}v$ have the same norms we obtain the statement of the lemma. □

For $N \in \mathcal{S}$ the norm of the matrix operator $M(I-(1/2)M)^{-1}$ in $X(D)$ can be estimated by

$$\left\| M\left(I - \frac{1}{2}M\right)^{-1} v \right\|_{X(D)} \le c||v||_{X(D)} \tag{3.23}$$

with some constant $c < 2$. This can be done by again using the diagonal matrix $M_D := I - N_D$. We have

$$\begin{aligned} M\left(1 - \frac{1}{2}M\right)^{-1}(x) \\ = U^*(x) \begin{pmatrix} \dfrac{d_1(x)}{1-(1/2)d_1(x)} & 0 \\ 0 & \dfrac{d_2(x)}{1-(1/2)d_2(x)} \end{pmatrix} U(x) \end{aligned} \tag{3.24}$$

with $1 - \operatorname{Re}(d_1) > 0$ and $1 - \operatorname{Re}(d_2) > 0$ for the continuous and compactly supported functions d_1 and d_2. For the diagonal matrix we have

$$\left\| \begin{pmatrix} \dfrac{d_1(x)}{1-(1/2)d_1(x)} & 0 \\ 0 & \dfrac{d_2(x)}{1-(1/2)d_2(x)} \end{pmatrix} \tilde{v} \right\|_{X(D)}$$
$$= \left(\left\| \frac{d_1(x)}{1-(1/2)d_1(x)} \tilde{v}_1(x) \right\|_{L^2(D)} + \left\| \frac{d_2(x)}{1-(1/2)d_2(x)} \tilde{v}_2(x) \right\|_{L^2(D)} \right)^{1/2}$$
$$\leq \max_{i=1,2} \sup_{x \in D} \left| \frac{d_i(x)}{1-(1/2)d_i(x)} \right| ||\tilde{v}||_{X(D)}$$
$$= \max_{i=1,2} \sup_{x \in D} 2 \left(\frac{(\operatorname{Re} d_i(x))^2 + (\operatorname{Im} d_i(x))^2}{(2-\operatorname{Re} d_i(x))^2 + (\operatorname{Im} d_i(x))^2} \right)^{1/2} ||\tilde{v}||_{X(D)},$$

from which we get the estimate. Therefore by restricting T_0 to $X(D)$ we obtain that the operator norm of $M(I-(1/2)M)^{-1}T_0$ in $X(D)$ is less than one. Looking at the power series of $\Phi(x,y)$, cf., [**3**, (3.51), (3.52)] we see that the kernel of $T-T_0$ is weakly singular. Hence the operator is compact in $X(D)$ for the bounded domain D. We have thus shown that in $X(D)$ the lefthand side of (3.18) consists of the sum of the identity operator, an operator with norm lower than one and a compact operator.

Lemma 3. *Assume that the matrix $N = I - M$ is semi-coercive and elliptic. Then the integral operator on the lefthand side of* (3.14) *is injective and the integral equation* (3.14) *has at most one solution in* $X(D)$.

Proof. Consider a solution $v \in X(D)$ of (3.14) with zero righthand side. We first obtain $v \equiv 0$ in $\Re^2 \setminus \operatorname{supp}(M)$. From the regularity results of the next chapter we obtain $v \in H^1(D) \times H^1(D)$. Then we get as shown above a solution $u \in H^2(\Re^2)$ to the scattering problem with vanishing incident field. The uniqueness of the scattering problem yields $u \equiv 0$. From this we obtain $v = M\nabla u \equiv 0$, which proves the injectivity of the integral equation in $X(D)$. $\square$

We now come to the existence theorem.

Theorem 4. *For $N = I - M \in \mathcal{S}$ the operator given by the lefthand side of* (3.14) *is invertible in $X(D)$ and the inverse operator is bounded.*

Proof. Since the integral operator is injective we obtain the invertibility and the boundedness of the inverse by the Neumann series theorem and the Riesz-Fredholm theory for compact operators, see for example [**7**, Corollary 3.8]. $\square$

4. Regularity properties of the solution. In this section we prove regularity properties of the solution of (3.14) by means of singular integral equations.

For a domain $D \subset \Re^2$ we introduce the Sobolev space $X^{(l)}(D) := H^{(l)}(D) \times H^{(l)}(D)$ equipped with the norm

$$||v||_{X^{(l)}(D)} := \sum_{|\alpha|\le l} (||D^\alpha v_1||^2_{L^2(D)} + ||D^\alpha v_2||^2_{L^2(D)})^{1/2} \tag{4.1}$$

for $v = (v_1, v_2) \in X^{(l)}(D)$. $X_0^{(l)}(D)$ denotes the subspace of functions in $X^{(l)}(D)$ with compact support in D. Note that the term in brackets is the $X(D)$-norm of the function $D^\alpha v$. We use the abbreviation $X^{(l)} := X^{(l)}(\Re^2)$ and $\mathcal{M}(x) := M(x)(I - (1/2)M(x))^{-1}$.

Let us first consider the solution of the equation

$$(I + \mathcal{M}(x)T_0)v = f. \tag{4.2}$$

We will show that for $\mathcal{M} \in C_0^l(D)$ and $f \in X_0^{(l)}(D)$ the solution of (4.2) is in $X_0^{(l)}(D)$. From the proof of Theorem 4 we obtain the invertibility of the operator $I + \mathcal{M}(x)T_0$ in $X(D)$ and the representation

$$\begin{aligned} v &= (I + \mathcal{M}(x)T_0)^{-1} f \\ &= \sum_{\nu=0}^{\infty} (-\mathcal{M}(x)T_0)^\nu f \end{aligned} \tag{4.3}$$

where we have $q := ||\mathcal{M}(x)T_0||_{X(D)} < 1$. From

$$||D^\alpha T_0 v||_X = ||\xi^\alpha \mathcal{F}(k_0)\mathcal{F}(v)||_X = \frac{1}{2}||\xi^\alpha \mathcal{F}(v)||_x = \frac{1}{2}||D^\alpha v||_X$$

we see that T_0 maps $X_0^{(l)}(D)$ into $X^{(l)}(D)$. Thus $\mathcal{M}(x)T_0$ maps $X_0^{(l)}(D)$ into itself. We conclude that every term in the expansion (4.3) is l times differentiable with respect to x. We want to show that the sum of the derivatives converges in X. For the derivative of T_0 we have

$$D^\alpha T_0 f = T_0(D^\alpha f), \tag{4.4}$$

see, for example [**8**, p. 303]. Therefore we obtain

$$\nabla_j(\mathcal{M}(x)T_0 f) = (\nabla_j \mathcal{M}(x))T_0 f + \mathcal{M}(x)T_0(\nabla_j f),$$

$$\begin{aligned} \nabla_j((\mathcal{M}(x)T_0)^2 f) = {} & (\nabla_j \mathcal{M}(x))T_0(\mathcal{M}(x)T_0)f \\ & + (\mathcal{M}(x)T_0)(\nabla_j \mathcal{M}(x))T_0 f \\ & + (\mathcal{M}(x)T_0)^2(\nabla_j f) \end{aligned} \tag{4.5}$$

and an analogous expression for the terms $(-\mathcal{M}(x)T_0)^\nu f$. Let us estimate the norm of $\nabla_j((-\mathcal{M}(x)T_0)^\nu f)$ in $X(D)$. Inductively we obtain

$$||\nabla_j((\mathcal{M}(x)T_0)^\nu f)||_{X(D)} \leq \nu \frac{1}{2} C q^{\nu-1} ||f||_{X(D)} + q^\nu ||f||_{X^1(D)}. \tag{4.6}$$

where C is a bound for the derivatives of $\mathcal{M}$. From (4.6) we clearly see that the sum of the derivatives of (4.3) is convergent in $X(D)$, i.e., the function v is in $X_0^{(1)}(D)$. An analogous estimate holds true for the higher derivatives up to the order l. We have proven that $v \in X_0^{(l)}(D)$.

Let us look at the term $M(x)(I-(1/2)M(x))^{-1}(T-T_0)$ of equation (3.18). In the Appendix it is shown that that the operator $M(I-(1/2)M)^{-1}(T-T_0)$ maps $X_0^{(j)}(D)$ continuously into $X_0^{(j+1)}(D)$ for all $j = 0, 1, 2, \dots, l-1$. By induction we now obtain the following theorem.

Theorem 5. *Assume that $M \in C_0^l(\Re^2)$ for some $l \in \mathbf{N}$ and that $N = I - M \in \mathcal{S}$ and let D be a domain with supp$(M) \subset D$. Then the solution of* (3.14) *is in $X_0^{(l)}(D)$, i.e., the solution u to the scattering problem* (2.8) *is in $H_{loc}^{(l+1)}(\Re^2)$. We have the estimate*

$$||u||_{H^{l+1}(D)} \leq C||u^i||_{H^l(D)} \tag{4.7}$$

with some constant C depending on M.

Remark. Note that we have avoided using both the classical regularity results for elliptic partial differential equations of second order with coefficients in C^l, cf., [**4**] as well as the use of the symbol of singular integral operators, cf., [**8**].

5. Fréchet differentiability with respect to M. We now come to the differentiability properties of the mapping $M \mapsto u^s(M)$. Let $\mathcal{S}_M := \{M : I - M \in \mathcal{S}\}$.

Theorem 6. *The mapping of the matrix M onto the solution u of* (3.9) *is infinitely Fréchet differentiable from $\mathcal{S}_M \subset C_0^1(\Re^2)$ into $H^1(D)$. The nth derivative is given by*

$$\frac{\partial^n u}{\partial M^n}(dM) = n!((I - L(M))^{-1} L(dM))^n (I - L(M))^{-1} u^{in} \tag{5.1}$$

Proof. Define $L(M) : H^1(D) \to H^1(D)$ by

$$L(M)u := \int_D \nabla\Phi(.,y) \cdot M\nabla u(y)\, dy \tag{5.2}$$

From its linearity in M we see that the mapping

$$\mathcal{S}_M \longrightarrow BL(H^1(D), H^1(D)), \quad M \longmapsto L(M) \tag{5.3}$$

is infinitely Fréchet differentiable on $\mathcal{S}_M$. Therefore the operator

$$I - L(M)$$

is infinitely Fréchet differentiable $\mathcal{S}_M \to BL(H^1(D), H^1(D))$. Since the operator is invertible for $M \in \mathcal{S}_M$ from Theorem 2 of [**9**] we obtain that its inverse is Fréchet differentiable as a mapping $\mathcal{S}_M \to BL(H^1(D), H^1(D))$, and we obtain the given form of the inverse. The statement for higher derivatives can be obtained by induction. $\square$

Analogously to [**6, 9**] or [**10**] we can obtain a characterization of the Fréchet derivative as the solution to a special scattering problem.

Theorem 7. *The nth Fréchet derivative $u^{(n)}$ of the scattering problem* (2.8) *with respect to the $M = I - N$ at the point M and direction δ satisfies the Sommerfeld radiation condition and solves the PDE*

$$(\nabla \cdot N\nabla)u^{(n)} + \kappa^2 u^{(n)} = n(\nabla \cdot \delta\nabla)u^{(n-1)}.$$

Proof. The statement is an immediate consequence of equation (5.1). $\square$

Acknowledgments. I would like to thank Prof. Dr. David Colton for many stimulating discussions during the course of this work. I also enjoyed discussions with Prof. Dr. Ralf Kleinman and his student Wen Lixin on the scalar version of the equation.

Appendix

Here we investigate the mapping properties of the operator $M(I - (1/2)M)^{-1}(T - T_0)$.

Theorem 8. *Assume that $M \in C_0^l(D)$. Then the operator $M(I - (1/2)M)^{-1}(T - T_0)$ maps the space $X_0^{(j)}(D)$ continuously into the space $X_0^{(j+1)}(D)$ for all $j + 1 \leq l$.*

Proof. We first examine the operator $T - T_0$. From the expansions of the Hankel function we obtain for the difference $\Phi - \Phi_0$ the form

$$\Phi(x, y) - \Phi_0(x, y) = c_1|x-y|^2 \ln|x-y| + c_2|x-y|^2 + O(|x-y|^3) \tag{5.4}$$

as $|x - y| \to 0$ with some constants c_1 and c_2. Since the expansions are absolutely convergent we can differentiate each term and we obtain for the kernel k of $T - T_0$ the form

$$k_{ij}(x,y) = 2c_1\delta_{ij}\ln|x-y| + 2c_1\frac{(x-y)_i(x-y)_j}{|x-y|^2} + 2c_2\delta_{ij} + R_{ij}(x,y) \tag{5.5}$$

where $R_{ij}(x,y) = O(|x-y|)$ for $|x-y| \to 0$. According to [**3**, Theorem 8.2], the leading term in (5.5) defines a bounded integral operator $V : L^2(D) \to H^2(D)$. Denote the second term of the right hand side of (5.5) by t_2. The term t_2 is continuously differentiable for $x \neq y$ with weakly singular partial derivatives

$$\frac{\delta_{i\mu}(x-y)_j}{|x-y|^2} + \frac{\delta_{j\mu}(x-y)_i}{|x-y|^2} - 2\frac{(x-y)_i(x-y)_j(x-y)_\mu}{|x-y|^4}, \quad \mu = 1,2. \tag{5.6}$$

Assume first that $u \in C_0(D)$. Then we get

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\int_{D\setminus B(\varepsilon,x)} t_2(x,y)u(y)\,dy &= \int_{D\setminus B(\varepsilon,x)} \frac{\partial t_2(x,y)}{\partial x_\mu}u(y)\,dy \\ &\quad - \int_{\partial B(\varepsilon,x)} t_2(x,y)u(y)\cos(y-x,x_\mu)\,ds(y).\end{aligned} \tag{5.7}$$

In the limit $\varepsilon \to 0$ the last term vanishes. We obtain that the integral

$$w(x) := \int_D \frac{(x-y)_i(x-y)_j}{|x-y|^2}u(y)\,dy, \quad x \in D$$

with the second term of the righthand side of (5.5) as kernel is a differentiable function w, that differentiation can be done by the differentiation of the kernel and that the first derivatives are bounded with respect to the norm on $L^2(D)$, i.e., the operator maps the space $C_0(D) \subset L^2(D)$ continuously into $H^1(D)$. Since $C_0(D)$ is dense in $L^2(D)$ there exists a unique continuous extension of the operator from $L^2(D)$ into $H^1(D)$.

For the function $2c_2\delta_{ij} + R_{ij}$ in (5.5) we now can proceed in the same way as for the second term. Together we obtain that the operator

$T - T_0$ maps $X(D) = X^{(0)}(D)$ continuously into $X^{(1)}(D)$. Therefore the operator $M(I - (1/2)M)^{-1}(T - T_0)$ maps $X_0^{(0)}(D)$ continuously into $X_0^{(1)}(D)$.

To prove the statement for higher derivatives we will use induction. Assume that we have $m \in C_0^l(D)$. For $u \in C_0^l(D)$, $l \geq 1$, we get, using Gauss's divergence theorem,

$$
\begin{aligned}
(5.8) \quad \frac{\partial}{\partial x_\mu}\Big(m(x) \int_D k_{ij}(x,y)u(y)\,dy\Big) \\
&= \frac{\partial m(x)}{\partial x_\mu} \int_D k_{ij}(x,y)u(y)\,dy \\
&\quad + m(x) \int_D k_{ij}(x,y) \frac{\partial u}{\partial y_\mu}(y)\,dy, \quad x \in D.
\end{aligned}
$$

Assume that the operators on the righthand side of (5.8) define bounded operators from $H_0^{l-1}(D)$ into $H_0^l(D)$. For $l = 1$ this was proven in the first part of our proof. Then from (5.8) we obtain that m times the integral over k_{ij} is bounded from $C_0^l(D) \cap H_0^l(D)$ into $H_0^{l+1}(D)$ with respect to the Sobolev norms. Since $C_0^l(D)$ is dense in $H_0^l(D)$ we now obtain the statement by induction. □

REFERENCES

1. D. Colton and C. Erbe, *Spectral theory for the magnetic far field operator in an orthotropic medium*, in *Nonlinear problems in applied mathematics* (T.S. Angell, et al., eds.), SIAM Publications, Philadelphia, 1996, 96–103.

2. D. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley-Interscience Publications, New York, 1983.

3. ——— *Inverse acoustic and electromagnetic scattering theory*, Springer-Verlag, Berlin, 1992.

4. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, New York, 1977.

5. L. Hörmander, *The analysis of linear partial differential operators*, III, Springer Verlag, New York, 1985.

6. A. Kirsch, *The domain derivative and two applications in inverse scattering theory*, Inverse Problems **9** (1993), 601–615.

7. R. Kress, *Linear integral equations*, Springer, Berlin, 1989.

8. S.G. Michlin and S. Prößdorf, *Singular integral equations*, Akademie Verlag, Berlin, 1986.

9. R. Potthast, *Fréchet differentiability of boundary integral operators in inverse acoustic scattering*, Inverse Problems **10** (1994), 431–447.

10. ———, *Domain derivatives in electromagnetic scattering*, Math. Meth. Appl. Sci. **19** (1996), 1157–1175.

Institute for Numerical and Applied Mathematics, University of Göttingen, Lotzestrasse 16-18, 37083 Göttingen, Germany
E-mail address: potthast@math.uni-goettingen.de