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FAST COLLOCATION SOLVERS FOR INTEGRAL EQUATIONS ON OPEN ARCS

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ABSTRACT. In this work we develop a unified approach for numerical approximation and fast solution of classical integral equations on open arcs. The approximation is obtained applying the cosine transform and fully discrete trigonometric collocation together with an asymptotic approximation of the operator. The computed approximation is of optimal accuracy order in a large scale of Sobolev norms, and it can be obtained in $\mathcal{O}(n \log n)$ arithmetical operations. Our results cover logarithmic singular integral equations, Cauchy singular integral equations, as well as hypersingular integral equations.

1. Introduction. In many applications the boundary integral method leads to solution of an integral equation on an open arc, when two-dimensional phenomena are considered. In the basic examples the arising integral equations can be covered by the following types: logarithmic singular integral equations, Cauchy singular integral equations and hypersingular integral equations. For the parametrized forms of the model equations see (2.1)–(2.3). Equations of these types come from various fields such as fracture mechanics, aerodynamics, electromagnetism and elasticity, for example. Except for some special cases the arising integral equation cannot be solved explicitly but requires an approximate solution by numerical methods. In the literature various numerical schemes have been proposed for particular examples. These schemes cover the standard spline based methods [**31, 8, 38, 12**] as well as trigonometric methods and polynomial approximation with their fully discrete variants including also other quadrature methods [**17, 2, 9, 25, 5, 19, 6, 7, 11**]. This list is not complete, in particular for Cauchy singular equations there are many earlier studies but they can be traced from the works mentioned here. Also fast solution has been considered: for the case of Cauchy singular equation with a polynomial approximation, see [**3, 4**].

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The purpose of this work is to develop a unified approach for numerical approximation and fast solution of classical integral equations on open arcs. For our approach we proceed as follows. Starting from the parametrized form we first apply the cosine transform and obtain for the original problem an equivalent formulation as a periodic problem of certain parity. It is well-known that the cosine transform has some obvious advantages. In particular, in the case of logarithmic singular and Cauchy singular equations on an open arc Γ, the solution may have a singularity of the order $\mathcal{O}(|x-c|^{-1/2})$ at the endpoint c of Γ even when the righthand term of the equation is smooth. The cosine transform removes this singularity. Moreover, for smooth $b_k(x,y)$ and $g(x)$, see (2.1)–(2.3), also the coefficients and the righthand term of the periodized problem are smooth, consequently so is the solution of the periodized problem. Thus the periodized forms of problems (2.1)–(2.3) are rather convenient for an approximate solution. We use a fully discrete version, cf. [**28**], of trigonometric collocation method for determining low frequencies of the solution. The computation of high frequencies is based on some asymptotic approximation of the operator. The approximation, see (8.10), (8.11) and their matrix form, Sections 9, 10, is organized so that the application of the $n\times n$-stiffness matrix to an n-vector costs $\mathcal{O}(n\log n)$ arithmetical operations. This allows us to solve the discrete problem in $\mathcal{O}(n\log n)$ operations using a suitable two grid iteration. The computed approximation is of optimal accuracy order in a scale of Sobolev norms. This scale is of maximal length in the situation where only grid values of the righthand term are used. We also point out that in case of analytic data and solution, [**28**] provides exponential convergence.

2. Parametrized equation. In our analysis we start from an integral equation which is given on the open interval $I=(0,1)$ and apply the cosine transform to obtain a more convenient form of the equation which allows application of an approach for fast solution based on [**29**]. For the parametrized equations we assume one of the following three types

$$(2.1)\qquad \begin{aligned}(B_L v)(x) :&= \int_0^1 (b_0(x,y)\log|x-y| + b_1(x,y))v(y)\,dy\\ &= g(x),\quad x\in I,\end{aligned}$$

(2.2)
$$(B_C v)(x) := \int_0^1 \left(\frac{b_0(x,y)}{x-y} + b_1(x,y) \log|x-y| + b_2(x,y) \right) v(y)\, dy$$
$$= g(x), \quad x \in I,$$

(2.3)
$$(B_H v)(x) := \int_0^1 \left(\frac{b_0(x,y)}{|x-y|^2} + b_1(x,y) \log|x-y| + b_2(x,y) \right) v(y)\, dy$$
$$= g(x), \quad x \in I.$$

Here we assume that $b_k \in C^\infty(\bar{I} \times \bar{I})$, $k = 0, 1, 2$ and $b_0(x,x) \neq 0$, $x \in \bar{I}$. The first integrals in (2.2) and (2.3) are understood in the sense of the principal value and the finite-part of Hadamard, respectively. Introduce the weighted spaces $L^2_\sigma(I)$, $L^2_{1/\sigma}(I)$ and $H^1_\sigma(I)$ of functions having a finite norm

$$\|v\|_\sigma = \left(\int_0^1 \sigma(y)|v(y)|^2\, dy \right)^{1/2},$$
$$\sigma(y) = y^{1/2}(1-y)^{1/2},$$
$$\|v\|_{1/\sigma} = \left(\int_0^1 \frac{1}{\sigma(y)}|v(y)|^2\, dy \right)^{1/2},$$
$$\|v\|_{1,\sigma} = (\|v\|^2_\sigma + \|v'\|^2_\sigma)^{1/2},$$

respectively. Define also $\overset{\circ}{H}{}^1_\sigma(I) = \{v \in H^1_\sigma(I) \mid v(0) = v(1) = 0\}$ with the norm induced from $H^1_\sigma(I)$. The following mapping properties of B_L, B_C and B_H can be obtained from our consideration:

$$\begin{aligned}
B_L : L^2_\sigma(I) &\longrightarrow H^1_\sigma(I) && \text{is a Fredholm operator of index } 0,\\
B_C : L^2_\sigma(I) &\longrightarrow L^2_\sigma(I) && \text{is a Fredholm operator of index } 1,\\
B_C : L^2_{1/\sigma}(I) &\longrightarrow L^2_{1/\sigma}(I) && \text{is a Fredholm operator of index } -1,\\
B_H : \overset{\circ}{H}{}^1_\sigma(I) &\longrightarrow L^2_\sigma(I) && \text{is a Fredholm operator of index } 0.
\end{aligned}$$

In the case of B_C the above properties are explicitly given, for the Cauchy operator, in [**16**]. For B_L and B_H also the mapping properties $B_L : \tilde{H}^{-1/2}(I) \to H^{1/2}(I)$ and $B_H : \tilde{H}^{1/2}(I) \to H^{-1/2}(I)$ are known, see [**31, 36**], but we will not use those (for the definitions of spaces $H^s(I)$ and $\widetilde{H}^s(I) = H^s_{00}(I)$, $s \in \mathbf{R}$, see, e.g., [**14**]).

Example 2.1. Equations with a logarithmic singular kernel. Let Γ be a smooth open arc in the plane. Consider the integral equation

$$(S_{k,\Gamma} v_\Gamma)(z) = \int_\Gamma G_k(z,\xi) v_\Gamma(\xi)\, ds_\xi g_\Gamma(z), \quad z \in \Gamma \tag{2.1a}$$

where ds_ξ denotes the integration with respect to the arc-length and $G_k(z,\xi)$ is the fundamental solution to the Laplace equation if $k=0$ and to the Helmholtz equation if $k \neq 0$,

$$\begin{aligned} G_0(z,\xi) &= -\frac{1}{2\pi} \log|z-\xi|, \\ G_k(z,\xi) &= \frac{i}{4} H_0^{(1)}(k|z-\xi|), \quad k \neq 0, \end{aligned} \tag{2.1b}$$

where $H_0^{(1)}(z)$ is the Hankel function of the first kind and order zero. With $k=0$ we have Symm's equation with the logarithmic kernel. It arises when solving the potential equation with the Dirichlet boundary condition in the exterior domain of Γ. With $k \neq 0$ equation (2.1a) appears in solution of the exterior Dirichlet problem for the Helmholtz equation which arises from the two-dimensional time-harmonic scattering problem at a soft screen. For a general k the singularity of the kernel G_k is also logarithmic. Let $x \mapsto z(x)$, $x \in I$ be a parametrization of the arc Γ. We transform the integral operator $S_{k,\Gamma}$ on Γ to the integral operator S_k on I where

$$(S_k v)(x) := \int_0^1 G_k(z(x), z(y)) |z'(y)| v(y)\, dy. \tag{2.1c}$$

Then we have $(S_k v)(x) = (S_{k,\Gamma} v_\Gamma)(z(x))$ if $v(x) = v_\Gamma(z(x))$. Now equation (2.1a) becomes

$$\begin{aligned} (S_k v)(x) &= -\frac{1}{2\pi} \int_0^1 (|z'(y)| \log|x-y| + b_k(x,y)) v(y)\, dy \\ &= g(x), \quad x \in I \end{aligned} \tag{2.1d}$$

where $b_k(x,y)$ is a smooth function and $g(x) = g_\Gamma(z(x))$. In some appropriate function spaces the equations (2.1a) and (2.1d) are uniquely solvable if $k = 0$ and $\operatorname{Cap}(\overline{\Gamma}) \neq 1$ [**24, 37**], or if $k \neq 0$ and $\operatorname{Im} k \geq 0$ [**27, 30, 31**]. Here $\operatorname{Cap}(\overline{\Gamma})$ is the logarithmic capacity or, equivalently, the transfinite diameter of $\overline{\Gamma}$. In fact, under these conditions, S_k defines an isomorphism from $L^2_\sigma(I)$ to $H^1_\sigma(I)$.

Example 2.2. Cauchy-singular integral equations. The singular integral equation

(2.2a)
$$\int_0^1 \left(\frac{b_0(x,y)}{x-y} + b_1(x,y) \log|x-y| + b_2(x,y) \right) v(y)\, dy = g(x), \quad x \in I,$$

appears in several applications concerning flow problems around airfoils. In particular, with the constant function $b_0(x,y) = b_0 \neq 0$ and $b_1 = b_2 = 0$, we have the basic airfoil equation. Equations including the logarithmic term and the function $b_2(x,y)$ appear also in applications; an example coming from modeling the circulation around freely moving weakly loaded propellers is discussed in [**20, 23**]. Another example which appears in solving the pressure distribution around thin oscillating airfoils in a ventilated wind tunnel is described in [**10**]. However, equation (2.2a) is not yet uniquely solvable in these examples, but the uniqueness is assured by imposing an additional condition of the form

$$\Phi_I v := \int_0^1 v(y)\, dy = \gamma \tag{2.2b}$$

which has the interpretation that the circulation around the profile is given. So, instead of (2.2a) we have to consider the system

$$\begin{aligned} (B_C v)(x) &= g(x), \quad x \in I, \\ \Phi_I v &= \gamma. \end{aligned} \tag{2.2c}$$

In the case of the special example given in [**23**] the system (2.2c) is uniquely solvable in $v \in L^2_\sigma(I)$ for any given data $g \in L^2_\sigma(I)$, $\gamma \in \mathbf{C}$.

Example 2.3. Hypersingular equations. Consider the hypersingular integral equation

$$\begin{aligned} (H_{k,\Gamma} v_\Gamma)(z) &= -\frac{\partial}{\partial n_z} \int_\Gamma \frac{\partial}{\partial n_\xi} G_k(z,\xi) v_\Gamma(\xi)\, ds_\xi \\ &= g_\Gamma(z), \quad z \in \Gamma \end{aligned} \tag{2.3a}$$

where $G_k(z,\xi)$ is the fundamental solution defined in Example 2.1. Here we additionally assume that Γ is an oriented arc, n_z being the unit normal vector at the point $z \in \Gamma$. Substituting $z = z(x)$, $\xi = z(y)$, the kernel can be decomposed as

$$\text{(2.3b)} \quad -\frac{\partial}{\partial n_z}\frac{\partial}{\partial n_\xi} G_k(z,\xi) = -\frac{2\pi}{|z'(x)||x-y|^2} + b_{1k}(x,y)\log|x-y| + b_{2k}(x,y)$$

where b_{1k}, b_{2k} are smooth functions, see, e.g., [**18**]. Now equation (2.3a) reduces to an integral equation on I,

(2.3c)
$$\int_0^1 \Big(-\frac{2\pi}{|z'(x)||x-y|^2} + b_{1k}(x,y)\log|x-y| + b_{2k}(x,y)\Big) v(y)\,dy = g(x), \quad x \in I$$

where $v(x) = v_\Gamma(z(x))$, $g(x) = g_\Gamma(z(x))$. Equations (2.3a) and (2.3c) are uniquely solvable if $k \neq 0$ and $\text{Im} \geq 0$ [**27, 36**]. Equation (2.3a) arises from the Neumann-type exterior problem for the potential equation, $k = 0$, and for the Helmholtz equation, $k \neq 0$, which appears in the time-harmonic scattering at a hard screen.

3. Periodization of the parametrized equation. Having an integral equation which is given by one of the forms (2.1)–(2.3) on $I = (0,1)$, we apply the cosine transform

$$\text{(3.1)} \qquad x = x(t) = (1 - \cos 2\pi t)/2, \quad t \in (0, 1/2).$$

After this transform we obtain a new integral equation for the unknown function u and the righthand side f on $(0,1/2)$. The new kernel is defined in a natural way already on the symmetric interval $(-1/2, 1/2)$ and, moreover, has a natural 1-biperiodic extension to $\mathbf{R}^2$. The final form of the equation is obtained by extending the functions u and f as even or odd functions to $\mathbf{R}$.

3.1 *Equations with a logarithmic singular kernel.* We recall the equations of the general form (2.1),

$$\text{(3.2)} \qquad \int_0^1 (b_0(x,y)\log|x-y| + b_1(x,y))v(y)\,dy = g(x), \quad x \in I$$

where $b_0, b_1 \in C^\infty(\bar{I} \times \bar{I})$. Applying the transform (3.1), extending $x(t)$ by the formula (3.1) for all $t \in \mathbf{R}$ and writing $u(t) = v(x(t))|x'(t)|$, $f(t) = g(x(t))$, $t \in \mathbf{R}$, we obtain

$$\text{(3.3)} \quad \int_0^{1/2} (b_0(x(t), x(s)) \log |x(t) - x(s)| + b_1(x(t), x(s)))u(s)\, ds = f(t), \quad t \in (0, 1/2).$$

Since u and f are even functions and the kernel is an even function with respect to both of the arguments t and s, equation (3.3) is equivalent to

$$\text{(3.4)} \quad \frac{1}{2} \int_{-1/2}^{1/2} (b_0(x(t), x(s)) \log |x(t) - x(s)| + b_1(x(t), x(s)))u(s)\, ds = f(t), \quad t \in \mathbf{R}.$$

Here we have

$$\begin{aligned} \frac{1}{2} \int_{-1/2}^{1/2} b_0(x(t), x(s)) \log |x(t) - x(s)| u(s)\, ds &= \frac{1}{2} \int_{-1/2}^{1/2} b_0(x(t), x(s)) \log |\sin \pi (t-s)| u(s)\, ds \\ &+ \frac{1}{2} \int_{-1/2}^{1/2} b_0(x(t), x(s')) \log |\sin \pi (t+s')| u(s')\, ds'. \end{aligned}$$

Substituting $s' = -s$ in the last term, we obtain

$$\frac{1}{2} \int_{-1/2}^{1/2} b_0(x(t), x(s)) \log |x(t) - x(s)| u(s)\, ds = \int_{-1/2}^{1/2} b_0(x(t), x(s)) \log |\sin \pi (t-s)| u(s)\, ds.$$

Hence we have reduced (3.4) to the more convenient form

$$\text{(3.5)} \quad (A_L u)(t) := \int_{-1/2}^{1/2} (a_0(t,s) \kappa_0(t-s) + a_1(t,s)) u(s)\, ds = f(t), \quad t \in \mathbf{R}$$

where u and f are 1-periodic even functions on $\mathbf{R}$ and $a_0(t,s) = b_0(x(t),x(s))$, $a_1(t,s) = b_1(x(t),x(s))/2$ are smooth 1-biperiodic even functions, and $\kappa_0(t) = \log|\sin\pi t|$.

3.2 *Cauchy-singular equations.* Here we consider equations of the general form (2.2),

(3.6)
$$\int_0^1 \left(\frac{b_0(x,y)}{x-y} + b_1(x,y)\log|x-y| + b_2(x,y)\right)v(y)\,dy = g(x), \quad x\in I$$

where $b_k \in C^\infty(\bar I\times\bar I)$. We introduce two different periodizations for (3.6). First let $u(t)$, $t\in\mathbf{R}$ be the 1-periodic even extension of the function $u(t) = v(x(t))x'(t)$, $t\in(0,1/2)$. Then, due to parity properties, $\int_{-1/2}^{1/2}(b_0(x(t),x(s))x'(s)u(s))/(\cos 2\pi s - \cos 2\pi t)\,ds = 0$ and equation (3.6) is equivalent with

$$\begin{aligned}(3.7)\quad & \frac{1}{2}\int_{-1/2}^{1/2}\frac{b_0(x(t),x(s))(x'(t)-x'(s))u(s)}{(\cos 2\pi s - \cos 2\pi t)/2}\,ds \\ &\quad + \frac{1}{2}\int_{-1/2}^{1/2} b_1(x(t),x(s))x'(t)\log|x(t)-x(s)|u(s)\,ds \\ &\quad + \frac{1}{2}\int_{-1/2}^{1/2} b_2(x(t),x(s))x'(t)u(s)\,ds \\ &= x'(t)g(x(t)), \quad t\in\mathbf{R}.\end{aligned}$$

For the first term T_0, we obtain

$$\begin{aligned}(3.8)\quad T_0 &= \int_{-1/2}^{1/2}\frac{b_0(x(t),x(s))(x'(t)-x'(s))u(s)}{(\cos 2\pi s - \cos 2\pi t)}\,ds \\ &= \pi\int_{-1/2}^{1/2} b_0(x(t),x(s))\,\cot\pi(t-s)u(s)\,ds.\end{aligned}$$

Here we have the formula $(\sin x - \sin y)/(\cos x - \cos y) = -\cot((x+y)/2)$. Putting $f(t) = x'(t)g(x(t))$, we see that (3.7) is of the form

$$\begin{aligned}(3.9\text{a})\quad (A_{1C}u)(t) := \int_{-1/2}^{1/2}&(a_0(t,s)\kappa_0(t-s) + a_{11}(t,s)\kappa_1(t-s) \\ &+ a_{12}(t,s))u(s)\,ds \\ = f(t),&\quad t\in\mathbf{R},\end{aligned}$$

where u is an even and f an odd 1-periodic function in $\mathbf{R}$. Moreover,

(3.9b)
$$\kappa_0(t) = \cot \pi t, \qquad \kappa_1(t) = \log|\sin \pi t|, \qquad a_0(t,s) = \pi b_0(s(t), x(s)),$$
$$a_{11}(t,s) = b_1(x(t), x(s))x'(t), \qquad a_{12}(t,s) = b_2(x(t), x(s))x'(t)/2.$$

In the applications connected to Example 2.2, we have to take the additional condition (2.2b) into account. By the cosine transform we obtain

$$\Phi_I v = \Phi u := \frac{1}{2}\int_{-1/2}^{1/2} u(s)\,ds = \gamma. \tag{3.9c}$$

Define the operator $A_{1C} \times \Phi$ by $(A_{1C} \times \Phi)u = [A_{1C}u, \Phi u]$. Now the system of the equations (3.9a) and (3.9c) is given by a single equation: for given $[f, \gamma]$ find the function u such that

$$(A_{1C} \times \Phi)u = [f, \gamma]. \tag{3.10}$$

In the other periodization of (3.6) the function $u(t)$ is chosen to be the odd 1-periodic extension of $v(x(t))$, $0 < t < 1/2$ and $f(t) = g(x(t))$, $t \in \mathbf{R}$ is even. Proceeding in a similar manner as above, we find that (3.6) is equivalent to

$$\begin{aligned}(A_{2C}u)(t) := \int_{-1/2}^{1/2} (a_0(t,s)\kappa_0(t-s) + a_{21}(t,s)\kappa_1(t-s) \\ + a_{22}(t,s))u(s)\,ds \\ = f(t), \quad t \in \mathbf{R}.\end{aligned} \tag{3.11}$$

Here the functions $a_0(t,s)$, $\kappa_0(t)$ and $\kappa_1(t)$ are the same as for A_{1C} but

$$a_{21}(t,s) = b_1(x(t), x(s))x'(s), \qquad a_{22}(t,s) = b_2(x(t), x(s))x'(s)/2.$$

In the case of the second formulation we do not use any additional condition for the uniqueness but we insert a new parameter ω in order to obtain a uniquely solvable equation for all righthand sides f. Thus we shall consider the solution of the equation

$$A_{2C}u + w = f. \tag{3.12}$$

3.3 *Hypersingular equations.* We recall the equation (2.3),

$$\int_0^1 \left(\frac{b_0(x,y)}{|x-y|^2} + b_1(x,y)\log|x-y| + b_2(x,y)\right)v(y)\,dy = g(x), \quad x \in I. \tag{3.13}$$

Applying the cosine transform and multiplying the resulting equation by $x'(t)$, we obtain for $u(t) = v(x(t))$, $f(t) = x'(t)g(x(t))$, the equation

$$\int_0^{1/2} K(t,s)u(s)\,ds = f(t), \quad 0 < t < \frac{1}{2}, \tag{3.14}$$

where $K = K_0 + K_1 + K_2$ with

$$\begin{aligned} K_0(t,s) &= \frac{b_0(x(t),x(s))x'(t)x'(s)}{|\cos 2\pi t - \cos 2\pi s|^2/4}, \\ K_1(t,s) &= b_1(x(t),x(s))x'(t)x'(s)\log|\sin\pi(t-s)\sin\pi(t+s)| \\ K_2(t,s) &= b_2(x(t),x(s))x'(t)x'(s). \end{aligned}$$

The function f is an odd function on $\mathbf{R}$, and we extend u as an odd function to the whole real axis. By the parity properties of the functions K_j, f and u, (3.14) is equivalent to

$$\frac{1}{2}\int_{-1/2}^{1/2} K(t,s)u(s)\,ds = f(t), \quad t \in \mathbf{R}. \tag{3.15}$$

By using the relation $(\cos 2\pi t - \cos 2\pi s)^2/4 = (\sin\pi(t-s)\sin\pi(t+s))^2$ and

$$\frac{\sin 2\pi t \sin 2\pi s}{(\sin\pi(t-s)\sin\pi(t+s))^2} = \frac{1}{\sin^2\pi(t-s)} - \frac{1}{\sin^2\pi(t+s)}$$

we obtain

$$\frac{1}{2}\int_{-1/2}^{1/2} K_0(t,s)u(s)\,ds = \pi^2\int_{-1/2}^{1/2}\frac{b_0(x(t),x(s))}{\sin^2\pi(t-s)}u(s)\,ds.$$

Moreover, we have

$$\begin{aligned} &\frac{1}{2}\int_{-1/2}^{1/2} K_1(t,s)u(s)\,ds \\ &\qquad = \pi^2\int_{-1/2}^{1/2} b_1(x(t),x(s))x'(t)x'(s)\log|\sin\pi(t-s)|u(s)\,ds, \end{aligned}$$

and (3.15) becomes for the 1-periodic odd functions u, f,

$$(3.16)\qquad \begin{aligned}(A_H u)(t) := \int_{-1/2}^{1/2} (a_0(t,s)\kappa_0(t-s) + a_1(t,s)\kappa_1(t-s)\\ + a_2(t,s))u(s)\,ds\\ = f(t), \quad t \in \mathbf{R},\end{aligned}$$

where

$$\begin{aligned}\kappa_0(t) &= (\sin^2 \pi t)^{-1}, \kappa_1(t) = \log|\sin \pi t|,\\ a_0(t,s) &= \pi^2 b_0(x(t), x(s)),\\ a_1(t,s) &= b_1(x(t), x(s))x'(t)x'(s),\\ a_2(t,s) &= b_2(x(t), x(s))x'(t)x'(s)/2.\end{aligned}$$

Remark 3.1. The cosine substitution was introduced by Multhopp in [**15**] for the airfoil equation of Prandtl, see also [**35, 22, 21**]. For Symm's equation it was applied by Yan and Sloan [**37**] and for the basic hypersingular integral equation on an interval by Bühring [**7**].

4. Even and odd operators. By means of the periodization we have transformed all the equations (2.1)–(2.3) to the form

$$(4.1)\qquad \begin{aligned}\int_{-1/2}^{1/2} (a_0(t,s)\kappa_0(t-s) + a_1(t,s)\kappa_1(t-s)\\ + a_2(t,s))u(s)\,ds = f(t), \quad t \in \mathbf{R},\end{aligned}$$

where u and f are 1-periodic and $a_p \in C^\infty_{1,1}(\mathbf{R}^2)$, the space of all 1-biperiodic smooth functions. Moreover, it holds that $a_0(t,t) \neq 0$, $t \in \mathbf{R}$. Equation (4.1) is a special example of equations analyzed in [**28**]. In order to include more applications than just those connected to equations (2.1)–(2.3), we assume the general form used in [**28**]. We consider the equation

$$(4.2)\qquad \mathcal{A}u = f,$$

where $\mathcal{A} = \sum_{p=0}^{q} A_p$ such that

$$(4.3)\qquad (A_p u)(t) = \int_{-1/2}^{1/2} \kappa_p(t-s) a_p(t,s) u(s)\,ds, \quad a_p \in C^\infty_{1,1}(\mathbf{R}^2).$$

Furthermore, we assume that κ_p, $0 \le p \le q$, are 1-periodic distributions on $\mathbf{R}$ such that the Fourier coefficients satisfy for $\alpha \in \mathbf{R}$,

$$|\Delta^k \hat{\kappa}_p(l)| \le c_k |l|^{\alpha-p-k}, \quad 0 \ne l \in \mathbf{Z}, \quad k \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}, \quad p = 0, 1, \dots, q. \tag{4.4}$$

Here Δ is the difference operator, $\Delta\hat{\kappa}_p(l) = \hat{\kappa}_p(l+1) - \hat{\kappa}_p(l)$. Due to (4.3) and (4.4), $A_p \in Op \sum^{\alpha-p}$, i.e., A_p is a periodic pseudodifferential operator of order $\alpha - p$. On the main part A_0 of the operator $\mathcal{A}$ we impose the following condition for a positive number c_{00}:

$$|\hat{\kappa}_0(l)| \ge c_{00}|l|^{\alpha}, \quad 0 \ne l \in \mathbf{Z}, \tag{4.5a}$$

$$a_0(t,t) \ne 0, \quad t \in \mathbf{R}. \tag{4.5b}$$

It follows from (4.3)–(4.5) that $\mathcal{A}$ is an elliptic periodic pseudodifferential operator of order α, see [**33**], and $\mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ for any $\lambda \in \mathbf{R}$. Here H^λ is the Sobolev space of 1-periodic distributions u with the norm

$$\|u\|_\lambda = \left(\sum_{k \in \mathbf{Z}} [\max(1, |k|)]^{2\lambda} |\hat{u}(k)|^2 \right)^{1/2},$$

$$\hat{u}(k) = \int_{-1/2}^{1/2} u(s) e^{-ik2\pi s}\, ds = \langle u, e^{-ik2\pi t} \rangle.$$

Moreover, $\mathcal{A} : H^\lambda \to H^{\lambda-\alpha}$ is a Fredholm operator of index zero for any $\lambda \in \mathbf{R}$, and $N(\mathcal{A}) = \{u \in H^\lambda \mid \mathcal{A}u = 0\} \subset C_1^\infty(\mathbf{R})$ is independent of λ; $N(\mathcal{A}) = \{u \in C_1^\infty(\mathbf{R}) \mid \mathcal{A}u = 0\}$. Therefore, if

$$\mathcal{A}u = 0, \quad u \in C_1^\infty(\mathbf{R}) \;\Rightarrow\; u = 0, \tag{4.6}$$

then $\mathcal{A} : H^\lambda \to H^{\lambda-\alpha}$, $\lambda \in \mathbf{R}$ is an isomorphism.

Next we discuss solvability of (4.2) in the special cases of (2.1)–(2.3). The transformed equations (3.5), (3.9) and (3.16) were derived by using special parity properties of the functions u, f and the operator A. These properties are also utilized when discussing the unique solvability of (4.2). For this we introduce the Sobolev spaces H_e^λ and H_o^λ of the even and odd functions

$$\begin{aligned} H_e^\lambda &= \{u \in H^\lambda \mid u(-t) = u(t)\} \\ &= \{u \in H^\lambda \mid \hat{u}(-n) = \hat{u}(n), n \in \mathbf{Z}\}, \\ H_o^\lambda &= \{u \in H^\lambda \mid u(-t) = -u(t)\} \\ &= \{u \in H^\lambda \mid \hat{u}(-n) = -\hat{u}(n), n \in \mathbf{Z}\}. \end{aligned}$$

These spaces are closed subspaces of H^λ, and H^λ is represented as the direct sum $H^\lambda = H_e^\lambda + H_o^\lambda$. Let $P_e : H^\lambda \to H_e^\lambda$ and $P_o : H^\lambda \to H_o^\lambda$ be the corresponding projections. For $u \in H^\lambda$ we write $u_e = P_e u$, $u_o = P_o u$. We say that the operator $\mathcal{A}$ is an *even operator* if $\mathcal{A}$ does not change the parity of the function, i.e., if $P_e \mathcal{A} u_{ee} = \mathcal{A} u_e$, $P_0 \mathcal{A} u_0 = \mathcal{A} u_0$ holds. The operator $\mathcal{A}$ is an *odd operator* if $\mathcal{A}$ changes the parity, i.e., we have $P_e \mathcal{A} u_o = \mathcal{A} u_o$, $P_o \mathcal{A} u_e = \mathcal{A} u_e$. Equivalently, $\mathcal{A}$ is even if $\mathcal{A} P_e = P_e \mathcal{A}$ and $\mathcal{A}$ is odd if $\mathcal{A} P_e = P_o \mathcal{A}$. Consider the solution of equation (4.2). If $\mathcal{A}$ is even, the equation (4.2) is equivalent to the system

$$\mathcal{A} u_e = f_e, \qquad \mathcal{A} u_o = f_o. \tag{4.7}$$

Similarly, if $\mathcal{A}$ is odd, the equation (4.2) is equivalent to the system

$$\mathcal{A} u_o = f_e, \qquad \mathcal{A} u_e = f_o. \tag{4.8}$$

In our applications which arise from applying the cosine transform to an equation on an open arc, we do not use the whole system (4.7) or (4.8) but just one or other of the equations appearing in these systems.

Now we characterize the parity properties of an integral operator through its kernel. Consider a general term $A_p = A$ in the representation of $\mathcal{A}$ defined by

$$(Au)(t) = \int_{-1/2}^{1/2} \kappa(t-s) a(t,s) u(s)\, ds, \tag{4.9}$$

$$a \in C_{1,1}^\infty(\mathbf{R}^2), \quad |\Delta^k \hat{\kappa}(l)| \le c|l|^{\alpha-k}, \quad 0 \ne l \in \mathbf{Z}, k \in \mathbf{N}_0. \tag{4.10}$$

Introduce the conditions

$$\kappa \text{ is even, i.e., } \hat{\kappa}(-l) = \hat{\kappa}(l), \quad l \in \mathbf{Z}, \tag{4.11}$$

$$\kappa \text{ is odd, i.e., } \hat{\kappa}(-l) = -\hat{\kappa}(l), \quad l \in \mathbf{Z}, \tag{4.12}$$

$$a \text{ is even, i.e., } a(-t-s) = a(t,s), \quad t, s \in \mathbf{R} \tag{4.13}$$

$$a \text{ is odd, i.e., } a(-t,-s) = -a(t,s), \quad t, s \in \mathbf{R}. \tag{4.14}$$

Observe that (4.13) is equivalent to the condition $\hat{a}(-k,-j) = \hat{a}(k,j)$, $k, j \in \mathbf{Z}$, and (4.14) is equivalent to $\hat{a}(-k,-j) = -\hat{a}(k,j)$, $k, j \in \mathbf{Z}$.

Lemma 4.1. (i) *A is even if conditions* {(4.11), (4.13)} *or* {(4.12),(4.14)} *are fulfilled.*

(ii) *A is odd if conditions* {(4.11),(4.14)} *or* {(4.12),(4.13)} *are fulfilled.*

Proof. By definition

$$A \text{ is even iff} \left\{ \begin{array}{l} \hat{u}(-j) = \hat{u}(j), j \in \mathbf{Z} \ \Rightarrow \ (\widehat{Au})(-k) = (\widehat{Au})(k), k \in \mathbf{Z}; \\ \hat{u}(-j) = -\hat{u}(j), j \in \mathbf{Z} \ \Rightarrow \ (\widehat{Au})(-k) = -(\widehat{Au})(k), k \in \mathbf{Z} \end{array} \right\}$$

$$A \text{ is odd iff} \left\{ \begin{array}{l} \hat{u}(-j) = \hat{u}(j), j \in \mathbf{Z} \ \Rightarrow \ \widehat{Au})(-k) = -(\widehat{Au})(k), k \in \mathbf{Z}; \\ \hat{u}(-j) = -\hat{u}(j), j \in \mathbf{Z} \ \Rightarrow \ (\widehat{Au})(-k) = (\widehat{Au})(k), k \in \mathbf{Z} \end{array} \right\}.$$

We have, see, e.g., [**33**, p. 90],

$$(\widehat{Au})(k) = \sum_{j,l \in \mathbf{Z}} \hat{a}(k-l, l-j)\hat{\kappa}(l)\hat{u}(j), \quad k \in \mathbf{Z}.$$

Respectively,

$$\begin{aligned} (\widehat{Au})(-k) &= \sum_{j,l \in \mathbf{Z}} \hat{a}(-k-l, l-j)\hat{\kappa}(l)\hat{u}(j) \\ &= \sum_{j,l \in \mathbf{Z}} \hat{a}(-(k-l), -(l-j))\hat{\kappa}(-l)\hat{u}(-j). \end{aligned}$$

Now the assertions of the lemma easily follow. □

Remark 4.1. Condition (4.13) holds true, if one of the following is valid

(4.15a)
$$a(-t,s) = a(t,s), \qquad a(t,-s) = a(t,s), \quad t, s \in \mathbf{R},$$

(4.15b)
$$a(-t,s) = -a(t,s), \qquad a(t,-s) = -a(t,s), \quad t, s \in \mathbf{R}.$$

Similarly, (4.14) holds true if one of the following is valid

(4.16a)
$$a(-t,s) = -a(t,s), \qquad a(t,-s) = a(t,s), \quad t,s \in \mathbf{R},$$
(4.16b)
$$a(-t,s) = a(t,s), \qquad a(t,-s) = -a(t,s), \quad t,s \in \mathbf{R}.$$

We introduce a linear functional Φ by

$$\Phi u = \int_{-1/2}^{1/2} u(s)\overline{\phi(s)}\, ds, \quad \text{where } \phi \in C_1^\infty(\mathbf{R}) \text{ is even.} \tag{4.17}$$

Furthermore, we define the operators $A \times \Phi$ and $A \dot{+} \Phi$ such that

$$(A \times \Phi)u = [Au, \Phi u], \quad u \in H_e^\lambda, \tag{4.18}$$
$$(A \dot{+} \Phi)[u,w] = Au + \omega\phi, \quad [u,w] \in H_o^\lambda \times \mathbf{C}. \tag{4.19}$$

Lemma 4.2. *In addition to (4.10), assume that $a(t,t) \neq 0$, $t \in \mathbf{R}$ and*
$$|\hat{\kappa}(l)| \geq c_0 |l|^\alpha, \quad 0 \neq l \in \mathbf{Z}, c_0 > 0.$$
Then, for any $\lambda \in \mathbf{R}$, the following holds true

(i) *Under conditions* (4.11) *and* (4.13), $A \in \mathcal{L}(H_e^\lambda, H_e^{\lambda-\alpha})$ *and* $A \in \mathcal{L}(H_o^\lambda, H_o^{\lambda-\alpha})$ *are Fredholm operators of index* 0,

(ii) *Under conditions* (4.12) *and* (4.13), $A \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha})$ *and* $A \in \mathcal{L}(H_o^\lambda, H_e^{\lambda-\alpha})$ *are Fredholm operators of index* 1 *and* -1, *respectively. Consequently, we have* $A \times \Phi \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha} \times \mathbf{C})$ *and* $A \dot{+} \Phi \in \mathcal{L}(H_o^\lambda \times \mathbf{C}, H_e^{\lambda-\alpha})$, *and these are Fredholm operators of index* 0.

Proof. Represent $A = A_0 + A_1$ with $A_0 = M_b B$, where M_b denotes multiplication by $b(t) = a(t,t)$ and B is defined by

$$(Bu)(t) = \int_{-1/2}^{1/2} \kappa(t-s)u(s)\, ds = \sum_{l \in \mathbf{Z}} \hat{u}(l)\hat{\kappa}(l)e^{il2\pi t}.$$

Clearly, under condition (4.11), $B \in \mathcal{L}(H_e^\lambda, H_e^{\lambda-\alpha})$ and $B \in \mathcal{L}(H_o^\lambda, H_o^{\lambda-\alpha})$ are Fredholm operators of index 0, and under condition (4.12),

$B \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha})$ and $B \in \mathcal{L}(H_o^\lambda, H_e^{\lambda-\alpha})$ are Fredholm of index 1 and -1, respectively. The same property has $A_0 = M_b B$, since b is even under condition (4.13), and, therefore, $M_b \in \mathcal{L}(H_e^{\lambda-\alpha}, H_e^{\lambda-\alpha})$ and $M_b \in \mathcal{L}(H_o^{\lambda-\alpha}, H_o^{\lambda-\alpha})$ are isomorphisms. Finally, $A_1 = A - A_0 \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+1})$, see, e.g., [**28**] or [**33**]; therefore, $A_1 \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ is compact. Of course, it has the same parity properties as A and A_0. From this we obtain the assertions of the lemma concerning the Fredholmness and index of A. The last assertion in (ii) concerning $A \times \Phi$ and $A \dot{+} \Phi$ follows from properties of A. $\square$

Notice that the condition $a(t,t) \neq 0$, $t \in \mathbf{R}$, cannot be fulfilled if $a(t,s)$ is an odd function.

5. Analysis of the periodic problem. In this section we analyze the solvability of the periodic problem for even and odd operators. As an application we obtain solutions of the periodic problems derived in Section 3. Consider first the case of even operators. Assuming the general conditions (4.3)–(4.5), an even operator $\mathcal{A}$ defines bounded mappings $\mathcal{A} \in \mathcal{L}(H_e^\lambda, H_e^{\lambda-\alpha})$, $\mathcal{A} \in \mathcal{L}(H_o^\lambda, H_o^{\lambda-\alpha})$. For our applications we need to specify the main part further. For this, let $C_{1e}^\infty(\mathbf{R})$ and $C_{1o}^\infty(\mathbf{R})$ be the space of all even, respectively, odd, functions in $C_1^\infty(\mathbf{R})$. We require on the main part the following properties:

(5.1a) $$\hat{\kappa}_0(-l) = \hat{\kappa}_0(l), \quad 0 \neq l \in \mathbf{Z},$$

(5.1b) $$a_0(-t,-s) = a_0(t,s), \quad t,s \in \mathbf{R},$$

(5.1c) $$a_0(t,t) \neq 0, \quad t \in \mathbf{R}.$$

Moreover, we impose the conditions

(5.2a) $$u \in C_{1e}^\infty(\mathbf{R}), \quad \mathcal{A}u = 0 \ \Rightarrow \ u = 0,$$

(5.2b) $$u \in C_{1o}^\infty(\mathbf{R}), \quad \mathcal{A}u = 0 \ \Rightarrow \ u = 0.$$

Theorem 5.1. *Let $\lambda \in \mathbf{R}$ be given. Assume that $\mathcal{A}$ is an even operator with the conditions* (4.3)–(4.5) *and* (5.1). *If* (5.2a) *is valid, then $\mathcal{A} : H_e^\lambda \to H_e^{\lambda-\alpha}$ is an isomorphism. Moreover, if* (5.2b) *is valid, then $\mathcal{A} : H_o^\lambda \to H_o^{\lambda-\alpha}$ is an isomorphism.*

Proof. The main part of the operator $\mathcal{A}$ satisfies the conditions of Lemma 4.2 part (i), and $\mathcal{A}$ is a compact perturbation of the main part. This with (5.2) yields the assertions. □

Consider the solvability of the equations (3.5) and (3.16). For these we set

$$v \in L^2_\sigma(I), \quad B_L v = 0 \;\Rightarrow\; v = 0, \tag{5.3a}$$

$$v \in \overset{\circ}{H}{}^1_\sigma(I), \quad B_H v = 0 \;\Rightarrow\; v = 0. \tag{5.3b}$$

Lemma 5.1. *The following assertions are valid*:

(i) *The mapping $v \mapsto u$ with $u(t) = v(x(t))$, $t \in \mathbf{R}$, defines a linear isomorphism between $L^2_{1/\sigma}(I)$ and H^0_e as well as between $H^1_\sigma(I)$ and H^1_e.*

(ii) *The mapping $v \mapsto u$ with $u(t) = v(x(t))\operatorname{sign} t$, $|t| \le 1/2$, extended to a 1-periodic function, defines a linear isomorphism between $L^2_{1/\sigma}(I)$ and H^0_o as well as between $\overset{\circ}{H}{}^1_\sigma(I)$ and H^1_o.*

(iii) *The mapping $v \mapsto u$ with $u(t) = v(x(t))x'(t)$, $t \in \mathbf{R}$, defines a linear isomorphism between $L^2_\sigma(I)$ and H^0_o.*

(iv) *The mapping $v \mapsto u$ with $u(t) = v(x(t))|x'(t)|$, $t \in \mathbf{R}$, defines a linear isomorphism between $L^2_\sigma(I)$ and H^0_e.*

Proof. For $u(t) = v(x(t))$ and $u(t) = v(x(t))\operatorname{sign} t$, we have

$$\begin{aligned}\|u\|_0^2 &= \int_{-1/2}^{1/2} |u(t)|^2\,dt = 2\int_0^{1/2} |u(t)|^2\,dt \\ &= 2\int_0^{1/2} \frac{|v(x(t))|^2}{x'(t)} x'(t)\,dt = 2\int_0^1 \frac{|v(x)|^2}{x'(t(x))}\,dx \\ &= \frac{1}{\pi}\int_0^1 \frac{|v(x)|^2}{x^{1/2}(1-x)^{1/2}}\,dx = \frac{1}{\pi}\|v\|^2_{1/\sigma},\end{aligned}$$

where $t(x) = (1/(2\pi))\arccos(1-2x)$ is the inverse function of $x = x(t) = [(1-\cos 2\pi t)/2]$, $0 \le t \le 1/2$. A similar calculation shows that, for $u(t) = v(x(t))x'(t)$ as well as for $u(t) = v(x(t))|x'(t)|$, $t \in \mathbf{R}$, we have $\|u\|_0^2 = 4\pi\|v\|_\sigma^2$. Hence, for $u(t) = v(x(t))$, we also have

$$(5.4) \qquad (\|u\|_0^2 + \|u'\|_0^2)^{1/2} = ((1/\pi)\|v\|_{1/\sigma}^2 + 4\pi\|v'\|_\sigma^2)^{1/2}.$$

On the lefthand side there is a norm equivalent to $\|u\|_1$ and on the righthand side there is a norm equivalent to $\|v\|_{1,\sigma}$. Now assertions (i), (iii), (iv) and the first part of (ii) easily follow. To obtain the second part of (ii), notice that (5.4) remains true also for $u(t) = v(x(t))\operatorname{sign} t$ provided that this function is continuous at $t = 0$, i.e., $v(0) = 0$. The continuity of the 1-periodic extension of u means that $u(1/2) = u(-1/2)$ or $v(1) = 0$.

Thus, $u \in H_o^1$ if and only if $v \in \overset{\circ}{H}{}_\sigma^1(I)$, and inequality (5.4) holds true for those functions. Hence the second part of (ii) is also proved. □

Now we obtain

Theorem 5.2. *Assume* (5.3). *Then the operators* $B_L : L_\sigma^2(I) \to H_\sigma^1(I)$, $B_H : \overset{\circ}{H}{}_\sigma^1(I) \to L_\sigma^2(I)$ *and* $A_L : H_e^\lambda \to H_e^{\lambda+1}$, $A_H : H_o^\lambda \to H_o^{\lambda-1}$ *are isomorphic for all* $\lambda \in \mathbf{R}$.

Remark 5.1. Assume that $v \in L_\sigma^2(I)$ is the solution of $B_L v = g$ such that $g \in C^\infty(\bar{I})$. Then it follows from Theorem 5.2 that v is of the form $v(x) = x^{-1/2}(1-x)^{-1/2}\psi(x)$, $\psi \in C(\bar{I})$. For this regularity result, the condition on g can be considerably relaxed. Similarly, if $v \in \overset{\circ}{H}{}_\sigma^1(I)$ is the solution of $B_H v = g$ where $g \in C^\infty(\bar{I})$, it follows from Theorem 5.2 that v is of the form $v(x) = x^{1/2}(1-x)^{1/2}\psi(x)$, $\psi \in C(\bar{I})$. In fact, for the latter case, one can show sharper result $\psi \in C^\infty(\bar{I})$. For the case of the basic hypersingular equation, see [**7, 6**].

Consider now the case of an odd operator $\mathcal{A}$ together with the operators $\mathcal{A} \times \Phi$ and $\mathcal{A} \dotplus \Phi$. We require on the main part A_0 the properties

$$(5.5a) \qquad \hat{\kappa}_0(-l) = -\hat{\kappa}_0(l), \quad 0 \neq l \in \mathbf{Z},$$

$$(5.5b) \qquad a_0(-t,-s) = a_0(t,s), \quad t,s \in \mathbf{R},$$

$$(5.5c) \qquad a_0(t,t) \neq 0, \quad t \in \mathbf{R}.$$

Moreover, for the linear functional $\Phi : H_e^\lambda \to \mathbf{C}$ we additionally impose

$$\Phi 1 = \int_{-1/2}^{1/2} \overline{\phi(s)}\, ds \neq 0. \tag{5.6}$$

We consider the solution of the equation

$$u \in H_e^\lambda : Au = f, \quad \Phi u = \gamma, \quad f \in H_o^{\lambda-\alpha}, \quad \gamma \in \mathbf{C} \tag{5.7}$$

and assume uniqueness for the homogeneous problem in the form

$$u \in C_{1e}^\infty(\mathbf{R}), \quad \mathcal{A}u = 0, \quad \Phi u = 0 \;\Rightarrow\; u = 0. \tag{5.8}$$

Theorem 5.3. *Assume* (4.3)–(4.5), (4.17), (5.5), (5.6) *and* (5.8). *Then, for any* $\lambda \in \mathbf{R}$, *the operator* $\mathcal{A} \times \Phi : H_e^\lambda \to H_o^{\lambda-\alpha} \times \mathbf{C}$ *is an isomorphism.*

We apply this result to the solution of the Cauchy singular integral equations on the interval I in the case where the periodization is carried out by the first method described in Section 3.2. The corresponding operator A_{1C} is given in (3.9a). We set the condition

$$v \in L_\sigma^2(I), \quad B_C v = 0, \quad \Phi_I v = 0 \;\Rightarrow\; v = 0. \tag{5.9}$$

Theorem 5.4. *Assume* (5.9) *and define* $\Phi u = (1/2)\int_{-1/2}^{1/2} u(s)\, ds$. *Then the mapping* $B_C \times \Phi_I : L_\sigma^2(I) \to L_\sigma^2(I) \times \mathbf{C}$ *and* $A_{1C} \times \Phi : H_e^\lambda \to H_o^\lambda \times \mathbf{C}$ *are isomorphic for all* $\lambda \in \mathbf{R}$.

Proof. It suffices to show that any function $u \in C_{1e}^\infty(\mathbf{R})$ satisfying $A_{1C}u = 0$, $\Phi u = 0$ vanishes identically on $\mathbf{R}$. We define $v(x) = (u(t(x))/(2\pi\sqrt{x(1-x)}))$. Since $u(t(x))$ is continuous, one easily verifies that $v \in L_\sigma^2(I)$. Moreover, $B_C v = 0$, $\Phi_I v = 0$, which implies $v = 0$ and $u = 0$. □

Remark 5.2. Consider the equation of the special form, b_0 is constant,

$$\begin{aligned}(B_C v)(x) &:= \int_0^1 \left(\frac{b_0}{x-y} + b_1(x)\log|x-y| + b_2(x,y)\right) v(y)\, dy \\ &= g(x), \quad x \in I,\end{aligned} \tag{5.10}$$

where b_k, $k = 0, 1, 2$, are real and $b_1 \in L^2_\sigma(I)$. Applying the results of [**23**, p. 86], one finds conditions on the functions b_k which guarantee that the problem $B_C v = g$, $\Phi_I v = \gamma$, $v \in L^2_\sigma(I)$ is uniquely solvable. Then (5.9) is valid, in particular.

Remark 5.3. Assume that g is sufficiently smooth and $v \in L^2_\sigma(I)$ such that $B_C v = g$. Then v is of the form $v(x) = x^{-1/2}(1-x)^{-1/2}\psi(x)$, $\psi \in C(\bar{I})$.

We now turn to describe how the second periodization in Section 3.2 can be utilized for solution of Cauchy singular equations on an interval. This leads to a problem of the general form

$$(5.11) \qquad u \in H^\lambda_o, \quad \omega \in \mathbf{C} : \mathcal{A}u + \omega\phi = f, \quad f \in H^{\lambda-\alpha}_e.$$

Problem (5.11) can be viewed as a "dual" problem of (5.7). As in the previous cases, the uniqueness for homogeneous equation is sufficient (and necessary) to solve the general equation (5.11) completely. We put the condition

$$(5.12) \quad \mathcal{A}u + \omega\phi = 0, \quad u \in C^\infty_{1o}(\mathbf{R}), \quad \omega \in \mathbf{C} \ \Rightarrow \ u = 0, \quad \omega = 0,$$

and have a solvability result for (5.11) given by the operator as $\mathcal{A} \dotplus \Phi$ as follows.

Theorem 5.5. *Assume* (4.3)–(4.5), (5.5), (5.6) *and* (5.12). *Then, for any* $\lambda \in \mathbf{R}$*, the operator* $\mathcal{A} \dotplus \Phi : H^\lambda_o \times \mathbf{C} \to H^{\lambda-\alpha}_e$ *is an isomorphism.*

For the Cauchy singular operator we impose the condition in terms of functions on I,

$$(5.13) \qquad v \in L^2_{1/\sigma}(I), \quad \omega \in \mathbf{C}, B_C v + \omega = 0 \ \Rightarrow \ v = 0, \quad \omega = 0.$$

Theorem 5.6. *Assume* (5.13). *Then* $B_C \dotplus \Phi_I : L^2_{1/\sigma}(I) \times \mathbf{C} \to L^2_{1/\sigma}(I)$ *is an isomorphism and* $A_{2C} \dotplus \Phi : H^\lambda_o \times \mathbf{C} \to H^\lambda_e$ *is an isomorphism for all* $\lambda \in \mathbf{R}$.

In real applications one has to check that (5.12) or (5.13) holds, and it may not be easy. In the following we introduce an alternative way to discuss the solvability properties of (5.11). The idea is to make an effective use of the "duality" between (5.11) and (5.7). To be more precise, we first show that (5.11) is an adjoint problem, in a strict sense, of another problem of the general form (5.7).

For given $\mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$, we have the adjoint $\mathcal{A}^* \in \mathcal{L}(H^{\alpha-\lambda}, H^{-\lambda})$ defined by

$$(5.14) \qquad (\mathcal{A}u, v) = (u, \mathcal{A}^*v), \quad u \in H^\lambda, v \in H^{\alpha-\lambda}.$$

If $\mathcal{A}$ is an odd operator additionally, the operator $\mathcal{A}^*$ is an odd operator, too. Moreover, the restriction operator $\mathcal{A} : H_o^\lambda \to H_e^{\lambda-\alpha}$ has the adjoint $\mathcal{A}^* : H_e^{\alpha-\lambda} \to H_o^{-\lambda}$ and $\mathcal{A} : H_e^\lambda \to H_o^{\lambda-\alpha}$ has the adjoint $\mathcal{A}^* : H_o^{\alpha-\lambda} \to H_e^{-\lambda}$. Furthermore, if $\mathcal{A}$ is a pseudodifferential operator satisfying the general conditions (4.3)–(4.5) and (5.5) for given $\alpha \in \mathbf{R}$, then the operator $\mathcal{A}^*$ also satisfies these conditions for the same number α. We introduce the duality pairing

$$\langle [u, w], [v, \mu] \rangle := (u, v) + \omega\bar{\mu},$$
$$[u, \omega] \in H^\lambda \times \mathbf{C}, [v, \mu] \in H^{-\lambda} \times \mathbf{C}.$$

Now we have

$$\langle (\mathcal{A}^* \times \Phi)u, [v, \omega] \rangle = (u, (\mathcal{A} \dotplus \Phi)[v, \omega]), u \in H_e^{\alpha-\lambda}, v \in H_o^\lambda, \omega \in \mathbf{C},$$

and therefore, $\mathcal{A} \dotplus \Phi = (\mathcal{A}^* \times \Phi)^*$. By the general results for Fredholm operators, we have

Theorem 5.7. *Assume* (4.3)–(4.5) *and* (5.1)*. Then the operator* $\mathcal{A} \dotplus \Phi : H_o^\lambda \times \mathbf{C} \to H_e^{\lambda-\alpha}$ *is an isomorphism for all* $\lambda \in \mathbf{R}$ *if and only if* $\mathcal{A}^* \times \Phi : H_e^\lambda \to H_o^{\lambda-\alpha} \times \mathbf{C}$ *is an isomorphism for all* $\lambda \in \mathbf{R}$.

As an application of Theorem 5.7, we recall the second periodization method described for Cauchy singular equations in Section 3.2. We consider the equation

$$(5.15) \qquad u \in H_o^\lambda, \quad \omega \in \mathbf{C} : A_{2C}u + (1/2)\omega = f, \quad f \in H_e^\lambda.$$

In order to apply Theorem 5.7, we need the adjoint problem

$$u \in H_e^\lambda : A_{2C}^* u = f, \quad \frac{1}{2}\int_{-1/2}^{1/2} u(s)\,ds = \gamma, \quad f \in H_o^\lambda, \quad \gamma \in \mathbf{C}. \tag{5.16}$$

Now operator A_{2C}^* has the kernel $a_{2c}^*(t,s) = \overline{a_{2C}}(s,t)$ which becomes

$$\begin{aligned} a_{2C}^*(t,s) = &-\pi \overline{b_0}(x(s),x(t))\kappa_0(t-s) \\ &+ \overline{b_1}(x(s),x(t))x'(t)\kappa_1(t-s) \\ &+ \frac{1}{2}\overline{b_2}(x(s),x(t))x'(t). \end{aligned} \tag{5.17}$$

Comparing (5.17) with the formula (3.9b) we find that the kernel a_{2C}^* coincides with the kernel obtained by the first method, if applied to the operator

$$(B_C^* v)(x) := \int_0^1 \left(\frac{\overline{b_0}(y,x)}{y-x} + \overline{b_1}(y,x)\log|x-y| + \overline{b_2}(y,x) \right) v(y)\,dy. \tag{5.18}$$

We impose the condition

$$B_C^* v = 0, \quad \Phi_I v = 0, \quad v \in L_\sigma^2(I) \;\Rightarrow\; v = 0. \tag{5.19}$$

Theorem 5.8. *Assume* (5.19). *Then the mapping* $A_{2C} \dotplus \Phi : H_o^\lambda \times \mathbf{C} \to H_e^\lambda$ *is an isomorphism for all* $\lambda \in \mathbf{R}$.

Remark 5.4. Recalling the operator in Remark 5.2, we have

$$(B_C^* v)(x) = \int_0^1 \left(\frac{b_0}{y-x} + b_1(y)\log|x-y| + b_2(y,x) \right) v(y)\,dy. \tag{5.20}$$

Now we can directly apply the results of [**23**] if b_1 is constant. Then we have the property (5.19) under a condition concerning b_0, b_1 and b_2.

Remark 5.5. The second periodization method for the singular integral equation gives a solution of the original equation $B_C v = g$ if the unknown parameter ω in (5.15) turns out to be zero. If, additionally,

f is sufficiently smooth, the function $v(x) = u(t(x))$ vanishes at the endpoints of I. Solutions which vanish at a given endpoint are of particular interest in physical applications. This condition is known as the Kutta condition, see, e.g., [**10**].

6. Asymptotic approximation of integral operators. For the following two lemmas, see [**33, 32**].

Lemma 6.1. *For $d \in \mathbf{N}$, approximate the operator A defined in (4.9)–(4.10) by A_d,*

$$(A_d u)(t) = \sum_{j=0}^{d-1} a_j(t) \int_{-1/2}^{1/2} \kappa_j(t-s) u(s)\, ds \tag{6.1}$$

with

$$a_j(t) = \partial_s^{(j)} a(t,s)|_{s=t}, \quad \partial_s^{(0)} = 1,$$
$$\partial_s^{(j)} = \prod_{k=0}^{j-1} \left(\frac{1}{2\pi i} \frac{\partial}{\partial s} - k \right), \quad j \geq 1, \tag{6.2}$$

$$\hat{\kappa}_j(l) = \frac{1}{j!} \Delta_j \hat{\kappa}(l), \quad l \in \mathbf{Z}, \quad j \in \mathbf{N}_0. \tag{6.3}$$

Then $A_d - A \in Op \sum^{\alpha - d}$.

Lemma 6.2. *Assume that $\hat{\kappa} : \mathbf{Z} \to \mathbf{C}$ is extended to $\hat{\kappa} : \mathbf{R} \to \mathbf{C}$ such that $\hat{\kappa}$ is C^∞-smooth and*

$$\left| \left(\frac{d}{d\xi} \right)^k \hat{\kappa}(\xi) \right| \leq C |\xi|^{\alpha - k}, \quad |\xi| \geq 1, \quad k \in \mathbf{N}_0. \tag{6.4}$$

Define the operator A_d by (6.1) with

$$a_j(t) = \left(\frac{1}{2\pi i} \frac{\partial}{\partial s} \right)^j a(t,s) \bigg|_{s=t},$$
$$\hat{\kappa}_j(l) = \frac{1}{j!} \left(\frac{d}{d\xi} \right)^j \hat{\kappa}(\xi) \bigg|_{\xi = l}, \quad l \in \mathbf{Z}, j \in \mathbf{N}_0. \tag{6.5}$$

Then $A_d - A \in Op\sum^{\alpha-d}$.

There is a standard linear procedure of extension of functions $\hat{\kappa}$: $\mathbf{Z} \to \mathbf{C}$ to $\hat{\kappa} \in C^\infty(\mathbf{R})$ such that (4.10) implies (6.4), see [**33**]. In problems from practice, $\hat{\kappa}(l)$ is usually of a form where the extension is obvious. We present the asymptotic expansions (6.1) and (6.4) for integral operators from Section 3. Actually we have to write the formulae for $\hat{\kappa}_j(l)$, $l \in \mathbf{Z}$, only.

(i) For $(Au)(t) = \int_{-1/2}^{1/2} a(t,s) \log|\sin \pi(t-s)| u(s)\, ds$ we have

$$\alpha = -1, \quad \hat{\kappa}(l) = \begin{cases} -\log 2, & l = 0 \\ -1/(2|l|), & 0 \neq l \in \mathbf{Z}. \end{cases}$$

The extension $\hat{\kappa} : \mathbf{R} \to \mathbf{R}$ can be defined so that $\hat{\kappa}(\xi) = -(2|\xi|)^{-1}$ for $|\xi| \geq 1$; the form of the extension for $-1 \leq \xi \leq 1$ has no influence on the validity of the property $A_d - A \in Op\sum^{-1-d}$, and we may put, e.g., $\hat{\kappa}_j(0) = 0$, $j \geq 1$. For $|l| \geq 1$, $j \geq 1$, we have

$$\begin{aligned} \hat{\kappa}_j(l) &= -\frac{1}{j!}\left(\frac{d}{d\xi}\right)^j (2|\xi|)^{-1}\bigg|_{\xi=l} \\ &= -\frac{1}{2}|l|^{-j-1} \cdot \begin{cases} (-1)^j & l \geq 1, \\ 1 & l \leq -1. \end{cases} \end{aligned}$$

(ii) For $(Au)(t) = (1/i)\int_{-1/2}^{1/2} a(t,s) \cot \pi(s-t) u(s)\, ds$, we have

$$\alpha = 0, \quad \hat{\kappa}(l) = \begin{cases} -1, & l < 0 \\ 0, & l = 0 \\ 1, & l > 0 \end{cases}, \quad l \in \mathbf{Z}.$$

The extension $\hat{\kappa} : \mathbf{R} \to \mathbf{R}$ can be defined so that

$$\hat{\kappa}(\xi) = \operatorname{sign} \xi = \begin{cases} 1, & \xi \geq 1 \\ -1, & \xi \leq -1. \end{cases}$$

Respectively, $\hat{\kappa}_j(l) = 0$, $l \in \mathbf{Z}$, $j \geq 1$. This means that already for $d = 1$ we have $A_1 - A \in Op\sum^{-\infty}$ where

$$(A_1 u)(t) = (1/i) a(t,t) \int_{-1/2}^{1/2} \kappa_0(t-s) u(s)\, ds, \ \hat{\kappa}_0(l) = \hat{\kappa}(l),$$
$$l \in \mathbf{Z}.$$

(iii) For $(Au)(t) = \int_{-1/2}^{1/2} (a(t,s)u(s))/(\sin^2 \pi(t-s))\, ds$ we have $\alpha = 1$,

$$\hat{\kappa}(l) = \begin{cases} 0, & l = 0 \\ -2|l|, & 0 \neq l \in \mathbf{Z}. \end{cases}$$

Now, already with $d = 2$, $A_2 - A \in Op \sum^{-\infty}$, where

$$\begin{aligned}(A_2u)(t) = a(t,t) &\int_{-1/2}^{1/2} \kappa_0(t-s)u(s)\, ds \\ &+ \frac{1}{2\pi i} \frac{\partial a(t,s)}{\partial s}\bigg|_{s=t} \int_{-1/2}^{1/2} \kappa_1(t-s)u(s)\, ds, \\ \hat{\kappa}_0(l) = \hat{\kappa}(l), &\quad \hat{\kappa}_1(l) = -2\operatorname{sign} l, \quad l \in \mathbf{Z}.\end{aligned}$$

Remark 6.1. If A is an even (odd) operator, then so is A_d defined by (6.1) and (6.4). The approximation (6.1), (6.2) does not preserve this property.

7. Trigonometric interpolation.

7.1 *Interpolation of functions $u \in H^\mu$.* For $n \in \mathbf{N}$, denote

$$\mathbf{Z}_n = \left\{ k \in \mathbf{Z} : -\frac{n}{2} < k \leq \frac{n}{2} \right\},$$
$$\mathcal{T}_n = \left\{ \sum_{k \in \mathbf{Z}_n} c_k e^{ik2\pi t} : c_k \in \mathbf{C}, k \in \mathbf{Z}_n \right\}.$$

Thus $\mathcal{T}_n$ consists of trigonometric polynomials, $\dim \mathcal{T}_n = n$. Further,

$$(P_n u)(t) = \sum_{k \in \mathbf{Z}_n} \hat{u}(k) e^{ik2\pi t}$$

is the orthogonal projection of $u \in H^\mu$, $\mu \in \mathbf{R}$, onto $\mathcal{T}_n$. Clearly,

$$\|u - P_n u\|_\lambda \leq \left(\frac{n}{2}\right)^{\lambda - \mu} \|u\|_\mu, \quad \lambda \leq \mu. \tag{7.1}$$

The interpolation projection Q_nu, $u \in H^\mu$, $\mu > 1/2$, is defined by the conditions

$$Q_nu \in \mathcal{T}_n, \quad (Q_nu)(jn^{-1}) = u(jn^{-1}), \quad j = 0, 1, \dots, n-1.$$

An explicit formula is given by

$$\begin{aligned} Q_nu &= \sum_{j=0}^{n-1} u(jn^{-1})\varphi_{n,j} = \sum_{j\in\mathbf{Z}_n} u(jn^{-1})\varphi_{n,j}, \\ \varphi_{n,j}(t) &= \frac{1}{n}\sum_{k\in\mathbf{Z}_n} e^{ik2\pi(t-jn^{-1})}; \end{aligned}$$

the Lagrange polynomials $\varphi_{n,j} \in \mathcal{T}_n$ satisfy $\varphi_{n,j}(kn^{-1}) = \delta_{jk}$. The error of trigonometric interpolations can be estimated by, see [**33, 29**] or, without a characterization of the constant, [**1, 26**],

$$\|u - Q_nu\|_\lambda \le \gamma_\mu \left(\frac{n}{2}\right)^{\lambda-\mu} \|u\|_\mu, \quad 0 \le \lambda \le \mu, \quad \mu > \frac{1}{2}, \tag{7.2}$$

where $\gamma_\mu = (1 + 2\sum_{j=1}^{\infty} j^{-2\mu})^{1/2}$.

7.2 *Interpolation of even and odd functions.* Denote

$$\begin{aligned} \mathcal{T}_n^e &= \left\{c_0 + \sum_{k=1}^{n} c_k \cos k2\pi t : c_k \in \mathbf{C}, k = 0, \dots, n\right\}, \\ \mathcal{T}_n^o &= \left\{\sum_{k=1}^{n} c_k \sin k2\pi t : c_k \in \mathbf{C}, k = 1, \dots, n\right\}. \end{aligned}$$

Thus, $\mathcal{T}_n^e \oplus \mathcal{T}_n^o = \mathcal{T}_{2n+1}$. For $u \in H_e^\mu$, respectively $v \in H_o^\mu$, $\mu > 1/2$, the even and odd interpolations are defined by

$$\begin{gathered} Q_n^e u \in \mathcal{T}_n^e, (Q_n^e u)(jh) = u(jh), \\ j = 0, \dots, n, \quad h = 1/(2n+1), \\ Q_n^o v \in \mathcal{T}_n^o, (Q_n^o v)(jh) = v(jh), \\ j = 1, \dots, n, \quad h = 1/(2n+1). \end{gathered}$$

It is very easy to see that

$$Q_n^e u = Q_{2n+1} u \quad \text{for } u \in H_e^\mu, \mu > 1/2,$$
$$Q_n^o v = Q_{2n+1} v \quad \text{for } v \in H_o^\mu, \mu > 1/2.$$

Therefore, the estimate (7.2) implies

$$\|u - Q_n^e u\|_\lambda \le \gamma_\mu \left(n + \frac{1}{2}\right)^{\lambda-\mu} \|u\|_\mu \quad (0 \le \lambda \le \mu) \quad \text{for } u \in H_e^\mu, \mu > 1/2, \tag{7.3}$$

$$\|v - Q_n^o v\|_\lambda \le \gamma_\mu \left(n + \frac{1}{2}\right)^{\lambda-\mu} \|v\|_\mu \quad (0 \le \lambda \le \mu) \quad \text{for } v \in H_o^\mu, \mu > 1/2. \tag{7.4}$$

Implicit formulae for the even and odd interpolations are given by

$$Q_n^e u = \sum_{j=0}^{n} u(jh)\varphi_{n,j}^e, \qquad Q_n^o v = \sum_{j=1}^{n} v(jh)\varphi_{n,j}^o,$$
$$h = 1/(2n+1),$$

where

$$\varphi_{n,j}^e(t) = \begin{cases} \varphi_{n,0}(t) = h(1 + 2\sum_{k=1}^n \cos k2\pi t), & j = 0 \\ \varphi_{n,j}(t) + \varphi_{n,-j}(t) & \\ = 2h(1 + 2\sum_{k=1}^n \cos(k2\pi jh)\cos(k2\pi t)), & j = 1\ldots, n, \end{cases}$$

$$\varphi_{n,j}^o(t) = \varphi_{n,j}(t) - \varphi_{n,-j}(t) = 4h \sum_{k=1}^{n} \sin(k2\pi jh)\sin(k2\pi t),$$
$$j = 1, \ldots, n.$$

7.3 *Discrete Fourier transform.* There are two possible representations of $v_n \in \mathcal{T}_n$: through its Fourier coefficients and through its nodal values,

$$v_n(t) = \sum_{k \in \mathbf{Z}_n} c_k e^{ik2\pi t} = \sum_{j \in \mathbf{Z}_n} d_j \varphi_{n,j}(t),$$
$$c_k = \hat{v}_n(k), d_j = v_n(jn^{-1}), k, j \in \mathbf{Z}_n.$$

The vectors $\mathbf{c}_n = \{c_k : k \in \mathbf{Z}_n\}$ and $\mathbf{d}_n = \{d_j : j \in \mathbf{Z}_n\}$ are related by the discrete Fourier transforms $\mathbf{c}_n = \mathcal{F}_n \mathbf{d}_n$, $\mathbf{d}_n = \mathcal{F}_n^{-1} \mathbf{c}_n$:

$$c_k = \frac{1}{n} \sum_{j \in \mathbf{Z}_n} e^{-ikj2\pi n^{-1}} d_j, \quad k \in \mathbf{Z}_n,$$

$$d_j = \sum_{k \in \mathbf{Z}_n} e^{ijk2\pi n^{-1}} c_k, \quad j \in \mathbf{Z}_n.$$

For $v_n \in \mathcal{T}_n^e$ we have the representations

$$v_n = c_0 + \sum_{k=1}^{n} c_k \cos k2\pi t = \sum_{j=0}^{n} d_j \varphi_{n,j}^e.$$

The vectors $\mathbf{c}_n = (c_0, \dots, c_n)$ and $\mathbf{d}_n = (d_0, \dots, d_n)$ are related by discrete cosine Fourier transforms $\mathbf{c}_n = \mathcal{C}_n \mathbf{d}_n$, $\mathbf{d}_n = \mathcal{C}_n^{-1} \mathbf{c}_n$:

$$c_0 = h\biggl(d_0 + 2\sum_{j=1}^{n} d_j\biggr),$$

$$c_k = 2h\biggl(d_0 + 2\sum_{j=1}^{n} \cos(kj2\pi h) d_j\biggr),$$
$$k = 1, \dots, n,$$

$$d_j = c_0 + \sum_{k=1}^{n} \cos(jk2\pi h) c_k,$$
$$j = 0, \dots, n, \ h = 1/(2n+1).$$

For $v_n \in \mathcal{T}_n^0$ we have the representations

$$v_n = \sum_{k=1}^{n} c_k \sin k2\pi t = \sum_{j=1}^{n} d_j \varphi_{n,j}^o.$$

The vectors $\mathbf{c}_n = (c_1, \dots, c_n)$ and $\mathbf{d}_n = (d_1, \dots, d_n)$ are related by discrete sine Fourier transforms $\mathbf{c}_n = \mathcal{S}_n \mathbf{d}_n$, $\mathbf{d}_n = \mathcal{S}_n^{-1} \mathbf{c}_n$:

$$c_k = 4h \sum_{j=1}^{n} \sin(2\pi kjh) d_j, \quad k = 1, \dots, n,$$

$$d_j = \sum_{k=1}^{n} \sin(2\pi jkh) c_k,$$
$$j = 1, \dots, n, \ h = 1/(2n+1).$$

By FFT, an application of $\mathcal{F}_n, \mathcal{F}_n^{-1}, \mathcal{C}_n, \mathcal{C}_n^{-1}, \mathcal{S}_n$ and $\mathcal{S}_n^{-1}$ costs $\mathcal{O}(n \log n)$ arithmetical operations instead of $\mathcal{O}(n^2)$ operations by the usual matrix-to-vector multiplications.

8. Some discrete versions of trigonometric collocation. Consider the problem

$$\mathcal{A}u := \sum_{p=0}^{q} A_p u = f, \tag{8.1}$$

where the operator $\mathcal{A}$ satisfies the conditions (4.3)–(4.6). Then $\mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ is an isomorphism for any $\lambda \in \mathbf{R}$. We also assume that

$$f \in H^{\mu-\alpha}, \quad \mu - \alpha > \frac{1}{2}, \tag{8.2}$$

then f is continuous and (8.1) may be solved by the trigonometric collocation method

$$u_n \in \mathcal{T}_n, \quad Q_n \mathcal{A} u_n = Q_n f. \tag{8.3}$$

The following result is proved in [**28**] and actually is elementary in the situation examined here.

Theorem 8.1. *Under conditions* (4.3)–(4.6), *and* (8.2), $Q_n\mathcal{A} \in \mathcal{L}(\mathcal{T}_n)$ *is for sufficiently large* n *invertible,*

$$\|(Q_n\mathcal{A})^{-1}\|_{L(H^{\lambda-\alpha}, H^\lambda)} \le c_\lambda, \quad n \ge n_0, \lambda \in \mathbf{R}, \tag{8.4}$$

$$\|u_n^c - u\|_\lambda \le c n^{\lambda-\mu} \|u\|_\mu, \quad \alpha \le \lambda \le \mu, \tag{8.5}$$

where $u_n^c = (Q_n\mathcal{A})^{-1} Q_n f$ *is the collocation approximation for* $u = \mathcal{A}^{-1} f \in H^\mu$.

To obtain fully discrete versions of the collocation method, we introduce some further approximations of $\mathcal{A}$. Let $l, m, n \in \mathbf{N}$ satisfy

$$2l \le m \le n, \ l \sim n^\rho, \ m \sim n^\sigma, \quad 0 < \rho \le \sigma \le 1. \tag{8.6}$$

For $u \in \mathcal{T}_l$ we approximate $\mathcal{A}u$ by $Q_m\mathcal{A}^{(m)}u$ with

$$\mathcal{A}^{(m)} = \sum_{p=0}^{q} A_p^{(m)}, \qquad (A_p^{(m)}u)(t) = \int_{-1/2}^{1/2} \kappa_p(t-s)Q_{m,s}[a_p(t,s)u(s)]\,ds \tag{8.7}$$

where subindex s stands for the interpolation with respect to argument s only. For $u \in \mathcal{T}_n \ominus \mathcal{T}_l$ we use asymptotic approximations $Q_n\mathcal{A}_d u$, $d \in \mathbf{N}$, of $\mathcal{A}u$ where $\mathcal{A}_d$ is of the form

$$\begin{aligned} \mathcal{A}_d &= \sum_{p=0}^{q} A_{p,d}, \\ (A_{p,d}u)(t) &= \sum_{j=0}^{d-p-1} a_{p,j}(t) \int_{-1/2}^{1/2} \kappa_{p,j}(t-s)u(s)\,ds \quad \text{for } p+1 \le d, \\ A_{p,d} &= 0 \quad \text{for } p+1 > d. \end{aligned} \tag{8.8}$$

The only condition we put on the approximation is

$$\mathcal{A} - \mathcal{A}_d \in Op\sum\nolimits^{\alpha-d}. \tag{8.9}$$

We refer to Lemmas 6.1 and 6.2 for two different possible constructions of $a_{p,j}$ and $\kappa_{p,j}$. In both cases $\kappa_{p,0} = \kappa_p$, $a_{p,0}(t) = a_p(t,t)$. Introduce the following modifications of the basic collocation method (8.3):

$$u_n \in \mathcal{T}_n, \quad A_{l,m,n,d}u_n = Q_n f \quad \text{where } A_{l,m,n,d} = Q_m\mathcal{A}^{(m)}P_l + Q_n\mathcal{A}_d(I-P_l), \tag{8.10}$$

$$u_n \in \mathcal{T}_n, \quad \tilde{A}_{l,m,n,d}u_n = Q_n f \quad \text{where } \tilde{A}_{l,m,n,d} = Q_nM_bQ_mM_{1/b}Q_m\mathcal{A}^{(m)}P_l + Q_n\mathcal{A}_d(I-P_l). \tag{8.11}$$

Here $b(t) := a_0(t,t)$ and

$$(M_b u)(t) = b(t)u(t), (M_{1/b}u)(t) = u(t)/b(t).$$

At first look, approximation (8.11) seems to be more complicated than (8.10), but actually it leads to more simple matrix schemes as we will see later.

Lemma 8.1. *For $v_l \in \mathcal{T}_l$, $2l \le m \le n$ and any $r > 0$, we have*

$$\|(Q_m \mathcal{A}^{(m)} - Q_n \mathcal{A}) v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda, \quad \lambda \in \mathbf{R}. \tag{8.12}$$

Proof. It is sufficient to prove (8.12) for large λ, say $\lambda > \alpha + 1/2$, $\lambda > 1/2$. We have

$$Q_m \mathcal{A}^{(m)} - Q_n \mathcal{A} = Q_m(\mathcal{A}^{(m)} - \mathcal{A}) - (I - Q_m)\mathcal{A} + (I - Q_n)\mathcal{A}.$$

(i) Notice that, due to the inequality $l \le m/2$,

$$(\mathcal{A}^{(m)} - \mathcal{A}) v_l \\ = -\sum_{p=0}^{q} \int_{-1/2}^{1/2} \kappa_p(t-s)(I - Q_{m,s})[(a_p - P_{l,s} a_p)(t,s) v_l(s)]\, ds.$$

Due to estimates (5.3) and (4.2) of [**13**],

$$\begin{aligned} \|(\mathcal{A}^{(m)} - \mathcal{A}) v_l\|_{\lambda-\alpha} &\le c \sum_{p=0}^{q} \|(I - Q_{m,s})(a_p - P_{l,s} a_p) v_l\|_{\lambda-\alpha,\lambda} \\ &\le c' \sum_{p=0}^{q} \|(a_p - P_{l,s} a_p) v_l\|_{\lambda-\alpha,\lambda} \\ &\le c'' \sum_{p=0}^{q} \|(a_p - P_{l,s} a_p)\|_{\lambda-\alpha,\lambda} \|v_l\|_\lambda \\ &\le c''' \sum_{p=0}^{q} \|a_p\|_{\lambda-\alpha,\lambda+r} l^{-r} \|v_l\|_\lambda \end{aligned}$$

with any $r > 0$, where

$$\|a\|_{\lambda_1,\lambda_2} = \left(\sum_{k_1,k_2 \in \mathbf{Z}} [\max\{1,|k_1|\}]^{2\lambda_1} [\max\{1,|k_2|\}]^{2\lambda_2} |\hat{a}(k_1,k_2)|^2 \right)^{1/2}.$$

Since $\|Q_m\|_{\mathcal{L}(H^{\lambda-\alpha},H^{\lambda-\alpha})} \le \text{const}$ for $\lambda-\alpha > 1/2$, we also have with any $r > 0$,

$$\|Q_m(\mathcal{A}^{(m)} - \mathcal{A})v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda.$$

(ii) Notice that, with $l' = [(m-l)/2]$,

$$(I - Q_m)Av_l = \sum_{p=0}^{q} (I - Q_m) \cdot \int_{-1/2}^{1/2} \kappa_p(t-s)[a_p(t,s) - (P_{l',l'}a_p)(t,s)]v_l(s)\, ds$$

where $P_{l,l}$ denotes the orthogonal projection of order l with respect to both arguments t and s. Estimating as in (i) we obtain

$$\|(I - Q_m)\mathcal{A}v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda \quad \text{with any } r > 0.$$

(iii) The treatment of the term $(I - Q_n)\mathcal{A}v_l$ is similar to (ii). □

Lemma 8.2. *For $v_l \in \mathcal{T}_l$, $2l \le m \le n$ and any $r > 0$, we have*

$$\|(Q_n M_b Q_m M_{1/b} - I) Q_m \mathcal{A}^{(m)} v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda, \quad \lambda \in \mathbf{R}.$$

Proof. Due to Lemma 8.1, we have, with $m' = [(m+l)/2]$,

$$\|(Q_m \mathcal{A}^{(m)} - Q_{m'} \mathcal{A}^{(m')}) v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda,$$

therefore it is sufficient to show that

$$\|(Q_n M_b Q_m M_{1/b} - I) Q_{m'} \mathcal{A}^{(m')} v_l\|_{\lambda-\alpha} \le c_{\lambda,r} l^{-r} \|v_l\|_\lambda.$$

Notice that $Q_n M_b Q_n M_{1/b} = Q_n$, therefore, with $w_{m'} = Q_{m'} \mathcal{A}^{(m')} v_l$,

$$\begin{aligned}(Q_n M_b Q_m M_{1/b} - I) w_{m'} &= Q_n M_b (Q_m - Q_n) M_{1/b} w_{m'} \\ &= Q_n M_b (Q_m - Q_n) M_{(1/b) - P_{m-m'}(1/b)} w_{m'}\end{aligned}$$

and, for $\lambda - \alpha > 1/2$,

$$\begin{aligned}\|(Q_n M_b Q_m M_{1/b} - I)Q_{m'}\mathcal{A}^{(m')}v_l\|_{\lambda-\alpha} &\le c\|(1/b) - P_{m-m'}(1/b)\|_{\lambda-\alpha}\|w_{m'}\|_{\lambda-\alpha}\\ &\le c_\lambda (m-m')^{-r}\|1/b\|_{\lambda-\alpha+r}\|v_l\|_\lambda\\ &\le c_{\lambda,r} l^{-r}\|v_l\|_\lambda. \qquad \square\end{aligned}$$

Lemma 8.3. *For $v_n \in \mathcal{T}_n$, $l \le n$, we have*

$$\|Q_n(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha} \le c_\lambda l^{-d}\|(I-P_l)v_n\|_\lambda, \quad \lambda \ge \alpha. \tag{8.13}$$

Proof. Due to (8.9), $\mathcal{A}_d - \mathcal{A} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+d})$, therefore

$$\|(\mathcal{A}_d - \mathcal{A})(I-P_l)\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \le c_\lambda l^{-d}.$$

From this (8.13) follows immediately for $\lambda - \alpha > 1/2$. For $0 \le \lambda - \alpha \le 1/2$, we estimate

$$\begin{aligned}\|Q_n(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha} \le \|(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha}\\ + \|(I-Q_n)(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha}\end{aligned}$$

and

$$\begin{aligned}\|(I-Q_n)(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha} &\le cn^{-1}\|(\mathcal{A}_d - \mathcal{A})(I-P_l)v_n\|_{\lambda-\alpha+1}\\ &\le c_\lambda n^{-1}n^{-d}\|(I-P_l)v_n\|_{\lambda+1}\\ &\le c_\lambda n^{-d}\|(I-P_l)v_n\|_\lambda,\end{aligned}$$

and this results in (8.13) again. $\square$

Theorem 8.2. *Assume* (4.3)–(4.6), (8.2) *and* (8.6). *Then the operators $\mathcal{A}_{l,m,n,d} \in \mathcal{L}(\mathcal{T}_n)$ and $\tilde{\mathcal{A}}_{l,m,n,d} \in \mathcal{L}(\mathcal{T}_n)$ are invertible for $n \ge n_0$ with some $n_0 \in \mathbf{N}$,*

$$\|\mathcal{A}^{-1}_{l,m,n,d}\|_{\mathcal{L}(H^{\lambda-\alpha},H^\lambda)} \le c_\lambda, \qquad \|\tilde{\mathcal{A}}^{-1}_{l,m,n,d}\|_{\mathcal{L}(H^{\lambda-\alpha},H^\lambda)} \le c_\lambda, \quad \lambda \ge \alpha, \tag{8.14}$$

and for the solution u_n of equation (8.10) *or* (8.11)*, we have with any $r \geq \mu - \alpha + d$,*

$$\begin{aligned}\|u_n - u_n^c\|_\lambda &\leq c_\lambda l^{-d} \|(I - P_l) u_n^c\|_\lambda + c_{\lambda,r} l^{-r} \|P_l u_n^c\|_\lambda \\ &\leq c'_\lambda l^{\lambda-\mu-d} \|u\|_\mu, \quad \alpha \leq \lambda \leq \mu,\end{aligned} \tag{8.15}$$

where $u_n^c = (Q_n \mathcal{A})^{-1} Q_n f$ is the collocation solution and $u = \mathcal{A}^{-1} f \in H^\mu$ is the exact solution of (8.1)*. If $d \geq ((1-\rho)/\rho)(\mu - \alpha)$, then*

$$\|u_n - u\|_\lambda \leq c n^{\lambda-\mu} \|u\|_\mu, \quad \alpha \leq \lambda \leq \mu. \tag{8.16}$$

Proof. We have

$$\begin{aligned}\mathcal{A}_{l,m,n,d} - Q_n \mathcal{A} &= (Q_m \mathcal{A}^{(m)} - Q_n \mathcal{A}) P_l + Q_n (\mathcal{A}_d - \mathcal{A})(I - P_l), \\ \tilde{\mathcal{A}}_{l,m,n,d} - \mathcal{A}_{l,m,n,d} &= (Q_n M_b Q_m M_{1/b} - I) Q_m \mathcal{A}^{(m)} P_l.\end{aligned}$$

With the help of Lemmas 8.1–8.3, we find, for $v_n \in \mathcal{T}_n$, $r \geq d$,

$$\begin{aligned}\|(\mathcal{A}_{l,m,n,d} - Q_n \mathcal{A}) v_n\|_{\lambda-\alpha} &\leq c_{\lambda,r} l^{-r} \|P_l v_n\|_\lambda + c_\lambda l^{-d} \|(I - P_l) v_n\|_\lambda \\ &\leq c'_\lambda l^{-d} \|v_n\|_\lambda, \\ \|(\tilde{\mathcal{A}}_{l,m,n,d} - \mathcal{A}_{l,n,n,d}) v_n\|_{\lambda-\alpha} &\leq c_{\lambda,r} l^{-r} \|P_l v_n\|_\lambda \\ &\leq c_{\lambda,r} l^{-r} \|v_n\|_\lambda.\end{aligned}$$

Together with (8.4), this implies (8.14). For $u_n = \mathcal{A}_{l,m,n,d}^{-1} Q_n f$, we have

$$\mathcal{A}_{l,m,n,d}(u_n^c - u_n) = (\mathcal{A}_{l,m,n,d} - Q_n \mathcal{A}) u_n^c$$

and

$$\begin{aligned}\|u_n^c - u\|_\lambda &\leq c_\lambda \|(\mathcal{A}_{l,m,n,d} - Q_n \mathcal{A}) u_n^c\|_{\lambda-\alpha} \\ &\leq c_{\lambda,r} l^{-r} \|P_l u_n^c\|_\lambda + c'_\lambda l^{-d} \|(I - P_l) u_n^c\|_\lambda \\ &\leq c''_\lambda l^{-d+\lambda-\mu} \|u_n^c\|_\mu \\ &\leq c'''_\lambda l^{\lambda-\mu-d} \|u\|_\mu\end{aligned}$$

(on the last step we used (8.5) with $\lambda = \mu$). We obtained estimates (8.15) for method (8.10). An obvious argument extends the result for method (8.11). Estimate (8.16) follows from (8.5) and (8.15). □

9. Matrix form of the method (8.10). Here we show that under conditions, cf. (8.6),

$$2l \le m \le n,\ l \sim n^{\rho},\ m \sim n^{\sigma},\quad 0 < \rho \le \sigma \le 1/2, \tag{9.1}$$

the computation of $\mathcal{A}_{l,m,n,d}v_n \in \mathcal{T}_n$ for $v_n \in \mathcal{T}_n$ costs $\mathcal{O}(n\log n)$ arithmetical operations. Consequently, iteration methods such as conjugate gradients or GMRES can be recommended to solve the corresponding n-system.

(i) Computation of $Q_m\mathcal{A}^{(m)}P_l v_n = \sum_{p=0}^{q} Q_m A_p^{(m)} P_l v_n,\ v_n \in \mathcal{T}_n$. For $w \in H^{\nu}$, $\nu > 1/2$, we have

$$Q_n w = \sum_{j=0}^{n-1} w(jn^{-1})\varphi_{n,j}, \quad \varphi_{n,j}(t) = \frac{1}{n}\sum_{k\in\mathbf{Z}_n} e^{ik2\pi(t-jn^{-1})}.$$

Thus,

$$Q_{m,s}(a_p(t,s)v_l(s)) = \sum_{j=0}^{m-1} a_p(t,jm^{-1})v_l(jm^{-1})\varphi_{m,j}(s),$$

$$\begin{aligned}
(A_p^{(m)}v_l)(t) \\
&= \int_{-1/2}^{1/2} \kappa_p(t-s)Q_{m,s}(a_p(t,s)v_l(s))\,ds \\
&= \sum_{j=0}^{m-1} a_p(t,jm^{-1})v_l(jm^{-1}) \int_{-1/2}^{1/2} \kappa_p(t-s)\varphi_{m,j}(s)\,ds \\
&= \sum_{j=0}^{m-1} a_p(t,jm^{-1})v_l(jm^{-1}) \frac{1}{m}\sum_{k\in\mathbf{Z}_m} \hat{\kappa}_p(k) e^{ik2\pi(t-jm^{-1})}
\end{aligned}$$

and

$$\begin{aligned}
(Q_m A_p^{(m)}v_l)(j'm^{-1}) &= \sum_{j=0}^{m-1} a_p(j'm^{-1},jm^{-1}) \\
&\quad \cdot \frac{1}{m}\left(\sum_{k\in\mathbf{Z}_m} e^{ik2\pi(j'-j)m^{-1}}\hat{\kappa}_p(k)\right) v_l(jm^{-1}), \\
\widehat{Q_m A_p^{(m)} P_l v_n} &= \mathcal{F}_m \mathbf{A}_{p,m} \mathcal{F}_m^{-1} \mathbf{P}_{m,n}^{(l)} \hat{v}_n
\end{aligned}$$

where

$$(\mathbf{P}^{(l)}_{m,n}\hat{v}_n)(k) = \begin{cases} \hat{v}_n(k), & k \in \mathbf{Z}_l, \\ 0, & k \in \mathbf{Z}_m \backslash \mathbf{Z}_l, \end{cases}$$

and $\mathbf{A}_{p,m}$ is an $m \times m$ matrix with the entries

$$\mathfrak{a}_{p,j',j} = a_p(j'm^{-1}, jm^{-1}) \frac{1}{m} \sum_{k \in \mathbf{Z}_m} e^{ik2\pi(j'-j)m^{-1}} \hat{\kappa}_p(k),$$
$$j', j = 0, \dots, m-1.$$

Clearly the computation of the entries of $\mathbf{A}_{p,m}$ costs $\mathcal{O}(m^2)$ arithmetical operations. Application of $\mathcal{F}_m^{-1}$, $\mathbf{A}_{p,m}$ and $\mathcal{F}_m$ also costs $\mathcal{O}(m^2)$ arithmetical operations (for $\mathcal{F}_m^{-1}$, $\mathcal{F}_m$ here the usual matrix application may be used).

(ii) Computation of $Q_n \mathcal{A}_d (I - P_l) v_n$, $v_n \in \mathcal{T}_n$. Recall that

$$\mathcal{A}_d = \sum_{p=0}^{q} A_{p,d},$$
$$A_{p,d} = M_{a_{p,j}} B_{p,j} \quad \text{for } p+1 \le d,$$
$$A_{p,d} = 0 \quad \text{for } p+1 > d,$$
$$(B_{p,j}u)(t) = \int_{-1/2}^{1/2} \kappa_{p,j}(t-s) u(s)\, ds.$$

For $a \in C_1^\infty(\mathbf{R})$, $w_n \in \mathcal{T}_n$, we have $\widehat{Q_n M_a w_n} = \mathcal{F}_n \mathbf{M}_{a,n} \mathcal{F}_n^{-1} \hat{w}_n$ with the diagonal $n \times n$-matrix $\mathbf{M}_{a,n} = \operatorname{diag}(a(jn^{-1}), j = 0, \dots, n-1)$. We see that

$$\widehat{Q_n M_{a_{p,j}} B_{p,j} (I - P_l) v_n} = \mathcal{F}_n \mathbf{M}_{a_{p,j},n} \mathcal{F}_n^{-1} \mathbf{B}_{p,j,n} \mathbf{R}_n^{(l)} \hat{v}_n, \quad p+1 \le d,$$

where

$$(\mathbf{R}_n^{(l)} \hat{v}_n)(k) = \begin{cases} 0, & k \in \mathbf{Z}_l \\ \hat{v}_n(k), & k \in \mathbf{Z}_n \backslash \mathbf{Z}_l, \end{cases}$$
$$\mathbf{B}_{p,j,n} = \operatorname{diag}(\hat{\kappa}_{p,j}(k), k \in \mathbf{Z}_n).$$

The computation of $Q_n \mathcal{A}_d (I - P_l) v_n$ costs $\mathcal{O}(n \log n)$ arithmetical operations provided that FFT is used for the application of $\mathcal{F}_n^{-1}$ and $\mathcal{F}_n$. Thus, under condition (9.1), $\mathcal{A}_{l,m,n,d}$ can be applied to $v_n \in \mathcal{T}_n$

in $\mathcal{O}(n\log n)$ arithmetical operations. The matrix form of the modified collocation method (8.10) is

$$\mathbf{P}_{n,m}\mathcal{F}_m\sum_{p=0}^{q}\mathbf{A}_{p,m}\mathcal{F}_m^{-1}\mathbf{P}_{m,n}^{(l)}\hat{v}_n + \mathcal{F}_n\sum_{p=0}^{q}\sum_{j=0}^{d-p-1}\mathbf{M}_{a_{p,j},n}\mathcal{F}_n^{-1}\mathbf{B}_{p,j,n}\mathbf{R}_n^{(l)}\hat{v}_n = \mathcal{F}_n\underline{f}_n \tag{9.2}$$

where $\underline{f}_n = (f(jn^{-1}), j = 0, 1, \dots, n-1)$.

$$(\mathbf{P}_{n,m}\hat{w}_m)(k) = \begin{cases}\hat{w}_m(k), & k \in \mathbf{Z}_m\\ 0, & k \in \mathbf{Z}_n\backslash\mathbf{Z}_m,\end{cases} \qquad m \le n,$$

and the convention $\sum_{j=0}^{h} = 0$ for $h < 0$ is used. The iteration methods such as GMRES or conjugate gradients, can be recommended to solve the m-system (9.2). The number of iterations for these methods to obtain the accuracy comparable with the approximation accuracy (8.16) seems to be $\mathcal{O}(\log n)$, see [**34**] for a more simple but multidimensional problem. We do not go into details. Instead, we propose an iteration method which needs a finite number of iterations which is independent of n. This method is based on the preconditioning of (8.11) by $\tilde{\mathcal{A}}_{l,m,n,1}^{-1}$.

Remark 9.1. Theorem 8.2 remains true for the further modification where, on construction of $\mathcal{A}_d$, we first approximate the coefficients $a_p(t,s)$, $p = 0, \dots, q$, by their two-dimensional interpolations $Q_{m,m}a_p$ and then differentiate these functions, cf. (6.2) and (6.5). Under condition (9.1), the computation of $\widehat{Q_{m,m}a_p}$ by FFT from the grid values $a_p(jm^{-1}, j'm^{-1})$, $j, j' = 0, \dots, m-1$, costs $\mathcal{O}(n\log n)$ arithmetical operations.

10. Preconditioning of (8.11) and iteration method. Equation (8.11) is equivalent to

$$u_n \in \mathcal{T}_n, \quad \tilde{\mathcal{A}}_{l,m,n,1}^{-1}(\tilde{A}_{l,m,n,d}u_n - Q_nf) = 0,$$

or

$$u_n = S_nu_n + g_n, \tag{10.1}$$

where

$$S_n = S_{l,m,n,d} = \tilde{\mathcal{A}}_{l,m,n,1}^{-1}(\tilde{\mathcal{A}}_{l,m,n,1} - \tilde{\mathcal{A}}_{l,m,n,d})$$
$$= \tilde{\mathcal{A}}_{l,m,n,1}^{-1} Q_n(\mathcal{A}_1 - \mathcal{A}_d)(I - P_l) \in \mathcal{L}(\mathcal{T}_n),$$
$$g_n = \tilde{\mathcal{A}}_{l,m,n,1}^{-1} Q_n f \in \mathcal{T}_n.$$

Due to (8.13) and (8.14), $\|S_n v_n\|_\lambda \leq cl^{-1}\|(I-P_l)v_n\|_\lambda$, $\lambda \geq \alpha$, $v_n \in \mathcal{T}_n$. Consequently, solving (10.1) by the iteration method

$$u_n^0 = g_n, \quad u_n^k = S_n u_n^{k-1} + g_n, \quad k = 1, 2, \dots, \tag{10.2}$$

we have with the solution u_n of (8.11) and (10.1), $d \geq ((1-\rho)/\rho)(\mu-\alpha)$,

$$u_n^k - u_n = S_n^k(u_n^0 - u_n),$$
$$\|u_n^k - u_n\|_\lambda \leq (cl^{-1})^k \|(I - P_l)(u_n^0 - u_n)\|_\lambda$$
$$\leq (cl^{-1})^k (l/2)^{\lambda-\mu} \|u_n^0 - u_n\|_\mu$$
$$\leq c'(cl^{-1})^{k+1} l^{\lambda-\mu} \|u\|_\mu,$$
$$\alpha \leq \lambda \leq \mu, \quad u = \mathcal{A}^{-1} f,$$

see (8.16). For $k + 1 \geq ((1 - \rho)/\rho)(\mu - \lambda)$, this provides

$$\|u_n^k - u_n\|_\lambda \leq cn^{\lambda-\mu}\|u\|_\mu, \quad 0 \leq \lambda \leq \mu.$$

Consequently, the following result holds true.

Theorem 10.1. *Fix $d \geq ((1 - \rho)/\rho)(\mu - \alpha)$. Then, under conditions of Theorem 8.2, for $k \geq ((1 - \rho)/\rho)(\mu - \alpha)^{-1}$, the iteration approximations u_n^k are of optimal accuracy order*

$$\|u_n^k - u\|_\lambda \leq cn^{\lambda-\mu}\|u\|_\mu, \quad 0 \leq \lambda \leq \mu \tag{10.3}$$

where $u = \mathcal{A}^{-1} f \in H^\mu$.

To present a matrix form of iteration method (10.2), we have to analyze the computation of $v_n = \tilde{\mathcal{A}}_{l,m,n,1}^{-1} w_n \in \mathcal{T}_n$ for a given $w_n \in \mathcal{T}_n$, i.e., the solution of equation $\tilde{\mathcal{A}}_{l,m,n,1} v_n = w_n$. Notice that $\mathcal{A}_1 = A_{0,0} = M_b B$ with $b(t) = a_{0,0}(t) = a_0(t,t)$, $B = B_{0,0}$ introduced in the previous section. Thus v_n satisfies

$$Q_n M_b Q_m M_{1/b} Q_m \mathcal{A}^{(m)} P_l v_n + Q_n M_b B(I - P_l)v_n = w_n.$$

Since $(Q_n M_{1/b})(Q_n M_b) = Q_n$, the equation takes the form

$$Q_m M_{1/b} Q_m \mathcal{A}^{(m)} P_l v_n + B(I - P_l) v_n = Q_n M_{1/b} w_n.$$

Applying to both sides P_l and taking into account that $P_l B = B P_l$ we obtain

$$P_l Q_m M_{1/b} Q_m \mathcal{A}^{(m)} P_l v_n = P_l Q_n M_{1/b} w_n.$$

Under conditions (4.3)–(4.6), the operator $P_l M_{1/b} \mathcal{A} \in \mathcal{L}(\mathcal{T}_l)$ is invertible for all sufficiently large l, and $\|(P_l M_{1/b} \mathcal{A})^{-1}\|_{\mathcal{L}(H^{\lambda-\alpha}, H^{\lambda})} \leq \text{const}$, see [**13**]. Using Lemma 8.1 it is easy to see that

$$\|P_l (Q_m M_{1/b} Q_m \mathcal{A}^{(m)} - M_{1/b} \mathcal{A}) P_l\|_{\mathcal{L}(H^{\lambda}, H^{\lambda-\alpha})} \to 0 \quad \text{as } l \to \infty;$$

therefore, the operator $P_l Q_m M_{1/b} Q_m \mathcal{A}^{(m)} \in \mathcal{L}(\mathcal{T}_l)$ is also invertible for sufficiently large l, and

$$\|(P_l Q_m M_{1/b} Q_m \mathcal{A}^{(m)})^{-1}\|_{\mathcal{L}(H^{\lambda-\alpha}, H^{\lambda})} \leq \text{const}, \quad l \geq l_0.$$

From this, we find for $v_n = P_n v_l + (I - P_l) v_n$, the representation

$$\begin{aligned} v_n &= B^{-1}(I - P_l) Q_n M_{1/b} w_n \\ &\quad + (I - B^{-1}(I - P_l) Q_m M_{1/b} Q_m \mathcal{A}^{(m)}) \\ &\quad \cdot (P_l Q_m M_{1/b} Q_m \mathcal{A}^{(m)} P_l)^{-1} P_l Q_n M_{1/b} w_n. \end{aligned}$$

The matrix form of this formula is given by

$$\begin{aligned} \hat{v}_n &= \mathbf{C}_n^{(l)} \mathcal{F}_n \mathbf{M}_{1/b,n} \mathcal{F}_n^{-1} \hat{w}_n \\ &\quad + \mathbf{P}_{n,m} \left(\mathbf{I}_m - \mathbf{C}_m^{(l)} \mathcal{F}_m \mathbf{M}_{1/b,m} \sum_{p=0}^{q} \mathbf{A}_{p,m} \mathcal{F}_m^{-1} \right) \\ &\quad \cdot \mathbf{P}_{m,l} \mathbf{D}_l^{-1} \mathbf{P}_{l,n} \mathcal{F}_n \mathbf{M}_{1/b,n} \mathcal{F}_n^{-1} \hat{w}_n \\ &=: \mathbf{E}_n \hat{w}_n \end{aligned}$$

where $\mathbf{I}_m$ is the identity matrix of dimension $m \times m$,

$$\mathbf{D}_l = \mathbf{P}_{l,m} \mathcal{F}_m \mathbf{M}_{1/b,m} \sum_{p=0}^{q} \mathbf{A}_{p,m} \mathcal{F}_m^{-1} \mathbf{P}_{m,l}$$

is an $l \times l$ matrix,

$$(\mathbf{P}_{l,n}\hat{v}_n)(k) = \hat{v}_n(k), \quad k \in \mathbf{Z}_l, \quad l \le n,$$
$$(\mathbf{C}_n^{(l)}\hat{v}_n)(k) = \begin{cases} \hat{v}_n(k)/\hat{\kappa}_0(k), & k \in \mathbf{Z}_n \backslash \mathbf{Z}_l \\ 0, & k \in \mathbf{Z}_l \end{cases}$$

and other matrices are introduced in the previous section. Clearly, under condition (9.1), the computation of $\hat{v}_n = \mathbf{E}_n \hat{w}_n$ costs $\mathcal{O}(n \log n)$ arithmetical operations, if the application of $\mathbf{D}_l^{-1}$ can be done in this limitation. Using FFT, the matrix products defining $\mathbf{D}_l$ need $\mathcal{O}(m^2 \log m) \le \mathcal{O}(n \log n)$ arithmetical operations. On the other hand, an application of $\mathbf{D}_l^{-1}$ to an l-vector by a Gauss-type method costs $\mathcal{O}(l^3)$ operations; therefore, we now strengthen the condition on l:

$$2l \le m \le n,\ l \sim n^\rho,\ n \sim n^\sigma, \quad 0 < \rho \le 1/3,\ \rho \le \sigma \le 1/2. \tag{10.4}$$

The matrix form of the iteration method (10.2) reads as follows:

$$\hat{u}_n^0 = \hat{g}_n = \mathbf{E}_n \mathcal{F}_n \underline{f}_n,$$
$$\hat{u}_n^k = -\mathbf{E}_n \mathcal{F}_n \Big(\sum_{j=1}^{d-1} \mathbf{M}_{a_{0,j},n} \mathcal{F}_n^{-1} \mathbf{B}_{p,j,n} + \sum_{p=1}^{q} \sum_{j=0}^{d-p-1} M_{a_{p,j},n} \mathcal{F}_n^{-1} \mathbf{B}_{p,j,n} \Big) \mathbf{R}_n^{(l)} \hat{u}_n^{k-1} + \hat{g}_n.$$

Under condition (10.4), an iteration costs $\mathcal{O}(n \log n)$ arithmetical operations and, since the number $k = [((1-\rho)/\rho)(\mu - \alpha)]$ of iterations is sufficient to achieve the optimal accuracy order (10.3), the total work remains in the amount of $\mathcal{O}(n \log n)$ arithmetical operations.

11. Modification for even/odd $\mathcal{A}$ and f. Now we modify the methods (8.10) and (8.11) for different cases where the solution of (8.1) belongs to H_e^μ or H_o^μ.

(a) $\mathcal{A}$ and f are even, e.g., problem (3.5). Here we assume that κ_0 and a_0 are even, whereas κ_p and a_p, $1 \le p \le q$, may be both even or both odd. More precisely, κ_0 and a_0 satisfy (4.11) and (4.13) whereas κ_p and a_p, $1 \le p \le q$ satisfy {(4.11),(4.13)} or {(4.12),(4.14)}. According to Lemmas 4.1 and 4.2, under these supplementary conditions together with (4.3)–(4.5), $\mathcal{A} = \sum_{p=0}^{q} A_p \in \mathcal{L}(H_e^\lambda, H_e^{\lambda-\alpha})$ is a Fredholm operator of index 0 for any $\lambda \in \mathbf{R}$. We assume that

$$u \in C_{1e}^\infty(\mathbf{R}), \quad \mathcal{A}u = 0 \;\Rightarrow\; u = 0, \tag{11.1}$$

then $\mathcal{A} \in \mathcal{L}(H_e^\lambda, H_e^{\lambda-\alpha})$ is isomorphic, and for

$$f \in H_e^{\mu-\alpha}, \quad \mu - \alpha > \frac{1}{2}, \tag{11.2}$$

equation (8.1) has a unique solution $u \in H_e^\mu$ which can be determined by the even trigonometric collocation method

$$u_n \in \mathcal{T}_n^e, \quad Q_n^e \mathcal{A} u_n = Q_n^e f. \tag{11.3}$$

The counterpart of (8.10) now reads as follows:

$$\begin{gathered} u_n \in \mathcal{T}_n^e, \quad \mathcal{A}_{l,m,n,d}^{e\leftarrow e} u_n = Q_n^e f, \\ \mathcal{A}_{l,m,n,d}^{e\leftarrow e} = Q_m^e \mathcal{A}_e^{(m)} P_l^e + Q_n^e \mathcal{A}_d (I - P_l^e). \end{gathered} \tag{11.4}$$

Here $\mathcal{A}_e^{(m)} = \sum_{p=0}^{q} A_{p,e}^{(m)}$ with

$$(A_{p,e}^{(m)} v_l)(t) = \begin{cases} \int_{-1/2}^{1/2} \kappa_p(t-s) Q_{m,s}^e (a_p(t,s) v_l(s))\, ds \\ \quad \text{if } a_p \text{ is even in } s, \\ \int_{-1/2}^{1/2} \kappa_p(t-s) Q_{m,s}^o (a_p(t,s) v_l(s))\, ds \\ \quad \text{if } a_p \text{ is odd in } s, \end{cases} \quad , \quad v_l \in \mathcal{T}_l^e$$

remains to be even. The approximation $\mathcal{A}_d$ is constructed following Lemma 6.2. Thus, $\mathcal{A}_d$ preserves the parity properties of $\mathcal{A}$, is of the form (8.8) and satisfies (8.9). The counterpart of (8.11) which reads as follows:

$$\begin{gathered} u_n \in \mathcal{T}_n^e, \quad \tilde{\mathcal{A}}_{l,m,n,d}^{e\leftarrow e} u_n = Q_n^e f, \\ \tilde{\mathcal{A}}_{l,m,n,d}^{e\leftarrow e} = Q_n^e M_b Q_m^e M_{1/b} Q_m^e \mathcal{A}_e^{(m)} P_l^e + Q_n^e \mathcal{A}_d (I - P_l^e). \end{gathered} \tag{11.5}$$

This equation is equivalent to (10.1) and can be solved by the iteration method (10.2) where now

$$S_n = (\tilde{\mathcal{A}}_{l,m,n,1}^{e\leftarrow e})^{-1} Q_n^e (\mathcal{A}_1 - \mathcal{A}_d)(I - P_l^e), \quad g_n = (\tilde{\mathcal{A}}_{l,m,n,1}^{e\leftarrow e})^{-1} Q_n^e f.$$

With obvious modifications, the results and analysis of previous sections holds for methods (11.3), (11.4), (11.5) and the iteration solution of (11.5). In the matrix form of the methods, sine and cosine transforms are used instead of $\mathcal{F}_n$.

(b) $\mathcal{A}$ is even and f is odd, e.g., problem (3.16). About κ_p and a_p, we make again the assumptions introduced in (a) but instead of (11.1) and (11.2) we assume

$$u \in C_{1o}^{\infty}(\mathbf{R}), \quad \mathcal{A}u = 0 \ \Rightarrow \ u = 0, \tag{11.6}$$

$$f \in H_o^{\mu-\alpha}, \quad \mu - \alpha > 1/2. \tag{11.7}$$

Then $\mathcal{A} \in \mathcal{L}(H_o^{\lambda}, H_o^{\lambda-\alpha})$ is isomorphic, and $u = \mathcal{A}^{-1}f \in H_o^{\mu}$ can be determined by the odd trigonometric collocation method

$$u_n \in \mathcal{T}_n^o, \quad Q_n^o \mathcal{A} u_n = Q_n^o f. \tag{11.8}$$

Its fully discrete modifications are

$$\begin{gathered} u_n \in \mathcal{T}_n^o, \quad \mathcal{A}_{l,m,n,d}^{o\leftarrow o} u_n = Q_n^o f, \\ \mathcal{A}_{l,m,n,d}^{o\leftarrow o} = Q_m^o \mathcal{A}_o^{(m)} P_l^o + Q_n^o \mathcal{A}_d (I - P_l^o), \end{gathered} \tag{11.9}$$

$$\begin{gathered} u_n \in \mathcal{T}_n^o, \quad \tilde{\mathcal{A}}_{l,m,n,d}^{o\leftarrow o} u_n = Q_n^o f, \\ \tilde{\mathcal{A}}_{l,m,n,d}^{o\leftarrow o} = Q_n^o M_b Q_m^o M_{1/b} Q_m^o \mathcal{A}_o^{(m)} P_l^o + Q_n^o \mathcal{A}_d (I - P_l^o) \end{gathered} \tag{11.10}$$

where $\mathcal{A}_o^{(m)} = \sum_{p=0}^{q} A_{p,o}^{(m)}$ with

$$(A_{p,o}^{(m)} v_l)(t) = \begin{cases} \int_{-1/2}^{1/2} \kappa_p(t-s) Q_{m,s}^o (a_p(t,s) v_l(s))\, ds \\ \quad \text{if } a_p \text{ is even in } s, \\ \int_{-1/2}^{1/2} \kappa_p(t-s) Q_{m,s}^e (a_p(t,s) v_l(s))\, ds \\ \quad \text{if } a_p \text{ is odd in } s, \end{cases}, \quad v_l \in \mathcal{T}_l^o.$$

Again, the results and analysis of the previous sections remains true for the methods (11.8), (11.9), (11.10) and the iterations (10.2) with

$$\begin{aligned} S_n &= (\tilde{\mathcal{A}}_{l,m,n,1}^{o\leftarrow o})^{-1} Q_n^o (\mathcal{A}_1 - \mathcal{A}_d)(I - P_l^o), \\ g_n &= (\tilde{\mathcal{A}}_{l,m,n,1}^{o\leftarrow o})^{-1} Q_n^o f. \end{aligned}$$

(c) $\mathcal{A}$ and f are odd, e.g., problem (3.10). Here we assume that a_0 is even and κ_0 is odd, whereas κ_p, $1 \le p \le q$ are even or odd with a_p, respectively, odd or even. Further we assume that

$$u \in C^\infty_{1e}(\mathbf{R}), \quad \hat{u}(0) = 0, \quad \mathcal{A}u = 0 \;\Rightarrow\; u = 0, \tag{11.1}$$

$$f \in H_o^{\mu-\alpha}, \quad \mu - \alpha > 1/2. \tag{11.12}$$

Then $\mathcal{A} \times \Phi \in \mathcal{L}(H_e^\lambda, H_o^{\lambda-\alpha} \times \mathbf{C})$ is an isomorphism, see Theorem 5.3, where $(\mathcal{A} \times \Phi)u = [\mathcal{A}u, \hat{u}(0)]$, $u \in H_e^\lambda$, and the problem

$$\mathcal{A}u = f, \quad \hat{u}(0) = \gamma,$$

has a unique solution $u = (\mathcal{A} \times \Phi)^{-1}[f, \gamma] \in H_e^\lambda$ which can be determined by the even-to-odd trigonometric collocation method

$$u_n \in \mathcal{T}_n^e, \quad Q_n^o \mathcal{A} u_n = Q_n^o f, \quad \hat{u}_n(0) = \gamma. \tag{11.13}$$

Its fully discrete modifications are given by

$$\begin{gathered} u_n \in \mathcal{T}_n^e, \quad \mathcal{A}^{o\leftarrow e}_{l,m,n,d} u_n = Q_n^o f, \hat{u}_n(0) = \gamma, \\ \mathcal{A}^{o\leftarrow e}_{l,m,n,d} = Q_m^o \mathcal{A}_e^{(m)} P_l^e + Q_n^o \mathcal{A}_d (I - P_l^e), \end{gathered} \tag{11.14}$$

$$\begin{gathered} u_n \in \mathcal{T}_n^e, \quad \tilde{\mathcal{A}}^{o\leftarrow e}_{l,m,n,d} u_n = Q_n^o f, \quad \hat{u}_n(0) = \gamma, \\ \tilde{\mathcal{A}}^{o\leftarrow e}_{l,m,n,d} = Q_n^o M_b Q_m^o M_{1/b} Q_m^o \mathcal{A}_e^{(m)} P_l^e + Q_n^o \mathcal{A}_d (I - P_l^e). \end{gathered} \tag{11.15}$$

The results of the previous sections hold true for the methods (11.13), (11.14), (11.15) and the iteration solution of (11.15).

(d) $\mathcal{A}$ is odd and f is even, e.g., problem (3.12). Here we put on $\mathcal{A}$ the same assumptions as in (c) but instead of (11.11) and (11.12),

$$u \in C^\infty_{1o}(\mathbf{R}), \quad \omega \in \mathbf{C}, \quad \mathcal{A}u + \omega = 0 \;\Rightarrow\; u = 0, \quad \omega = 0, \tag{11.16}$$

$$f \in H_e^{\mu-\alpha}, \quad \mu - \alpha > 1/2. \tag{11.17}$$

Then $\mathcal{A} \dotplus \Phi \in \mathcal{L}(H_o^\lambda \times \mathbf{C}, H_e^{\lambda-\alpha})$ defined by $(\mathcal{A} \dotplus \Phi)[u,\omega] = \mathcal{A}u + \omega$ is isomorphic, see Theorem 5.5, and the unique solution $[u,\omega] = (\mathcal{A} \dotplus \Phi)^{-1} f \in H_o^\lambda \times \mathbf{C}$ of the problem

$$\mathcal{A}u + \omega = f$$

can be determined by the odd-to-even trigonometric collocation method

$$u_n \in \mathcal{T}_n^o, \quad \omega \in \mathbf{C}, \quad Q_n^e(\mathcal{A}u_n + \omega) = Q_n^e f, \tag{11.18}$$

and its fully discrete modifications

$$\begin{gathered} u_n \in \mathcal{T}_n^o, \quad \omega \in \mathbf{C}, \quad \mathcal{A}_{l,m,n,d}^{e\leftarrow o} u_n + \omega = Q_n^e f, \\ \mathcal{A}_{l,m,n,d}^{e\leftarrow o} = Q_m^e \mathcal{A}_o^{(m)} P_l^o + Q_n^e \mathcal{A}_d (I - P_l^o), \end{gathered} \tag{11.19}$$

and

$$\begin{gathered} u_n \in \mathcal{T}_n^o, \quad \omega \in \mathbf{C}, \quad \tilde{\mathcal{A}}_{l,m,n,d}^{e\leftarrow o} u_n + \omega = Q_n^e f, \\ \tilde{\mathcal{A}}_{l,m,n,d}^{e\leftarrow o} = Q_n^e M_b Q_m^e M_{1/b} Q_m^e \mathcal{A}_o^{(m)} P_l^o + Q_n^e \mathcal{A}_d (I - P_l^o), \end{gathered} \tag{11.20}$$

with $\mathcal{A}_o^{(m)}$ as in case (b). The results of previous sections hold true for the methods (11.18), (11.19), (11.20) and the iteration solution of (11.20).

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