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THE DISCRETE PETROV-GALERKIN METHOD FOR WEAKLY SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. We propose discrete Petrov-Galerkin methods for Fredholm integral equations of the second kind with weakly singular kernels. To study the convergence of these methods, we develop a theoretical framework for analysis of a large class of numerical schemes including the discrete Galerkin, Petrov-Galerkin, collocation methods and quadrature methods. The theory is then applied to establish convergence results of the discrete Petrov-Galerkin methods. We also suggest a discrete iterated Petrov-Galerkin approximation for the solutions of these equations and prove a superconvergence property when the kernels are assumed to be smooth. Numerical examples are presented to illustrate the theoretical estimate for the error of approximation of these methods.

1. Introduction. We begin our presentation with a brief review of the literature. The Petrov-Galerkin method and the iterated Petrov-Galerkin method for Fredholm integral equations of the second kind were studied in [**7**], where the notions of the generalized best approximation and the regular pair of a trial space sequence and a test space sequence were developed to serve as an approach for the analysis of the methods. Several specific constructions of the Petrov-Galerkin elements in 1-D and 2-D were also designed in the paper. A general construction of the univariate Petrov-Galerkin elements of the piecewise polynomials were proposed in [**6**] and it was used to develop wavelet Petrov-Galerkin methods for integral equations of one dimension. Some early work on the Petrov-Galerkin method was found in [**10**]. Several Petrov-Galerkin elements constructed in [**7**] and [**6**] were proved to be useful in numerical solutions of integral equations.

Before describing the discrete Petrov-Galerkin method, a few remarks are in order on a comparison between the standard Galerkin method

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and the underlying method. Unlike the standard Galerkin method, the Petrov-Galerkin method allows us to have a trial space different from the test space. This provides us with great freedom in choosing a pair of space sequences to improve the computational efficiency while preserving the convergence order of the standard Galerkin method. It is revealed in [**7**] that for the Petrov-Galerkin methods the roles of the trial space and test space are to approximate the solution space of the equation and the range of the integral operator (or, in other words, the image space), respectively. Therefore, the convergence order of the Petrov-Galerkin method is the same as the approximation order of the trial space and it is independent of the approximation order of the test space. This leads to the following strategy of choosing the trial and test spaces. We may choose the trial space as piecewise polynomials of a higher degree and the test space as piecewise polynomials of a lower degree but keep them having the same dimension. This choice of the trial and test spaces results in a significantly less expensive numerical algorithm in comparison to the standard Galerkin method with the same convergence order which uses the same piecewise polynomials as those for the trial space. The saving comes from computing the entries of the matrix and the righthand side vector of the linear system that results from the corresponding discretization. Note that an entry of the *Galerkin matrix* is the inner product of the integral operator applied to a basis function for the trial space against a basis function for the same space, which is a piecewise polynomial of a *higher* degree, while an entry of the *Petrov-Galerkin matrix* is that against a basis function for the test space, which is a piecewise polynomial of a *lower* degree. Computing the latter is less expensive than computing the former due to the use of lower degree polynomials for the test space. In fact, the Petrov-Galerkin method *interpolates* between the Galerkin method and the collocation method.

However, to use the Petrov-Galerkin method in practical computation, we have to be able to efficiently compute the singular integrals occurring in the method. The discrete Galerkin method for integral equations of the second kind with continuous kernels has been studied in the literature. In [**3**], a discretized Galerkin method is obtained using numerical integration to evaluate the integrals occurring in the Galerkin method and in [**5**], by considering discrete inner product and discrete projections, the authors treated more appropriately kernels with discontinuous derivatives. A discrete convergence theory and its applications to the numerical solution of weakly singular integral equations were presented in [**11**]. A traditional construction of discrete projection methods for integral equations normally consists of two steps: using the projection to establish a finite dimensional projection equation and then applying the quadrature formula to discretize the entries to form the discrete projection methods. This way of development may result in some unnecessary hypotheses. In our current construction, we will take a one-step approach to discretizing a given integral equation by a discrete projection and a discrete inner product. By doing everything in one step, we do not assume the hypotheses for the convergence of the Petrov-Galerkin method which are imposed in [**7**] and a new assumption for the solvability and convergence of the discrete Petrov-Galerkin method. We only need to verify one condition on a reference element, condition (4.1). The further study of the condition (4.1) will be left to forthcoming research.

The iterated solution suggested in this paper is also fully discrete, see equation (2.13). Unlike a conventional iterated approximation in which we iterate an approximate solution of the corresponding numerical method by the exact integral operator $\mathcal{K}$, here we define our iterated solution using the approximate operator $\mathcal{K}_n$. This results in a discrete algorithm in which we do not need additional discretization. This method also covers the iterated Petrov-Galerkin method presented in [**7**], since we are allowed to choose $\mathcal{K}_n = \mathcal{K}$. Super-convergence for this iterated method is fulfilled.

We organize this paper as follows. In Section 2 we develop the discrete Petrov-Galerkin method and its iterated scheme. We first describe the method in an "abstract" sense and then we use three piecewise polynomial spaces to construct a concrete method. These three spaces are used for different purposes. They are used to approximate the solution space, the image space and the inner product, respectively. In Section 3, in preparing for the analysis of the methods proposed in Section 2, we extend the theory of collectively compact operators to a somewhat more general setting which covers the discrete Petrov-Galerkin method and its iterated scheme as special cases. We set up a theoretical framework which is not only convenient for the analysis of the discrete Petrov-Galerkin method, but is also suitable for analysis of other discrete numerical methods. In our development we benefit from various ideas shown in [**3, 5**]. In Section 4 we apply the theory established in Section 3 to the discrete Petrov-Galerkin method to obtain a convergence theorem. In Section 5 we prove a superconvergence result of the iterated discrete Petrov-Galerkin method. Section 6 is devoted to a presentation of numerical experiments, where we illustrate with two numerical examples the theoretical estimates obtained in the previous sections.

2. Discrete Petrov-Galerkin method. This section is devoted to a description of discrete Petrov-Galerkin methods for Fredholm integral equations of the second kind with weakly singular kernels. For this purpose, we consider the equation

$$(\mathcal{I} - \mathcal{K})u = f, \tag{2.1}$$

where $\mathcal{K} : L^\infty(D) \to C(D)$ is a compact linear integral operator defined by

$$(\mathcal{K}u)(s) = \int_D k(s,t)u(t)\,dt, \quad s \in D, \tag{2.2}$$

$D \subset \mathbf{R}^d$ is a bounded closed domain and $k(s,t)$ is a function defined on $D \times D$ which is allowed to have weak singularities. We assume that 1 is not an eigenvalue of the operator $\mathcal{K}$ to guarantee the existence of a unique solution $u \in C(D)$. Some additional specific assumptions will be imposed later in this section.

We first recall the Petrov-Galerkin method for equation (2.1) following [**7**]. In this description, we let X be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of finite dimensional subspaces of X such that

$$\dim X_n = \dim Y_n = d_n,$$

$$X_n = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{d_n}\},$$

and

$$Y_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_{d_n}\}.$$

We assume that $\{X_n, Y_n\}$ is a *regular pair* in the sense that there is a linear operator $\Pi_n : X_n \to Y_n$ with $\Pi_n X_n = Y_n$ and satisfying the conditions $\|x\| \leq C\langle x, \Pi_n x\rangle^{1/2}$ and $\|\Pi_n x\| \leq C_2\|x\|$, for all $x \in X_n$, where C_1 and C_2 are positive constants independent of n. The notion of regular pairs is closely related to the *generalized best approximation* which we review below. Given $x \in X$, an element $\mathcal{P}_n x \in X_n$ is called a generalized best approximation from X_n to x with respect to Y_n if it satisfies the equation

$$\langle x - \mathcal{P}_n x, y\rangle = 0, \quad \text{for all } y \in Y_n.$$

It is known that the necessary and sufficient condition for a generalized best approximation from X_n to $x \in X$ with respect to Y_n to exist uniquely is $Y_n \cap X_n^{\perp} = \{0\}$. If this condition holds, then $\mathcal{P}_n$ is a projection and $\{X_n, Y_n\}$ forms a regular pair if and only if $\mathcal{P}_n$ is uniformly bounded [**7**]. The Petrov-Galerkin method for solving equation (2.1) is a numerical scheme to find a function

$$u_n(s) := \sum_{j=1}^{d_n} \alpha_j \phi_j(s) \in X_n$$

such that

$$\langle(\mathcal{I} - \mathcal{K})u_n, y\rangle = \langle f, y\rangle, \quad \text{for all } y \in Y_n, \tag{2.3}$$

or, equivalently,

$$\sum_{j=1}^{d_n} \alpha_j[\langle\phi_j, \psi_i\rangle - \langle\mathcal{K}\phi_j, \psi_i\rangle] = \langle f, \psi_i\rangle, \quad i = 1, 2, \ldots, d_n. \tag{2.4}$$

Using the generalized best approximation $\mathcal{P}_n : X \to X_n$, we write equation (2.3) in the operator form as

$$(\mathcal{I} - \mathcal{P}_n\mathcal{K})u_n = \mathcal{P}_n f. \tag{2.5}$$

It is also proved in [**7**] that if $\{X_n, Y_n\}$ is a regular pair, then for a sufficiently large n, equation (2.5) has a unique solution $u_n \in X_n$ which satisfies the estimate

$$\|u_n - u\| \leq C \inf_{x \in X_n} \|u - x\|.$$

Solving equation (2.5) requires solving the linear system (2.4). Of course, the entries of the coefficient matrix of (2.4) involve the integrals $\langle \mathcal{K}\phi_j, \psi_i \rangle$ which are normally evaluated by a numerical quadrature formula. Roughly speaking, the discrete Petrov-Galerkin method is the scheme (2.4) with the integrals appearing in the method computed by quadrature formulas. However, we will develop our discrete Petrov-Galerkin method independent of the Petrov-Galerkin method (2.5). In other words, we do not assume that the Petrov-Galerkin method (2.5) has been previously constructed, to avoid the 'regular pair' assumption which is crucial for the solvability and convergence of the Petrov-Galerkin. We will take a one-step approach to fully discretize equation (2.1) directly. We will first describe the method in "abstract" terms without specifying the bases and the concrete quadrature formulas. Later we will specialize them using the piecewise polynomial spaces. The only assumption that we have to impose later to guarantee the solvability and convergence of the resulting concrete method is condition (4.1). It is a *local* condition on the reference element only and, thus, it is easy to verify it.

In our description we will use function values $f(t)$ at given points $t \in D$ for an L^∞ function f. We now follow [**4**] to define them precisely. Let $\tilde{C}(D)$ denote the subspace of $L^\infty(D)$ which consists of functions each of which is equal to an element in $C(D)$ almost everywhere. The point evaluation functional δ_t on the space $\tilde{C}(D)$ is defined by

$$\delta_t(f) := f(t), \quad t \in D, \quad f \in \tilde{C}(D),$$

where f on the righthand side is chosen to be the representative function $f \in \tilde{C}(D)$ which is continuous. By the Hahn-Banach theorem, the point evaluation function δ_t can be extended from $\tilde{C}(D)$ to the whole $L^\infty(D)$ in such a way that the norm is preserved. We will use d_t to denote such an extension and define

$$f(t) := d_t(f) \quad \text{for} \quad f \in L^\infty(D).$$

We remark that the extension is not unique but it is usually immaterial. What is important is that it exists and preserves many of the properties naturally with point evaluation. For example, at a point of continuity of f, the extended point evaluation is uniquely defined and has the natural value and, moreover, the point value is continuous. The reader is referred to [**4**] for more details in this extension.

We now return to our description of the discrete Petrov-Galerkin method. As in the description of the (continuous) Petrov-Galerkin method, we choose two subspaces $X_n = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_{d_n}\}$ and $Y_n = \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_{d_n}\}$ of the space $L^\infty(D)$ such that $\dim X_n = \dim Y_n = d_n$. We choose m_n points $t_i \in D$ and a set of weights $w_{1,i}$, $i = 1, 2, \dots, m_n$, and for $x, y \in L^\infty(D)$, we define the *discrete inner product*

$$(x, y)_n := \sum_{i=1}^{m_n} w_{1,i} x(t_i) y(t_i), \quad x, y \in L^\infty(D), \tag{2.6}$$

which will be used to approximate the inner product

$$\langle x, y \rangle := \int_D x(t) y(t)\, dt.$$

We also introduce a set of weight functions $w_{2,i}$, $i = 1, 2, \dots, m_n$, and define discrete operators by

$$(\mathcal{K}_n u)(s) := \sum_{i=1}^{m_n} w_{2,i}(s) u(t_i), \quad u \in L^\infty(D). \tag{2.7}$$

These operators will be used to approximate the operator $\mathcal{K}$. With the notation as above, the discrete Petrov-Galerkin method for equation (2.1) is a numerical scheme to find

$$u_n(s) := \sum_{j=1}^{d_n} \alpha_{n,j} \phi_j(s) \tag{2.8}$$

such that

$$((\mathcal{I} - \mathcal{K}_n) u_n, y)_n = (f, y)_n, \quad \text{for all } y \in Y_n. \tag{2.9}$$

In terms of basis functions, equation (2.9) is written as

(2.10)
$$\sum_{j=1}^{d_n} \alpha_{n,j} \left[\sum_{\ell=1}^{m_n} w_{1,\ell} \phi_j(t_\ell) \psi_i(t_\ell) - \sum_{\ell=1}^{m_n} w_{1,\ell} \sum_{m=1}^{m_n} w_{2,m}(t_\ell) \phi_j(t_m) \psi_i(t_\ell) \right] = \sum_{\ell=12}^{m_n} w_{1,\ell} f(t_\ell) \psi_i(t_\ell), \quad i = 1, 2, \dots, d_n.$$

Upon solving the linear system (2.10), we obtain d_n values $\alpha_{n,j}$. Substituting them into (2.8) yields an approximation to the solution u of equation (2.1). Equation (2.9) can also be written in the operator form by a *discrete generalized best approximation* $\mathcal{Q}_n$, which we define next. Let $\mathcal{Q}_n : X \to X_n$ be defined by

$$(\mathcal{Q}_n x, y)_n = (x, y)_n \quad \text{for all } y \in Y_n. \tag{2.11}$$

If $\mathcal{Q}_n x$ is uniquely defined for every $x \in X$, equation (2.9) can be written in the form

$$(\mathcal{I} - \mathcal{Q}_n \mathcal{K}_n) u_n = \mathcal{Q}_n f. \tag{2.12}$$

We postpone a discussion of the unique existence of $\mathcal{Q}_n x$ until later.

The iterated Petrov-Galerkin method has been shown in [**7**] to have a superconvergence property, where the additional order of convergence gained from an iteration is attributed by approximation of the kernel from the test space. The convergence order of the iterated Petrov-Galerkin method is equal to the approximation order of space X_n plus the approximation order of space Y_n. It is of interest to study the superconvergence of the *iterated discrete Petrov-Galerkin method*, which we define by

$$u_n' = f + \mathcal{K}_n u_n. \tag{2.13}$$

Equation (2.13) is a fully discrete algorithm, which can be implemented very easily, involving only multiplications and additions. It can be shown that u_n' satisfies the operator equation

$$(\mathcal{I} - \mathcal{K}_n \mathcal{Q}_n) u_n' = f. \tag{2.14}$$

This form of equations allows us to treat the iterated discrete Petrov-Galerkin method as an operator equation whose analysis is covered by the theory developed in the next section.

Up to now, the discrete Petrov-Galerkin method is described in abstract terms without specifying the spaces X_n and Y_n. In the remainder of this section, we specialize the discrete Petrov-Galerkin method by specifying the spaces X_n and Y_n and defining operators $\mathcal{Q}_n$ and $\mathcal{K}_n$ in terms of piecewise polynomials. We assume that D is a polyhedral region and construct a triangulation for D by dividing it into N_n simplices $T_n := \{E_{n,1}, \dots, E_{n,N_n}\}$ such that

$$h := \max\{\operatorname{diam} E_{n,i} : i = 1, 2, \dots, N_n\} \longrightarrow 0, \quad \text{as } n \to \infty, \tag{2.15}$$

$$D = \bigcup_{i=1}^{N_n} E_{n,i},$$

and

$$\operatorname{meas}(E_{n,i} \cap E_{n,j}) = 0, \quad i \neq j.$$

When the dependence of the simplex $E_{n,i}$ on n is well understood, we drop the first index n in the notation and simply write it as E_i. For each positive integer n, the set T_n forms a partition for the domain D. We also require that the triangulation be *regular* in the sense that any vertex of a simplex in T_n is not in the interior of an edge of another simplex in the set. It is well known that, for each simplex, there exists a unique bijective affine mapping which maps the simplex onto a unit simplex E called a *reference element*.

Let F_i, $i = 1, 2, \dots, N_n$, denote the affine mappings that map the simplices E_i bijectively onto the reference element E. Then the affine mappings F_i have the form

$$F_i(t) = B_i t + b_i, \quad t \in E, \tag{2.16}$$

where B_i is a $d \times d$ invertible matrix and b_i a vector in $\mathbf{R}^d$, and they satisfy

$$E_i = F_i(E).$$

On the reference element E, we choose two piecewise polynomial spaces $S_{1,k_1}(E)$ and $S_{2,k_2}(E)$ of total degree $k_1 - 1$ and $k_2 - 1$, respectively, such that

$$\dim S_{1,k_1}(E) = \dim S_{2,k_2}(E) = \mu.$$

The partitions Δ_1 and Δ_2 of E associated, respectively, with $S_{1,k_1}(E)$ and $S_{2,k_2}(E)$ may be different; they are arranged according to the integers k_1, k_2 and d. Assume that the numbers of the sub-triangles contained in the partitions Δ_1 and Δ_2 are denoted by ν_1 and ν_2. We have to choose these pair of integers k_1, ν_1 and κ_2, ν_2 such that

$$\binom{k_1 - 1 + d}{d} \nu_1 = \binom{k_2 - 1 + d}{d} \nu_2 = \mu,$$

because the dimension of the space of polynomials of total degree k is $\binom{k+d}{d}$. We will not provide a detailed discussion on how the partitions Δ_1 and Δ_2 are constructed. Instead, we assume that we have chosen bases for these two spaces so that

$$S_{1,k_1}(E) := \operatorname{span}\{\phi_1, \phi_1, \dots, \phi_\mu\},$$

and

$$S_{2,k_2}(E) := \operatorname{span}\{\psi_1, \psi_2, \dots, \psi_\mu\}.$$

We next map these piecewise polynomial spaces on E to each simplex E_i by letting

$$\phi_{i,j}(t) := \begin{cases} \phi_j \circ F_i^{-1}(t) & t \in E_i, \\ 0 & t \notin E_i, \end{cases}$$

and

$$\psi_{i,j}(t) := \begin{cases} \psi_j \circ F_i^{-1}(t) & t \in E_i, \\ 0 & t \notin E_i, \end{cases}$$

for $i = 1, 2, \dots, N_n$ and $j = 1, 2, \dots, \mu$. Using these functions as bases, we define the trial space and the test space, respectively, by

$$X_n = \operatorname{span}\{\phi_{i,j} : i = 1, 2, \dots, N_n, j = 1, 2, \dots, \mu\},$$

and

$$Y_n = \operatorname{span}\{\psi_{i,j} : i = 1, 2, \dots, N_n, j = 1, 2, \dots, \mu\}.$$

It follows from (2.15) that

$$C(D) \subseteq \overline{\bigcup X_n}$$

and

$$C(D) \subseteq \overline{\bigcup Y_n}.$$

Moreover, we have that, if $x \in W_\infty^{k_1}(D)$, then there exists a constant $C > 0$ independent of n such that

$$\inf_{\phi \in X_n} \|x - \phi\| \leq Ch^{k_1},$$

and if $x \in W_\infty^{k_2}(D)$, then

$$\inf_{\phi \in Y_n} \|x - \phi\| \leq Ch^{k_2}.$$

However, the space $\tilde{X} := \overline{\cup X_n}$ does not equal $L^\infty(D)$; it is a proper subspace of $L^\infty(D)$ because the space $L^\infty(D)$ is not separable. Due to this fact, the existing theory of collectively compact operators, cf. [**1**], does not apply directly to this setting. Some modifications of the theory are required.

We next specialize the definition of the discrete inner product (2.6) and describe a concrete construction of the approximate operators $\mathcal{K}_n$. To this end, we introduce a third piecewise polynomial space $S_{3,k_3}(E)$ of total degree $k_3 - 1$ on E. We divide the reference element E into ν_3 subtriangles

$$\Delta_3 := \{e_i : i = 1, 2, \ldots, \nu_3\}$$

and also assume that the triangulation Δ_3 is regular. On each of the triangles, e_i, we choose $m := \binom{k_3-1+d}{d}$ points $\tau_{i,j},\ j = 1, 2, \ldots, m$, such that they admit a unique Lagrange interpolating polynomial of total degree $k_3 - 1$ on e_i. For multivariate Lagrange interpolation by polynomials of total degree, see [**8**] and the references cited therein. Let $p_{i,j}$ be the polynomial of total degree $k_3 - 1$ on e_i satisfying the interpolation conditions

$$p_{i,j}(\tau_{i',j'}) = \delta_{i,i'}\delta_{j,j'}, \quad i, i' = 1, 2, \ldots, \nu_3,\ j, j' = 1, 2, \ldots, m,$$

where

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

We assemble these polynomials to form a basis for the space $S_{3,k_3}(E)$ by letting

$$\zeta_{(i-1)m+j}(t) := \begin{cases} p_{i,j}(t) & t \in e_i, \\ 0 & t \notin e_i, \end{cases}, \quad i = 1, 2, \dots, \nu_3, \ j = 1, 2, \dots, m.$$

Set $\gamma = m\nu_3$, which is equal to the dimension of $S_{3,k_3}(E)$ and

$$t_{(i-1)m+j} = \tau_{i,j}, \quad i = 1, 2, \dots, \nu_3, \ j = 1, 2, \dots, m.$$

Then $\zeta_i \in S_{3,k_3}(E)$ and satisfy the interpolation conditions

$$\zeta_i(t_j) = \delta_{i,j}, \quad i, j = 1, 2, \dots, \gamma.$$

This set of functions forms a basis for the space $S_{3,k_3}(E)$. It can be used to introduce a piecewise polynomial space on D by mapping the basis ζ_j, $j = 1, 2, \dots, \gamma$ for $S_{3,k_3}(E)$ from E into each E_i. Specifically, we define

$$\zeta_{i,j}(t) := \begin{cases} \zeta_j \circ F_i^{-1}(t) & t \in E_i, \\ 0 & t \notin E_i, \end{cases}$$

where F_i is the affine map defined by (2.16) and let

$$Z_n := \operatorname{span}\{\zeta_{i,j} : i = 1, 2, \dots, N_n, \ j = 1, 2, \dots, \gamma\}.$$

Hence, Z_n is a piecewise polynomial space of dimension γN_n. For each i, we define

$$t_{i,j} := F_i(t_j) = B_i t_j + b_i,$$

where B_i and b_i are the matrix and vector appearing in the definition of the affine map F_i. Furthermore, we define the linear projection $\mathcal{Z}_n : X \to Z_n$ by

$$\mathcal{Z}_n g = \sum_{i=1}^{N_n} \sum_{j=1}^{\gamma} d_{t_{i,j}}(g) \zeta_{i,j},$$

where d_t is the extension of the point evaluation functional δ_t satisfying $\|d_t\| = 1$, which was discussed earlier and satisfies the condition

$$d_{t_{i,j}}(\zeta_{i',j'}) = \delta_{i,i'} \delta_{j,j'},$$
$$i, i' = 1, 2, \dots, N_n, \quad j, j' = 1, 2, \dots, \gamma.$$

Moreover, we have

$$\|\mathcal{Z}_n\| = \text{ess}\sup_{t\in D}\sum_{i=1}^{N_n}\sum_{j=1}^{\gamma}|\zeta_{i,j}(t)| = \text{ess}\sup_{t\in E}\sum_{j=1}^{\gamma}|\zeta_j(t)|.$$

That is, $\|\mathcal{Z}_n\|$ is uniformly bounded independent of n. It follows from the uniform boundedness of $\|\mathcal{Z}_n\|$ that, for any $y \in W_\infty^{k_3}(D)$, the following estimate holds

$$\|y - \mathcal{Z}_n y\| \le C \inf_{\phi\in Z_n}\|y-\phi\| \le Ch^{k_3}. \tag{2.17}$$

Using the projection $\mathcal{Z}_n$ defined above, we have a quadrature formula

$$\int_D g(t)\,dt = \sum_{i=1}^{N_n}\sum_{j=1}^{\gamma} w_{i,j} d_{t_{i,j}}(g) + \mathcal{O}(h^{k_3}),$$

where

$$w_{i,j} := \int_D \zeta_{i,j}(t)\,dt.$$

If we set

$$w_i := \int_E \zeta_i(t)\,dt, \quad i = 1,2,\dots,\gamma,$$

then we have

$$w_{i,j} = \int_{E_i}\zeta_j(F_i^{-1}(t))\,dt = \det(B_i)\int_E \zeta_j(t)\,dt = \det(B_i) w_j.$$

Without loss of generality, we assume that

$$\det(B_i) > 0, \quad i = 1,2,\dots,\gamma.$$

Employing this formula, we introduce the following discrete inner product

$$(x,y)_n = \sum_{i=1}^{N_n}\sum_{\ell=1}^{\gamma} w_{i,\ell}\, x(t_{i,\ell}) y(t_{i,\ell}). \tag{2.18}$$

Formula (2.18) is a concrete form for (2.6). When $x, y \in W_\infty^{k_3}(D)$, we have the error estimate

$$|\langle x, y\rangle - (x, y)_n| \leq Ch^{k_3}.$$

With this specific definition of the spaces X_n, Y_n and the discrete inner product, we obtain a construction of the operators $\mathcal{Q}_n$ by using equation (2.11).

Finally, to describe a concrete construction of the approximate operators $\mathcal{K}_n$, we impose a few additional assumptions on the kernel $k(s,t)$ of the integral operator $\mathcal{K}$. Roughly speaking, we assume that $k(s,t)$ is a product of two kernels: one of them is continuous but perhaps involves a complicated function, and the other has a simple form but has a singularity. In particular, we let

$$k(s,t) = k_1(s,t)k_2(s,t),$$

where k_1 is continuous on $D \times D$ and k_2 has a singularity and satisfies the conditions

$$k_2(s,\cdot) \in L^1(D), \quad s \in D, \quad \sup_{s\in D} \int_D |k_2(s,t)|\, dt < +\infty, \tag{2.19}$$

$$\|k_2(s,\cdot) - k_2(s',\cdot)\|_1 \longrightarrow 0, \quad \text{as } s' \longrightarrow s. \tag{2.20}$$

Moreover, we assume that the integration of the product of $k_2(s,t)$ and a polynomial $p(t)$ with respect to the variable t can be evaluated exactly. Many integral operators $\mathcal{K}$ that appear in practical applications are of this type.

Using the linear projection $\mathcal{Z}_n$, we define $\mathcal{K}_n : X \to X$ by

$$(\mathcal{K}_n x)(s) = \int_D \mathcal{Z}_n(k_1(s,t)x(t))k_2(s,t)\, dt,$$

which approximates the operator $\mathcal{K}$. For $u_n \in X_n$, we have

$$(\mathcal{K}_n u_n)(s) = \sum_{i=1}^{N_n}\sum_{j=1}^{\gamma} w_{i,j}(s)k_1(s,t_{i,j})u_n(t_{i,j}),$$

where

$$w_{i,j}(s) = \int_{E_i} \zeta_{i,j}(t) k_2(s,t)\, dt.$$

This concrete construction of the trial space X_n, the test space Y_n, and operators $\mathcal{Q}_n, \mathcal{K}_n$ yields a specific discrete Petrov-Galerkin method which is described by equation (2.12). This is the method which we will analyze in Section 4.

3. An abstract framework. To analyze the discrete Petrov-Galerkin method described in the last section, we develop an abstract framework in this section. This framework covers the discrete Petrov-Galerkin method and many other cases as well. It is well known that the theory of collectively compact operators presented in [**1**] or [**2**] provides us with an abstract setting for analysis of many numerical schemes for integral equations of compact operators. The known theory requires that the approximate finite dimensional spaces X_n be dense in the original function space, X. However, as indicated in the last section, in the current method, this is not the case. Therefore, we need to extend the theory of collectively compact operators to a somewhat more general setting. This will be done in this section.

Let X be a Banach space with norm $\|\cdot\|$ and V its subspace. Assume $\mathcal{K} : X \to V$ is a compact linear operator. We consider the Fredholm equation of the second kind

$$u - \mathcal{K}u = f. \tag{3.1}$$

We assume that (3.1) is uniquely solvable in V, or X, for all $f \in V$, or X. This is equivalent to assuming that 1 is not an eigenvalue of the operator $\mathcal{K}$.

We first describe the setting for our discrete approximate scheme. For this purpose, we let $\{X_n\}$ denote a sequence of finite-dimensional subspaces of X, which will be used to approximate the solution u of equation (3.1). We let

$$\tilde{X} := \overline{\bigcup_{n=1}^{\infty} X_n}.$$

We also assume that

$$V \subseteq \tilde{X} \subseteq X, \quad \tilde{X} \neq X.$$

In our later application to the discrete Petrov-Galerkin method, the space X will be chosen as $L^\infty(D)$ and the subspace V will be chosen as $C(D)$, and X_n will be chosen as the piecewise polynomial space described in the last section. These spaces satisfy the above inclusion relation.

In order for our setting to cover the discrete approximate scheme, we require two sequences of approximate operators $\mathcal{K}_n$ and $\mathcal{Q}_n$. Specifically, we suppose that the operator $\mathcal{K}$ is approximated by operators $\mathcal{K}_n : X \to V$ and the identity operator $\mathcal{I}$ by operators $\mathcal{Q}_n : X \to X_n$. In other words, the operator $\mathcal{K}$ is approximated by the composite operator $\mathcal{Q}_n\mathcal{K}_n$. Using these two sequences of approximate operators, we define an approximation scheme for solving equation (3.1) by

$$(\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n)u_n = \mathcal{Q}_n f. \tag{3.2}$$

This approximate scheme includes the discrete Galerkin method, the discrete Petrov-Galerkin method and the discrete collocation method as special cases, and it also covers the (nondiscrete) Galerkin method, collocation method, and Petrov-Galerkin method and quadrature method as well. To see this, we define $\mathcal{K}_n$ by a given quadrature formula or product integration formula. When $\mathcal{Q}_n$ is chosen to be the orthogonal projection, the generalized best approximation projection introduced in [**7**] and the interpolation projection, respectively, equation (3.2) gives the discrete Galerkin method, the discrete Petrov-Galerkin method and the discrete collocation method. When $\mathcal{Q}_n$ is chosen to be the identity operator, equation (3.2) corresponds to the quadrature method. On the other hand, if $\mathcal{K}_n = \mathcal{K}$, then equation (3.2) defines the Galerkin method, the Petrov-Galerkin method and the collocation method when the operator $\mathcal{Q}_n$ is chosen to be the orthogonal projection, the generalized best approximation projection and the interpolation projection, respectively. The approximate scheme (3.2) gives us a unified framework for studies of different methods.

We are also interested in the iterated approximation associated with the approximate solution u_n of (3.2) defined by

$$u_n' := f + \mathcal{K}_n u_n. \tag{3.3}$$

It can be shown that the iterated approximation u_n' satisfies the following new operator equation

$$(\mathcal{I} - \mathcal{K}_n\mathcal{Q}_n)u_n' = f. \tag{3.4}$$

We will prove that u_n' approximates u faster than u_n does under certain conditions.

In this section we will establish a theoretical framework for the analysis of the approximate scheme (3.2) and its iterated approximations. To do this, we are required to analyze the existence and uniform boundedness of the inverse operators

$$\mathcal{A}_n^{-1} := (\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n)^{-1} \quad \text{and} \quad \tilde{\mathcal{A}}_n^{-1} := (\mathcal{I} - \mathcal{K}_n\mathcal{Q}_n)^{-1}.$$

We begin with a set of assumptions on the operators $\mathcal{K}_n$ and $\mathcal{Q}_n$. We assume that $\mathcal{K}_n : X \to V$ and $\mathcal{Q}_n : X \to X_n$ are bounded linear operators satisfying the following conditions.

(H1) The set of operators $\{\mathcal{K}_n : n = 1, 2, \dots\}$ is collectively compact, i.e., the set

$$\hat{B} := \bigcup_n \mathcal{K}_n(B)$$

is relatively compact whenever $B \subset X$ is bounded.

(H2) The approximate operators $\mathcal{K}_n$ converge pointwise to $\mathcal{K}$ on the set $\tilde{X}$, denoted by $\mathcal{K}_n \to \mathcal{K}$ on $\tilde{X}$, i.e., for each $x \in \tilde{X}$ the following holds

$$\|\mathcal{K}_n x - \mathcal{K}x\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

(H3) The set of operators $\{\mathcal{Q}_n : n = 1, 2, \dots\}$ is uniformly bounded, i.e., there is a constant $q > 0$ such that

$$\|\mathcal{Q}_n\| \le q \quad \text{for all } n.$$

(H4) The approximate operators $\mathcal{Q}_n$ converge pointwise to $\mathcal{I}$ on V, i.e., for each $x \in V$, the following holds

$$\|\mathcal{Q}_n x - x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

With the set of assumptions, we will extend the theory of the collectively compact operators presented in [**1**] to the setting which is needed for a study of the operators $\mathcal{A}_n$ and $\tilde{A}_n$ and their inverses. As a first step, a slight modification of the proof for Proposition 1.7 in [**1**] yields the following result.

Lemma 3.1. *Let X be a Banach space and $S \subset X$ a relatively compact set. Assume that $\mathcal{T}, \mathcal{T}_n$ are bounded linear operators from X to X satisfying*

$$\|\mathcal{T}_n\| \leq C \quad \text{for all } n,$$

and, for each $x \in S$,

$$\|\mathcal{T}_n x - \mathcal{T} x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty,$$

where C is a constant independent of n. Then $\|\mathcal{T}_n x - \mathcal{T} x\| \to 0$ uniformly for all $x \in S$.

This result generalizes Proposition 1.7 of [**1**] in the sense that Lemma 3.1 only requires the pointwise convergence of $\mathcal{T}_n$ to $\mathcal{T}$ on a relatively compact set $S \subseteq X$. The next result concerns the convergence of several approximate operators.

Lemma 3.2. *Assume that conditions* (H1)–(H4) *hold. Then*

(i) $\|(\mathcal{Q}_n - I)\mathcal{K}_n\| \to 0$ *as* $n \to \infty$,

(ii) $\|(\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n\mathcal{K}_n\| \to 0$ *as* $n \to \infty$,

(iii) $\|(\mathcal{K}_n\mathcal{Q}_n - \mathcal{K})\mathcal{K}_n\mathcal{Q}_n\| \to 0$ *as* $n \to \infty$.

Proof. (i) Let B denote the closed unit ball in X, that is,

$$B := \{x \in X : \|x\| \leq 1\}.$$

We also let

$$A := \{\mathcal{K}_n x : x \in B, n = 1, 2, \dots\}.$$

Because of condition (H1), we conclude that A is a relatively compact set in V. Using the hypotheses (H3) and (H4), we see that the conditions of Lemma 3.1 are satisfied. Hence, $\|\mathcal{Q}_n x - x\| \to 0$ uniformly for $x \in A$. It follows that

$$\begin{aligned}\|(\mathcal{Q}_n - \mathcal{I})\mathcal{K}_n\| &= \sup_{x \in B} \|(\mathcal{Q}_n - \mathcal{I})\mathcal{K}_n x\| \\ &= \sup_{x \in A} \|(\mathcal{Q}_n - \mathcal{I}) x\| \longrightarrow 0.\end{aligned}$$

(ii) For a fixed $x \in V$, it follows from (H4) that $\{\mathcal{Q}_n x : n = 1, 2, \dots\}$ is a relatively compact set in X. By Lemma 3.1 and the hypotheses (H1)–(H2), we conclude that

$$\|(\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Hence we have

$$\begin{aligned} \|(\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n \mathcal{K}_n\| &= \sup_{y \in B} \|(\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n \mathcal{K}_n y\| \\ &= \sup_{x \in A} \|(\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n x\|. \end{aligned}$$

Using Lemma 3.1 with $\mathcal{T} = 0$, $\mathcal{T}_n = (\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n$ and $S = A$, we conclude the validity of this result.

(iii) It can be shown by the hypotheses (H1) and (H3) that

$$A' := \{\mathcal{K}_n \mathcal{Q}_n x : x \in B, n = 1, 2, \dots\}$$

is a relatively compact set in V. On the other hand, note that

$$\mathcal{K}_n \mathcal{Q}_n - \mathcal{K} = (\mathcal{K}_n \mathcal{Q}_n - \mathcal{K}\mathcal{Q}_n) + (\mathcal{K}\mathcal{Q}_n - \mathcal{K}).$$

We then conclude from statement (ii) and (H4) that, for any $x \in V$,

$$\|(\mathcal{K}_n \mathcal{Q}_n - \mathcal{K})x\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Thus, the statement (iii) follows immediately from the above equation and the relative compactness of the set A'. $\square$

We next study the existence of the inverse operator of $\mathcal{A}_n$ and $\tilde{A}_n$. For this purpose, we recall a known result about the existence and the boundedness of inverse operators, e.g., [**1**, Proposition 1.2].

Lemma 3.3. *Let X be a Banach space and $\mathcal{B}, \mathcal{T}$ bounded linear operators from X to X. If*

$$\mathcal{B}\mathcal{T} = \mathcal{I} - \mathcal{A}, \quad \|\mathcal{A}\| < 1,$$

then $\mathcal{T}^{-1}$ *exists as an operator defined on* $\mathcal{T}(X)$, *and*

$$\|\mathcal{T}^{-1}\| \leq \frac{\|\mathcal{B}\|}{1 - \|\mathcal{A}\|}.$$

We now prove the main result of this section.

Theorem 3.4. *Assume that conditions* (H1)–(H4) *hold. Then there exists* $N_0 > 0$ *such that, for all* $n > N_0$, *the inverse* $(\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n)^{-1}$ *and* $(\mathcal{I} - \mathcal{K}_n\mathcal{Q}_n)^{-1}$ *exist as linear operators defined on* X, *and there exists a constant* C *independent of* n *such that*

$$\|(\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n)^{-1}\| \leq C \quad \textit{and} \quad \|(\mathcal{I} - \mathcal{K}_n\mathcal{Q}_n)^{-1}\| \leq C.$$

Moreover, let u, u_n *and* u_n' *be the solution of equations* (3.1), (3.2) *and* (3.4), *respectively. Then we have the estimates*

$$\|u - u_n\| \leq C(\|u - \mathcal{Q}_n u\| + q\|\mathcal{K}u - \mathcal{K}_n u\|), \tag{3.5}$$

and

$$\|u - u_n'\| \leq C(\|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)u\| + \|(\mathcal{K} - \mathcal{K}_n)\mathcal{Q}_n u\|). \tag{3.6}$$

Proof. We first note that a straightforward computation leads to the identities

$$[\mathcal{I} + (\mathcal{I} - \mathcal{K})^{-1}\mathcal{K}_n](\mathcal{I} - \mathcal{Q}_n\mathcal{K}_n) = \mathcal{I} - (\mathcal{I} - \mathcal{K})^{-1}[(\mathcal{Q}_n - \mathcal{I})\mathcal{K}_n + (\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n\mathcal{K}_n],$$

and

$$[\mathcal{I} + (\mathcal{I} - \mathcal{K})^{-1}\mathcal{K}_n\mathcal{Q}_n](\mathcal{I} - \mathcal{K}_n\mathcal{Q}_n) = \mathcal{I} - (\mathcal{I} - \mathcal{K})^{-1}(\mathcal{K}_n\mathcal{Q}_n - \mathcal{K})\mathcal{K}_n\mathcal{Q}_n.$$

By Lemma 3.2 there exists $N_0 > 0$ such that, for all $n > N_0$,

$$\Delta_1 := \|(\mathcal{I} - \mathcal{K})^{-1}[(\mathcal{Q}_n - \mathcal{I})\mathcal{K}_n + (\mathcal{K}_n - \mathcal{K})\mathcal{Q}_n\mathcal{K}_n]\| \leq \frac{1}{2}$$

and

$$\Delta_2 := \|(\mathcal{I}-\mathcal{K})^{-1}(\mathcal{K}_n\mathcal{Q}_n-\mathcal{K})\mathcal{K}_n\mathcal{Q}_n\| \leq \frac{1}{2}.$$

It follows from Lemma 3.3 that the inverse operators $(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)^{-1}$ and $(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)^{-1}$ exist, and the following holds

$$\|(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)^{-1}\| \leq \frac{1}{1-\Delta_1}(1+\|(\mathcal{I}-\mathcal{K})^{-1}\|\,\|\mathcal{K}_n\|),$$

and

$$\|(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)^{-1}\| \leq \frac{1}{1-\Delta_2}(1+q\|(\mathcal{I}-\mathcal{K})^{-1}\|\,\|\mathcal{K}_n\|).$$

Since the set of operators $\{\mathcal{K}_n\}$ is collectively compact, the norms $\|\mathcal{K}_n\|$ are uniformly bounded. Thus $(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)^{-1}$ and $(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)^{-1})$ are uniformly bunded. Moreover, it is easily seen that $\mathcal{Q}_n\mathcal{K}_n$ and $\mathcal{K}_n\mathcal{Q}_n$ are compact. Thus the Fredholm theory allows us to conclude that $(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)^{-1}$ and $(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)^{-1}$ are defined on X; that is, the equations (3.2) and (3.4) have unique solutions for every $f \in X$.

It remains to prove the second statement. To this end, we note that from equations (3.1), (3.2) and (3.4), we have

$$(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)u_n = \mathcal{Q}_n(\mathcal{I}-\mathcal{K})u,$$

and

$$(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)u_n' = (\mathcal{I}-\mathcal{K})u.$$

Using these equations, we obtain that

$$(\mathcal{I}-\mathcal{Q}_n\mathcal{K}_n)(u-u_n) = (u-\mathcal{Q}_n u)+\mathcal{Q}_n(\mathcal{K}u-\mathcal{K}_n u),$$

and

$$\begin{aligned}(\mathcal{I}-\mathcal{K}_n\mathcal{Q}_n)(u-u_n') &= \mathcal{K}u-\mathcal{K}_n\mathcal{Q}_n u\\ &= \mathcal{K}(u-\mathcal{Q}_n u)+(\mathcal{K}-\mathcal{K}_n)\mathcal{Q}_n u.\end{aligned}$$

By the first part of this theorem, we obtain the estimates. □

We remark that, from estimate (3.5), the convergence order of u_n depends completely on the orders of $\mathcal{Q}_n$ and $\mathcal{K}_n$ approximating the identity operator and $\mathcal{K}$, respectively. On the other hand, it is seen from (3.6) that if $\mathcal{K}_n$ approximates $\mathcal{K}$ in an order higher than the convergence order of $\mathcal{Q}_n$, superconvergence of the iterated solution will be exhibited since the significant error term is the first term which has superconvergence property due to iteration.

4. The analysis of discrete Petrov-Galerkin methods. In this section we follow the general theory developed in Section 3 to prove convergence results of the discrete Petrov-Galerkin method when piecewise polynomial approximation is used. Throughout the remaining part of this paper, we let $X = L^\infty(D)$, $V = C(D)$, X_n and Y_n be the piecewise polynomial spaces defined in Section 2, and $\tilde{X} = \overline{\cup_n X_n}$. Our main task in this section is to verify that the operators $\mathcal{Q}_n$ and $\mathcal{K}_n$ with the spaces X_n, Y_n defined in Section 2 by the piecewise polynomials satisfy the hypotheses (H1)–(H4), so that Theorem 3.4 can be applied. For this purpose, we define the necessary notation. Let

$$\Phi := [\phi_i(t_j)]_{\mu\times\gamma} \quad \text{and} \quad \Psi := [\psi_i(t_j)]_{\mu\times\gamma},$$

where $\{\phi_i\}$ and $\{\psi_i\}$ are the basis which we have chosen for the piecewise polynomial spaces $S_{1,k_1}(E)$ and $S_{2,k_2}(E)$, and $\{t_j\}$ are the interpolation points in the reference element E chosen in Section 2. Noting that w_i are the weights of the quadrature formula on the reference element developed in Section 2, we set

$$W := \operatorname{diag}(w_1, \dots, w_\gamma) \quad \text{and} \quad M := \Psi W \Phi^T.$$

The next proposition presents a necessary and sufficient condition for the discrete generalized best approximation to exist uniquely.

Proposition 4.1. *For each $x \in L^\infty(D)$, the discrete generalized best approximation $\mathcal{Q}_n x$ from X_n to x with respect to Y_n defined by* (2.11) *exists uniquely if and only if*

$$\det(M) \neq 0. \tag{4.1}$$

Under this condition, $\mathcal{Q}_n$ is a projection, i.e., $\mathcal{Q}_n^2 = \mathcal{Q}_n$.

Proof. Let $x \in L^\infty(D)$ be given. Showing that there is a unique $\mathcal{Q}_n x \in X_n$ satisfying equation (2.11) is equivalent to proving that the linear system

$$(4.2)\qquad \sum_{i=1}^{N_n}\sum_{j=1}^{\mu} c_{i,j}(\phi_{i,j},\psi_{i',j'})_n = (x,\psi_{i',j'})_n,$$
$$i' = 1,2,\dots,N_n,\ j' = 1,2,\dots,m,$$

has a unique solution $[c_{1,1},\dots,c_{1,\mu},\dots,c_{N_n,1},\dots,c_{N_n,\mu}]$. This is in turn equivalent to that the coefficient matrix $\tilde{M}$ of this system is nonsingular. It is easily seen that

$$\tilde{M} = \operatorname{diag}(\det(B_1)M,\dots,\det(B_{N_n})M).$$

Thus the first result of this proposition follows from the hypothesis (4.1).

It remains to show that $\mathcal{Q}_n$ is a projection. By definition, we have that for every $x \in L^\infty(D)$,

$$(\mathcal{Q}_n x, y)_n = (x,y)_n \quad \text{for all } y \in Y_n.$$

In particular, this equation holds when x is replaced by $\mathcal{Q}_n x$. That is,

$$(\mathcal{Q}_x^2, y)_n = (\mathcal{Q}_n x, y)_n \quad \text{for all } y \in Y_n.$$

It follows that, for each $x \in X$,

$$\mathcal{Q}_n^2 x = \mathcal{Q}_n x.$$

That is, $\mathcal{Q}_n$ is a projection. □

Condition (4.1) is a condition on the choice of the points $\{t_j\}$ on the reference element. They have to be selected in a careful manner so that they match with the choice of the bases $\{\phi_i\}$ and $\{\psi_i\}$. This condition has to be verified before a concrete construction of the projection $\mathcal{Q}_n$ is given. This is not a difficult task since the condition is on the reference element, it is independent of n and in practical applications the numbers μ and γ are not too large.

The next proposition gives two useful properties of the projection $\mathcal{Q}_n$.

Proposition 4.2. *Assume that condition* (4.1) *is satisfied. Let* $\mathcal{Q}_n$ *be defined by* (2.11) *with the spaces* X_n, Y_n *and the discrete inner product constructed in terms of the piecewise polynomials described in Section* 2. *Then*

(i) $\mathcal{Q}_n$ *is uniformly bounded, i.e., there exists a constant* $C > 0$ *such that* $\|\mathcal{Q}_n\| \le C$ *for all* n.

(ii) *There exists a constant* $C > 0$ *independent of* n *such that the estimate*

$$\|\mathcal{Q}_n x - x\|_\infty \le C \inf_{\phi \in X_n} \|x - \phi\|_\infty$$

holds for all $x \in L^\infty(D)$. *Thus, for each* $x \in C(D)$, $\|Q_n x - x\|_\infty \to 0$ *holds as* $n \to \infty$.

Proof. (i) For any $x \in L^\infty(D)$, we have the expression

$$\mathcal{Q}_n x = \sum_{i=1}^{N_n} \sum_{j=1}^{\mu} c_{i,j} \phi_{i,j}, \tag{4.3}$$

where the coefficients $c_{i,j}$ satisfy equation (4.2). It follows that

$$\begin{aligned} \|\mathcal{Q}_n x\|_\infty &\le \|\mathbf{c}\|_\infty \operatorname{ess\,sup}_{s \in D} \sum_{i=1}^{N_n} \sum_{j=1}^{\mu} |\phi_{i,j}(s)| \\ &= \|\mathbf{c}\|_\infty \max_{s \in E} \sum_{j=1}^{\mu} |\phi_j(s)|, \end{aligned} \tag{4.4}$$

where

$$\mathbf{c} := [c_{1,1}, \ldots, c_{1,\mu}, \ldots, c_{N_n,1}, \ldots, c_{N_n,\mu}]^T,$$

and the discrete norm of $\mathbf{c}$ is defined by $\|\mathbf{c}\|_\infty := \max_{i,j} |c_{i,j}|$. By definition, the vector $\mathbf{c}$ is dependent on n although we do not specify it in the notation. However, we will prove that $\|\mathbf{c}\|_\infty$ is in fact independent of n. To this end, we use system (4.2) and the hypothesis (4.1) to conclude that

$$\|\mathbf{c}\|_\infty = \|\tilde{M}^{-1} \mathbf{d}\|_\infty, \tag{4.5}$$

where

$$\mathbf{d} := [(x, \psi_{1,1})_n, \ldots, (x, \psi_{1,\mu})_n, \ldots, (x, \psi_{N_n,1})_n, \ldots, (x, \psi_{N_n,\mu})_n]^T,$$

and

$$\tilde{M}^{-1} = \operatorname{diag}(\det(B_1)^{-1}M^{-1}, \dots, \det(B_{N_n})^{-1}M^{-1}).$$

Let

$$\mathbf{d}_i = [(x, \psi_{i,1})_n, \dots, (x, \psi_{i,\mu})_n]^T \in \mathbf{R}^\mu.$$

Then, it follows from (4.5) that the following estimate of $\|\mathbf{c}\|_\infty$ holds in terms of blocks $\mathbf{d}_i$ and M^{-1}

$$\|\mathbf{c}\|_\infty \le \max_{1\le i\le N_n} \|\det(B_i)^{-1}M^{-1}\mathbf{d}_i\|_\infty. \tag{4.6}$$

This inequality reduces the estimating $\|\mathbf{c}\|_\infty$ to bounding each block $\mathbf{d}_i$. By the definition of the discrete inner product, we have an estimate for the norm of $\mathbf{d}_i$,

$$\begin{aligned} \|\mathbf{d}_i\|_\infty &\le \|x\|_\infty \max_{1\le j\le\mu} \sum_{\ell=1}^{\gamma} w_{i,\ell}|\psi_{i,j}(t_{i,\ell})| \\ &= \det(B_i)\|x\|_\infty \max_{1\le j\le\mu} \sum_{\ell=1}^{\gamma} w_\ell|\psi_j(t_\ell)|. \end{aligned} \tag{4.7}$$

From (4.4)–(4.7), we conclude that

$$\|\mathcal{Q}_n x\|_\infty \le C\|x\|_\infty \quad \text{for all } n,$$

where C is a constant independent of n with the value

$$C := \|M^{-1}\|_\infty \max_{s\in E} \sum_{j=1}^{\mu} |\phi_j(s)| \max_{1\le j\le\mu} \sum_{\ell=1}^{\gamma} w_\ell|\psi_j(t_\ell)|.$$

(ii) Let $\phi \in X_n$. Since $\mathcal{Q}_n$ is a projection, we have that, for each $x \in L_\infty$,

$$\|\mathcal{Q}_n x - x\|_\infty \le \|x-\phi\|_\infty + \|\mathcal{Q}_n\phi - \mathcal{Q}_n x\|_\infty \le (1+C)\|x-\phi\|_\infty.$$

Thus, we obtain the estimate

$$\|\mathcal{Q}_n x - x\|_\infty \le C \inf_{\phi\in X_n} \|x-\phi\|_\infty.$$

This estimate with the relation $C(D) \subseteq \overline{\cup_n X_n}$ implies that $\|\mathcal{Q}_n x - x\|_\infty \to 0$ as $n \to \infty$ for each $x \in C(D)$. □

In the next proposition we verify that the operators $\mathcal{K}_n$ defined in Section 2 by the piecewise polynomial approximation satisfy the hypotheses (H1)–(H2).

Proposition 4.3. *Suppose that $\mathcal{K}_n$ is defined as in Section 2 by the piecewise polynomial approximation. Then*

(i) *The set of operators $\{\mathcal{K}_n\}$ is collectively compact.*

(ii) *For each $x \in \tilde{X}$, $\|\mathcal{K}_n x - \mathcal{K}x\|_\infty \to 0$ holds as $n \to \infty$.*

(iii) *If $x \in W_\infty^{k_3}(D)$ and $k_1 \in C(D) \times W_\infty^{k_3}(D)$, the following estimate holds*

$$\|\mathcal{K}x - \mathcal{K}_n x\|_\infty \le Ch^{k_3}.$$

Proof. (i) By the continuity of the kernel $k_1(s,t)$ and condition (2.20), there exist constants C_1 and C_2 such that

$$\|k_1(s,\cdot)\|_\infty \le C_1 \quad \text{and} \quad \|k_2(s,\cdot)\|_1 \le C_2.$$

Thus we have

$$\begin{aligned} |(\mathcal{K}_n x)(s)| &= \left| \int_D \mathcal{Z}_n(k_1(s,t)x(t))k_2(s,t)\,dt \right| \\ &\le C_0 C_1 C_2 \|x\|_\infty. \end{aligned} \tag{4.8}$$

On the other hand,

$$\begin{aligned} |(\mathcal{K}_n x)(s) - (\mathcal{K}_n x)(s')| &= \left| \int_D \mathcal{Z}_n(k_1(s,t)x(t))k_2(s,t)\,dt \right. \\ &\qquad \left. - \int_D \mathcal{Z}_n(k_1(s',t)x(t))k_2(s',t)\,dt \right| \\ &\le \left| \int_D \mathcal{Z}_n(k_1(s,t)x(t))[k_2(s,t) - k_2(s',t)]\,dt \right| \\ &\quad + \left| \int_D [\mathcal{Z}_n(k_1(s,t)x(t)) \right. \end{aligned}$$

$$- \mathcal{Z}_n(k_1(s',t)x(t))]k_2(s',t)\,dt\Big|$$
$$\le C_0\|x\|_\infty(C_1\|k_2(s,\cdot)-k_2(s',\cdot)\|_1 + C_2\|k_1(s,\cdot)-k_1(s',\cdot)\|_\infty).$$

Since $\|k_2(s,\cdot)-k_2(s',\cdot)\|_1$ and $\|k_1(s,\cdot)-k_1(s',\cdot)\|_\infty$ are uniformly continuous on D, we observe that $\{\mathcal{K}_n x\}$ is equicontinuous on D. By the Arzela-Ascoli theorem we conclude that $\{\mathcal{K}_n\}$ is collectively compact.

(ii) For any $x \in \tilde{X}$,

$$\begin{aligned}|(\mathcal{K}_n x)(s) - (\mathcal{K}x)(s)| &= \left|\int_D [\mathcal{Z}_n(k_1(s,t)x(t)) - k_1(s,t)x(t)]k_2(s,t)\,dt\right| \\ &\le C_2\|\mathcal{Z}_n(k_1(s,t)x(t)) - k_1(s,t)x(t)\|_\infty.\end{aligned}$$

Note that $k_1(s,t)x(t)$ is piecewise continuous as is $x(t)$. By the definition of $\mathcal{Z}_n$, we have that the righthand side of the above inequality converges to zero as $n \to \infty$. We conclude that the lefthand side converges uniformly to zero on the compact set D. That is, $\|\mathcal{K}_n x - \mathcal{K}x\| \to 0$ as $n \to \infty$.

(iii) If $x \in W_\infty^{k_3}(D)$, by the approximate order of the interpolation projection $\mathcal{Z}_n$, we have

$$\begin{aligned}\|\mathcal{K}_n x - \mathcal{K}x\|_\infty &\le C \sup_{s\in D} \|(\mathcal{Z}_n(k_1(s,\cdot)x(\cdot)))(\cdot) - k_1(s,\cdot)x(\cdot)\|_\infty \\ &\le Ch^{k_3}.\end{aligned}$$

The estimate above follows immediately from the fact that $k_1 \in C(D) \times W_\infty^{k_3}(D)$ and inequality (2.17). □

Using Propositions 4.2 and 4.3 and Theorem 3.4, we obtain the following theorem.

Theorem 4.4. *The following statements are valid*:

(i) *There exists $N_0 > 0$ such that, for all $n > N_0$, the Petrov-Galerkin method using the piecewise polynomial approximation described in Section* 2 *has a unique solution $u_n \in X_n$.*

(ii) *If* $u \in W_\infty^\alpha(D)$ *with* $\alpha := \min\{k_1, k_3\}$, *then the following estimate holds.*

$$\|u - u_n\|_\infty \le Ch^\alpha.$$

Proof. By Propositions 4.2 and 4.3, we conclude that conditions (H1)–(H4) are satisfied. Hence, from Theorem 3.4, we have that statement (i) follows immediately and the following estimate holds

$$\|u - u_n\|_\infty \le C(\|u - \mathcal{Q}_n u\|_\infty + \|\mathcal{K}u - \mathcal{K}_n u\|_\infty). \tag{4.9}$$

Now let $u \in W_\infty^\alpha(D)$. Again, Proposition 4.2 ensures that

$$\|u - \mathcal{Q}_n u\|_\infty \le C \inf_{\phi \in X_n} \|u - \phi\|_\infty \le Ch^\alpha. \tag{4.10}$$

By (iii) of Proposition 4.3, we have that

$$\|\mathcal{K}u - \mathcal{K}_n u\|_\infty \le Ch^\alpha. \tag{4.11}$$

Substituting estimates (4.10) and (4.11) into inequality (4.9) yields the estimate (ii). □

5. Superconvergence of the iterated approximation. We present in this section a superconvergence property of the iterated discrete Petrov-Galerkin method when the kernel is smooth. Superconvergence of the iterated discrete Petrov-Galerkin approximation is also anticipated. It may be obtained by a similar analysis provided by [**9**] for the superconvergence of the iterated Galerkin method when the kernels are weakly singular. However, it is out of the scope of this paper; we will leave it to a future project.

To obtain superconvergence, we require, furthermore, that the partitions Δ_1 and Δ_3 of E, associated with spaces $S_{1,k}(E)$ and $S_{3,k_3}(E)$, respectively, be exactly the same. In the main theorem of this section we will prove that the corresponding iterated discrete Petrov-Galerkin approximation has a superconververgence property when the kernels are smooth. In particular, we assume that the kernel $k = k_1$ and $k_2 = 1$ in the notation of Section 2. We first establish a technical lemma.

Lemma 5.1. *Let $x \in L^\infty(D)$ and $k_1 \in C(D) \times W^{k_3}_\infty(D)$. Assume that $\Delta_1 = \Delta_3$. Then there exists a positive constant C independent of n such that*

$$\|(\mathcal{K} - \mathcal{K}_n)\mathcal{Q}_n x\|_\infty \le Ch^{k_3}.$$

Proof. Since $\mathcal{Q}_n x$ is not even a continuous function, Proposition 4.3 (iii) does not apply to this case. However, it follows from the proof of Proposition 4.3 (ii) that

$$|(\mathcal{K}_n \mathcal{Q}_n x)(s) - (\mathcal{K}\mathcal{Q}_n x)(s)| \le C\|r_s\|_\infty,$$

where

$$r_s(t) := (\mathcal{Z}_n(k_1(s,\cdot)(\mathcal{Q}_n x)(\cdot))(t) - k_1(s,t)(\mathcal{Q}_n x)(t).$$

Hence, it suffices to estimate $r_s(t)$.

Using the definition of the projection $\mathcal{Q}_n$, we write

$$(\mathcal{Q}_n x)(t) = \sum_{i=1}^{N_n}\sum_{j=1}^{\mu} c_{i,j}\phi_{i,j}(t), \quad t \in D, \tag{5.1}$$

where $\phi_{i,j}$ are the basis function for X_n given in Section 2, and the coefficients $c_{i,j}$ satisfy the linear system (4.2). Consequently, we have

$$(\mathcal{Z}_n(k_1(s,\cdot)(\mathcal{Q}_n x)(\cdot))(t) = \sum_{i=1}^{N_n}\sum_{j=1}^{\mu} c_{i,j}(\mathcal{Z}_n(k_1(s,\cdot)\phi_{i,j}(\cdot))(t). \tag{5.2}$$

By the construction of the functions $\phi_{i,j}$, we have that $\phi_{i,j}(t_{i',j'}) = 0$ if $i \ne i'$. Thus, it follows that

$$\begin{aligned}(\mathcal{Z}_n(k_1(s,\cdot)\phi_{i,j}(\cdot))(t) &= \sum_{i'=1}^{N_n}\sum_{j'=1}^{\gamma} k_1(s,t_{i',j'})\phi_{i,j}(t_{i',j'})\zeta_{i',j'}(t)\\ &= \sum_{j'=1}^{\gamma} k_1(s,t_{i,j'})\phi_{i,j}(t_{i,j'})\zeta_{i,j'}(t).\end{aligned}$$

Substituting this equation into (5.2) yields
(5.3)

$$(\mathcal{Z}_n(k_1(s,\cdot)(\mathcal{Q}_n x)(\cdot))(t) = \sum_{i=1}^{N_n}\sum_{j=1}^{\mu} c_{i,j} \sum_{j'=1}^{\gamma} k_1(s, t_{i,j'})\phi_{i,j}(t_{i,j'})\zeta_{i,j'}(t), \quad t \in D.$$

We now assume that, for some point $\hat{t} \in E_{i'}$, $\|r_s\|_\infty = |r_s(\hat{t})|$. For this point $\hat{t}$, there exists a point τ in the reference element E such that $\hat{t} = F_{i'}(\tau)$. Hence,

$$\|r_s\|_\infty = \left|\sum_{j=1}^{\mu} c_{i',j}\left[\sum_{j'=1}^{\gamma} k_1(s, F_{i'}(t_{j'}))\phi_j(t_{j'})\zeta_{j'}(\tau) - k_1(s, F_{i'}(\tau))\phi_j(\tau)\right]\right|.$$

Because $E = \cup_{i=1}^{\nu_3} e_i$, the point τ must be in some $\bar{e}_i$. For each integer $j' = 1, 2, \ldots, \gamma$, assume that positive integers i_0 and j_0 with $1 \le i_0 \le \nu_3$, $1 \le j_0 \le m$, are such that $(i_0 - 1)m + j_0 = j'$. Therefore, we have

$$\zeta_{j'(t)} = \begin{cases} p_{i_0,j_0}(t) & t \in e_{i_0} \\ 0 & t \notin e_{i_0}, \end{cases} \quad \text{and} \quad t_{j'} = \tau_{i_0,j_0},$$

so that

$$\begin{aligned} \|r_s\|_\infty &= \left|\sum_{j=1}^{\mu} c_{i,j}\left[\sum_{i_0=1}^{\nu_3}\sum_{j_0=1}^{m} k_1(s, F_{i'}(\tau_{i_0,j_0}))\phi_j(\tau_{i_0,j_0})p_{i_0,j_0}(\tau) \right.\right. \\ &\qquad \left.\left. - k_1(s, F_{i'}(\tau))\phi_j(\tau)\right]\right| \\ &= \left|\sum_{j=1}^{\mu} c_{i,j}\left[\sum_{j_0=1}^{m} k_1(s, F_{i'}(\tau_{i,j_0}))\phi_j(\tau_{i,j_0})p_{i,j_0}(\tau) \right.\right. \\ &\qquad \left.\left. - k_1(s, F_{i'}(\tau))\phi_j(\tau)\right]\right|. \end{aligned}$$

We identify that the function in the blanket of the last term is the error of polynomial interpolation of the function $k_1(s, F_{i'}(\tau))\phi_j(\tau)$ on e_i, which we call the error term on e_i. Since $\Delta_1 = \Delta_3$, $k_1(s, F_{i'}(\tau))\phi_j(\tau)$ as a function of τ is in the space $W_\infty^{k_3}(e_i)$. We conclude that the error term on e_i is bounded by a constant time $\|D^{k_3}(k_1(s, F_{i'}(\cdot))\phi_j(\cdot)\|_\infty$. The latter is bounded by a constant time $|\det(B_{i'})|^{k_3} \le Ch^{k_3}$. Hence, we obtain

$$\|r_s\|_\infty \le C\|\mathbf{c}\|_\infty h^{k_3}.$$

By the proof of Proposition 4.2, we know that $\|\mathbf{c}\|_\infty \le C$. Therefore, we have $\|r_s\|_\infty \le Ch^{k_3}$. □

We are now ready to establish the main result of this section concerning the superconvergence of the iterated solution.

Theorem 5.2. *Let $\beta := \min\{k_1 + k_2, k_3\}$, $u \in W_\infty^\beta(D)$ and $k \in C(D) \times W_\infty^{k_3}(D)$. Then the following estimate holds*

$$\|u - u_n'\|_\infty \le Ch^\beta.$$

Proof. It follows from Theorem 3.4 that

$$\|u - u_n'\|_\infty \le C(\|(\mathcal{K} - \mathcal{K}_n)\mathcal{Q}_n u\|_\infty + \|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)u\|_\infty). \tag{5.4}$$

Because $\Delta_1 = \Delta_3$, by applying Lemma 5.1, we have that

$$\|(\mathcal{K} - \mathcal{K}_n)\mathcal{Q}_n u\|_\infty \le Ch^{k_3}. \tag{5.5}$$

Moreover, since $k(s, \cdot) \in W_\infty^{k_3}(D)$ and $\Delta_1 = \Delta_3$, we conclude that

$$\|\mathcal{K}(u - \mathcal{Q}_n u)\|_\infty \le \|(k(s,t), u(t) - (\mathcal{Q}_n u)(t))_n\|_\infty + Ch^{k_3}. \tag{5.6}$$

It remains to estimate $\|(k(s,t), u(t) - (\mathcal{Q}_n u)(t))_n\|_\infty$. For this purpose, we note that, for any $y \in Y_n$, the following holds

$$(y, u - \mathcal{Q}_n u)_n = 0.$$

It follows that

$$\begin{aligned} |(k(s,t), u(t) - (\mathcal{Q}_n u)(t))_n| &= |(k(s,t) - y(t), u(t) - (\mathcal{Q}_n u)(t))_n| \\ &\le \inf_{y \in Y_n} \|k(s,t) - y(t)\|_\infty \|u - \mathcal{Q}_n u\|_\infty. \end{aligned}$$

This implies that

$$\|(k(s,t), u(t) - (\mathcal{Q}_n u)(t))_n\|_\infty \le Ch^{k_2} h^{k_1} = Ch^{k_1+k_2}. \tag{5.7}$$

Combining inequalities (5.4)–(5.7), we establish the estimate of this theorem. □

We remark that, when $k_1 < k_3 < k_1 + k_2$, the optimal order of convergence of u_n is $\mathcal{O}(h^{k_1})$ while the iterated solution u_n' has an order of convergence $\mathcal{O}(h^{k_3})$. This phenomenon is called superconvergence.

6. Numerical examples. In this section, we present two numerical examples to illustrate the theoretical estimates obtained in the previous sections. The kernel in the first example is weakly singular while the kernel in the second example is smooth. The second example is presented to show the superconvergence property of the iterated solution. Since examples are used to illustrate the performance of the method, we restrict ourselves to simple one-dimensional equations whose exact solutions are known.

In both examples we will use piecewise linear functions and piecewise constant functions for the spaces X_n and Y_n, respectively. Specifically, we define the trial space by

$$X_n = \operatorname{span}\{\phi_1, \dots, \phi_{2n}\}$$

where

$$\phi_{2j+1}(t) := \begin{cases} nt - j & (j/n) \le t \le ((j+1)/n), \\ 0 & \text{otherwise}, \end{cases}, \quad j = 0, 1, \dots, n-1,$$

and

$$\phi_{2j+2}(t) := \begin{cases} j+1-nt & (j/n) \le t \le ((j+1)/n), \\ 0 & \text{otherwise}, \end{cases} \quad j = 0, 1, \dots, n-1.$$

The test space is then defined by

$$Y_n = \operatorname{span}\{\psi_1, \dots, \psi_{2n}\},$$

where

$$\psi_i(t) := \begin{cases} 1 & (i-1/(2n)) \leq t \leq (i/(2n)), \\ 0 & \text{otherwise}, \end{cases} \qquad i = 1, 2, \dots, 2n.$$

Example 1. Consider the integral equation with a weakly singular kernel

$$x(s) - \int_0^\pi \log|\cos s - \cos t| x(t)\, dt = 1, \quad 0 \leq s \leq \pi.$$

This equation is a reformulation of a third boundary value problem of the 2-D Laplace equation, and it has the exact solution given by

$$x(s) = \frac{1}{1 + \pi \log 2},$$

see [**2**] for more details about this example. By changes of variables $t = \pi t'$, $s = \pi s'$, we have an equivalent equation

$$x(\pi s) - \pi \int_0^1 \log|\cos(\pi s) - \cos(\pi t)| x(\pi t)\, dt = 1, \quad 0 \leq s \leq 1.$$

We write the kernel

$$\log|\cos(\pi s) - \cos(\pi t)| = \sum_{i=1}^{4} k_{i,1}(s,t) k_{i,2}(s,t)$$

where

$$\begin{aligned} k_{1,1}(s,t) &= \log\left(\frac{\sin((\pi(t-s))/2)) \sin((\pi(t+s))/2))}{\pi^3((t-s)/2)(t+s)(2-t-s)} \right), \\ k_{1,2}(s,t) &= k_{2,1}(s,t) = k_{3,1}(s,t) = k_{4,1}(s,t) = 1, \\ k_{2,2}(s,t) &= \log|\pi(s-t)|, \quad k_{3,2}(s,t) = \log(\pi(2-s-t)), \end{aligned}$$

and

$$k_{4,2}(s,t) = \log(\pi(s+t)).$$

In Table 1 we present the error e_n of the approximate solution and the error e'_n of the iterated approximate solution, where we use q and

q' to represent the corresponding orders of approximation, respectively. In our computation, we choose $k_3 = 2$.

TABLE 1.

n	4	8	16	32
e_n	1.504077E-06	3.879971E-07	9.877713E-08	2.957718E-08
q		1.954761	1.973797	1.739639
e'_n	3.186220E-06	8.005914E-07	1.973337E-07	5.153006E-08
q'		1.992708	2.020429	1.93715

The order of approximation agrees with our theoretical estimate. The iteration does not improve the accuracy of the approximate solution for this example due to the nonsmoothness of the kernel.

Example 2. We consider the integral equation with a smooth kernel

$$x(s) - \int_0^1 \sin s \cos t x(t)\, dt = \sin s(1 - e^{\sin 1}) + e^{\sin s}, \quad 0 \le s \le 1.$$

It is not difficult to verify that $x(s) = e^{\sin s}$ is the unique solution of this equation. In the notation of Section 2, we have $k_1(s,t) = \sin s \cos t$ and $k_2(s,t) = 1$. In this case we choose $k_3 = 3$ for the quadrature formula. The notation in Table 2 is the same as that in Table 1.

TABLE 2.

n	4	8	16	32
e_n	1.68156E-02	4.10275E-03	1.01615E-03	3.00353E-04
q		2.035137	2.013478	1.75839
e'_n	6.78911E-05	4.16056E-06	2.58946E-07	1.61679E-08
q'		4.028373	4.006054	4.001447

In this example the iteration improves the accuracy of the approximation by the order as estimated in Theorem 5.2.

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