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ATTRACTING SOLUTIONS OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. This paper is devoted to studying the uniqueness and the attracting character of nontrivial solutions for some nonlinear Volterra integral equations. Also, a simple method to approximate the nontrivial solution is provided.

0. Introduction. We are interested in the nonlinear Volterra integral equations

$$u(x) = \int_0^x k(x-s)g(u(s))\,ds, \tag{k,g}$$

from now on referred to as equation (k,g). The operator

$$T_{k,g}u(x) = \int_0^x k(x-s)g(u(s))\,ds \tag{$T_{k,g}$}$$

shall be referred to as the *associated operator* to equation (k,g).

These equations appear in connection with several physical models: diffusion problems such as percolation from a reservoir [**7**] or fabrication of microchips [**8**], nonlinear models about the behavior of the shock-wave front in gas-filled tubes [**6**], etc. The physical models considered impose some restrictions on the *kernel* k and the *nonlinearity* g. The kernel k is always a nonnegative locally integrable function, and g is a continuous increasing function such that $g(0) = 0$. A solution of those equations is the function zero; so, our interest is centered on nontrivial solutions, i.e., positive solutions different from zero in every neighborhood of 0. When the equation (k,g) admits nontrivial solutions we say that (k,g) is *admissible*.

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Following [**2, 3**], a positive function f is said to be *like* a positive function h if there exist two positive constants α and β such that $\alpha h \leq f \leq \beta h$. The function f is said to be *weakly homogeneous* if there exists a positive function l such that, for all λ, $x \geq 0$, one has

$$f(\lambda x) \geq l(\lambda) f(x);$$

the function l is called a *lower transfer function* for f.

Examples of weakly homogeneous functions are: homogeneous functions, polynomials with nonnegative coefficients, the function $\log(x+1)$, compositions or linear combinations of weakly homogeneous functions. More examples and properties of these functions can be found in [**2**].

Assuming the kernel k is like a positive increasing function and the equation (k, g) is admissible, our purpose is to show that the solution, u, is unique and *attracts* all positive functions, f, in the following sense:

$$u = \lim_{n\to\infty} T^n_{k,g}(f)$$

where lim denotes a pointwise limit. All throughout the paper lim denotes a pointwise limit, unless otherwise stated.

We shall prove that the majority of weakly homogeneous functions are like a positive increasing function. In this way, when the kernel k is weakly homogeneous, one admissible equation (k, g) has a unique solution. And this solution is increasing and attracts all positive functions.

To approximate the solution of a nonlinear Volterra integral equation by Picard's iterates is hard. Considering the attracting character of nontrivial solutions, we obtain, in the second part of the paper, a simple way to approximate the solution of (k, g) when the associated operator $T_{k,g}$ is monotone.

1. Attracting solutions and uniqueness. All throughout the paper the positive kernel k is assumed to be like some increasing kernel μ. This means that there are two positive constants α and β such that $\alpha\mu \leq k \leq \beta\mu$. The nonlinearity g is assumed to be nonnegative, strictly increasing and smooth on the open half-axis $(0, \infty)$. Both kernel and nonlinearity vanish at 0.

Szwarc proves in [**9**, Proposition 1 and Theorem 1] that if the equation

$$u(x) = \int_0^x a(x,y)\phi(u(y))\,dy$$

has a nontrivial solution, then it is unique and attracts all positive functions f, assuming that the kernel $a(x,y)$ satisfies

$$a(x,y) \geq a(s,t), \quad \text{for } 0 \leq s \leq x, \quad 0 \leq t \leq y, \quad \text{and} \quad x-y > s-t,$$

and $a(x,x) = 0$; moreover, the functions u and ϕ have to be smooth, nonnegative and strictly increasing on the half-axis $[0,+\infty)$ and $u(0) = \phi(0) = 0$.

The kernel $a(x,y) = \mu(x-y)$ with μ increasing is a particular case of a kernel satisfying Szwarc's hypothesis. We shall study equations having convolution kernels $k(x-y)$; however, k shall only be assumed to be like some increasing kernel μ, and therefore our results are not a proper subset of those of Szwarc.

Our main results are Theorems 3 and 5, and the nontrivial part of the proof is to find a way to take advantage of such a weak assumption on k. That way passes through the technical power of the main result sketched in [**1**] that we describe now.

Theorem. *Let (ω, g) be an equation having a positive increasing kernel and a strictly increasing continuous N-function nonlinearity. The equation (ω, g) is admissible if and only if for every $\lambda > 0$ the equation $(\lambda\omega, g)$ is admissible.*

Recall that g is an N-function if it transforms null sets into null sets. Note that, for a strictly increasing and continuous function, to be an N-function is equivalent to being a locally absolutely continuous function. This is Banach's result [**4**]. The original proof of this result is presented in [**5**, p. 288].

Let T_α and T_β be the associated operators to the equations $(\alpha\mu, g)$ and $(\beta\mu, g)$, respectively. Considering the main result of Szwarc in [**9**], if $(\alpha\mu, g)$ and $(\beta\mu, g)$ are admissible, there is uniqueness of solution for these equations and they attract (in the above sense) all positive functions.

With this set up we are ready to pass to the proof of a technical result.

Proposition 1. *Let g be a smooth strictly increasing function. If the equation (k, g) admits a nontrivial solution then it admits a maximal solution and a minimal solution. Moreover, both maximal and minimal solutions are increasing functions.*

Proof. Note that the nonlinearity is a smooth and strictly increasing function and therefore it is a strictly increasing locally absolutely continuous function.

By the general assumptions on k, for some positive constants α and β and some positive increasing kernel μ, one has $\alpha\mu \leq k \leq \beta\mu$. If (k, g) is admissible, then $(\beta\mu, g)$ and $(\alpha\mu, g)$ are admissible by the preceding theorem. Let u_α be the nontrivial solution for $(\alpha\mu, g)$ and let u_β be the nontrivial solution for $(\beta\mu, g)$ given by Szwarc's theorem.

Now we shall construct the maximal solution for (k, g). It is clear that if we set $T = T_{k,g}$ then $Tu_\beta \leq u_\beta$ holds. Thus, since T is a monotone operator,

$$(T^n u_\beta)_n$$

is a monotone decreasing sequence.

Claim. $u_\alpha \leq u_\beta$.

Proof of the claim. Since $u_\alpha = T_\alpha u_\alpha \leq T_\beta u_\alpha$, then $T_\beta^n u_\alpha \leq T_\beta^{n+1} u_\alpha$ for $n \geq 0$. The equation $(\beta\mu, g)$ satisfies the hypothesis of Szwarc's paper, so u_β attracts all positive functions and then

$$u_\beta = \lim_{n\to\infty} T_\beta^n u_\alpha.$$

Therefore, $T^n u_\alpha \leq u_\beta$ for $n \geq 0$. In particular, $u_\alpha \leq u_\beta$. $\square$

The monotone sequence $(T^n u_\beta)_n$ is bounded from below by u_α because

$$u_\alpha = T_\alpha u_\alpha \leq T u_\beta.$$

We can define

$$u_1 = \lim_{n\to\infty} T^n u_\beta.$$

A standard application of Lebesgue's dominated convergence theorem yields that u_1 is a solution for (k, g). Our aim is to see that u_1 is a maximal solution for (k, g). To prove this, we shall show that u_1 attracts all functions $f \geq u_1$, i.e.,

$$u_1 = \lim_{n\to\infty} T^n f.$$

Case 1. $f \geq u_\beta$. In this case

$$u_\beta = \lim_{n\to\infty} T_\beta^n f$$

and $T_\beta^n f \geq T^n f \geq u_1$ for each $n > 0$. Since T transforms bounded sets into relatively compact sets (T is a totally continuous operator) and $\{T^n f\}_n$ is a bounded set, $\{T^n f\}_n$ is a relatively compact set and, by the last inequality it is easy to see that the set of its accumulation points, that we denote by Ω_f, is bounded from above by u_β and bounded from below by u_1. Therefore, $\Omega_f = \{u_1\}$ because Ω_f is invariant by T and

$$u_1 = \lim_{n\to\infty} T^n u_\beta.$$

Case 2. Otherwise. Consider the function

$$f_\beta(x) = \max\{f(x), u_\beta(x)\}.$$

Since $u_1 \leq f \leq f_\beta$ and f_β is under the assumption of case 1 we have that $\Omega_{f_\beta} = \{u_1\}$ and

$$u_1 \leq \inf \Omega_f \leq \sup \Omega_f \leq \sup \Omega_{f_\beta} = u_1.$$

Thus $\Omega_f = \{u_1\}$ and the proof is complete.

To obtain a minimal solution for (k, g) one should work with $u_2 = \lim_{n\to\infty} T^n u_\alpha$ in an analogous way.

It is clear that u_1 and u_2 are increasing functions since they are limits of monotone sequences of increasing functions. □

Proposition 2. *The equation (k, g) has at most one positive increasing solution.*

The proof follows that of Proposition 1 in [**9**] by replacing Lemma 2 in [**9**] by

Lemma. *Let u be an increasing function satisfying $Tu(x) \geq u(x)$, and let*

$$v(x) = \begin{cases} u(x), & \text{if } 0 \leq x \leq c, \\ u(c), & \text{if } c < x. \end{cases}$$

There exists $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} T^n v(x) \geq u(x), \quad \text{for } c < x < c + \varepsilon,$$

and Lemma 3 in [**9**] by

Lemma. *Let u be a solution of T, and let*

$$v(x) = \begin{cases} u(x), & \text{if } 0 \leq x \leq c, \\ u(c), & \text{if } c < x. \end{cases}$$

There exists $\varepsilon > 0$ such that

$$\lim_{n\to\infty} T^n v(x) = u(x), \quad \text{for } c < x < c + \varepsilon.$$

This is possible because a close inspection of the proofs in [**9**] reveals that one does not need the full strength of a "strictly increasing kernel": one only needs that the kernel be bounded in each closed subinterval, a condition satisfied by kernels like positive increasing kernels.

If one compares Szwarc's statements with ours, then one shall see that we have replaced stronger assumptions on the equation (k, g) by some weaker ones at the cost of imposing stronger restrictions on the functions u ("increasing" in the first lemma instead of "any function"; and "nontrivial increasing solution" instead of "nontrivial solution" in the second).

Thus, we arrive at our main result.

Theorem 3. *Let k be a kernel like an increasing kernel and g a smooth strictly increasing nonlinearity, both vanishing at 0. If the equation (k, g) admits a nontrivial solution, then it is unique and attracts all positive functions.*

Proof. Let u be the solution of (k, g) and v a positive function. It is sufficient to see that u attracts v. Let us consider the functions

$$v_1(x) = \max\{v(x), u(x)\} \quad \text{and} \quad v_2(x) = \min\{v(x), u(x)\}.$$

One has $u = \lim_{n\to\infty} T^n v_i$, $i = 1, 2$, and therefore, $u = \lim_{n\to\infty} T^n v$. □

The following result proves that most nonnegative weakly homogeneous functions are like an increasing function.

Proposition 4. *Let f be a weakly homogeneous function such that $f(0) = 0$ with a lower transfer function bounded from below by a strictly positive constant in the interval $[1, \infty)$. Then f is like a positive increasing function.*

Proof. We are going to prove that f is like $\tilde{f}$, where

$$\tilde{f}(x) = \max_{s\in[0,x]} f(s).$$

It is immediate that $\tilde{f}(x) \geq f(x)$ for $x \geq 0$. We show that there exists a positive constant θ for which $\theta\tilde{f}(x) \leq f(x)$ for $x \geq 0$, which is equivalent to

$$f(x)/\tilde{f}(x) \geq \theta > 0, \quad \forall\, x > 0.$$

To calculate θ it is sufficient to observe the inequalities

$$\begin{aligned} \tilde{f}(x) &= \max_{s\in[0,x]} f(s) = \max_{\lambda\in[0,1]} f(\lambda, x) \\ &\leq f(x) \sup_{\lambda\in[0,1]} d(\lambda) \leq f(x)\rho, \end{aligned}$$

where $d(\lambda) = 1/c(1/\lambda)$. Therefore, $\theta = 1/\rho$ is a valid selection. $\square$

The hypothesis about the bound for the lower transfer function in the previous proposition is not so strong; it essentially means that there exists a positive constant α bounding f from below when x is large or else that $f(\lambda x)$ is not remarkably smaller than $f(x)$ for large x.

The results proved in this section allow us to assert

Theorem 5. *Let k be a weakly homogeneous kernel as in Proposition 4, g a smooth strictly increasing nonlinearity, $k(0) = 0$ and $g(0) = 0$. If the equation (k, g) admits a nontrivial solution, then it is unique and attracts all positive functions.*

2. Approximation of solutions. The idea for this section is to describe a simple way to approximate solutions of equations $u = Tu$ in some interval $[0, \delta]$, assuming the existence and uniqueness of a nontrivial solution attracting the positive functions and the monotonicity of the operator T.

We construct a monotone decreasing sequence of continuous increasing functions converging to the solution u.

In what follows M denotes a positive constant function bounding u from above in a closed interval $[0, \delta_0]$. Since TM is an increasing continuous function such that $TM(0) = 0$, there is a closed interval $[0, \rho]$ where $TM(x) \leq M$. We assume that $[0, \rho] = [0, \delta_0]$ since our aim is to describe an iterative method defined on a zero closed interval. The characteristic function of an interval I shall be denoted by χ_I.

Let $\alpha < 1$ be a constant. Since TM is uniformly continuous on $[0, \delta_0]$, there exists $\delta_1 > 0$:

$$|x - y| < \delta_1 \Longrightarrow |TM(x) - TM(y)| < \alpha.$$

Take a finite partition $P_1 = \{x_i\}_i$ of the interval $[0, \delta_0]$ for which $x_i - x_{i-1} < \delta_1$.

Let M_1 be the function

$$M_1(x) = \sum_{x_i \in P_i} TM(x_i)\chi_{[x_{i-1}, x_i)}(x)$$

on $[0, \delta_0]$. It is easy to see that

$$0 \leq M_1(x) - TM(x) < \alpha \quad \text{and} \quad M(x) \geq M_1(x) \geq TM_1(x),$$

on $[0, \delta_0]$. Taking into account the monotonicity of T, one has

$$M_1(x) \geq TM(x) \geq TM_1(x), \quad \forall\, x \in [0, \delta_0].$$

Repeating the previous constructions with TM_1 and α^2 instead of TM and α, a positive δ_2 and a finite partition $P_2 = \{x_i\}_i$ can be found of the interval $[0, \delta_0]$ such that $x_i - x_{i-1} < \delta_2$, and we can define the function

$$M_2(x) = \sum_{x_i \in P_2} TM_1(x_i)\chi_{[x_{i-1}, x_i)}(x),$$

on $[0, \delta_0]$. In this case

$$0 \leq M_2(x) - TM_1(x) < \alpha^2 \quad \text{and} \quad M_1(x) \geq M_2(x) \geq TM_1(x),$$

on $[0, \delta_0]$. By the monotonicity of T one has $M_2(x) \geq TM_2(x)$ on $[0, \delta_0]$.

Proceeding inductively one obtains a sequence (M_n) of simple functions such that

$$M_n(x) \geq TM_n(x) \quad \text{and} \quad M_{n-1}(x) \geq M_n(x) \geq TM_{n-1}(x)$$

hold on $[0, \delta_0]$. Moreover, $(T^m M_n)$ converges to u on $[0, \delta_0]$. The second group of inequalities implies that the monotone sequences (M_n) and (TM_n) are convergent and their limit is a solution of (k, g). Let f be the limit of (M_n). One sees that f is a solution of $u = Tu$ since

$$\begin{aligned} |f(x) - Tf(x)| \leq |f(x) - M_n(x)| &+ |M_n(x) - TM_n(x)| \\ &+ |TM_n(x) - Tf(x)| \leq \varepsilon. \end{aligned}$$

Therefore, (TM_n) converges to u. So, one has

Theorem. *Let (k, g) be a nonlinear Volterra integral equation with a monotone associated operator T and a unique attracting nontrivial solution. There exist sequences (M_n) of simple functions such that (M_n) and (TM_n) converge to the nontrivial solution.*

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