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A WAVELET ALGORITHM FOR THE SOLUTION OF A SINGULAR INTEGRAL EQUATION OVER A SMOOTH TWO-DIMENSIONAL MANIFOLD

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ABSTRACT. In this paper we consider a piecewise bilinear collocation method for the solution of a singular integral equation over a smooth surface. Using a fixed set of parametrizations, we introduce special wavelet bases for the spaces of test and trial functions. The trial wavelets have two vanishing moments only if their supports do not intersect the lines belonging to the common boundary of two subsurfaces defined by different parameter representations. Nevertheless, analogously to well-known results on wavelet algorithms, the stiffness matrices with respect to these bases can be compressed to sparse matrices such that the iterative solution of the matrix equations becomes fast. Finally we present a fast quadrature algorithm for the computation of the compressed stiffness matrix.

1. Introduction. It is a well-known fact that usual finite element discretizations of linear integral equations, e.g., of boundary integral equations, lead to systems of linear equations with fully populated matrices. Thus, even an iterative solution method requires a huge number of arithmetic operations and a large storage capacity. In order to improve these finite element approaches, several new algorithms have been developed. For a relatively wide class of boundary integral equations, Rokhlin and Greengard [**37, 20**] have introduced their methods of multipole expansion, Hackbusch and Nowak [**21**], cf. also [**38**], have considered panel clustering algorithms, and Brandt and Lubrecht [**3**] have set up multilevel schemes. Another approach for saving storage and computation time consists in employing wavelet bases of the finite element spaces. This idea goes back to Beylkin, Coifman and Rokhlin [**2**] and has been thoroughly investigated by Dahmen, Petersdorff, Prößdorf, Schneider and Schwab [**13, 14, 12, 15, 32, 31, 30, 39**], cf. also the contributions by Alpert, Harten, Yad-Shalom, Dorobantu, Kleemann and the author [**1, 22, 19, 9, 10,**

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36]. Note that all the different algorithms from multipole expansion to wavelets seem to have a common multilevel background.

The subject of the present paper is to apply the wavelet technique from [**2**] to the collocation solution of two-dimensional singular integral equations. The two-dimensional singular integral equations and the bilinear collocation methods will be introduced in Section 2. In particular, the collocation for the singular boundary integral equation corresponding to the oblique derivative problem for Laplace's equation, cf. Miranda [**27**, Section 23], Klees, Engels [**25, 24**] or the similar equation for the Molodensky problem in Moritz [**28**, Section 43] is included.

If the underlying surface is smooth (continuously differentiable up to a certain order) and diffeomorphic to the torus, then it is clear that the wavelet algorithms, cf. [**12, 31**], admit high order compressions. For general smooth surfaces represented by a set of parametrizations, similar results hold if the wavelet functions are suitably chosen. Supposing that the parameter domains are squares, one can define the wavelets of the trial space as tensor products of the orthogonal wavelets and scaling functions over the interval [**7, 5**]. However, due to the orthogonality, these wavelets are not optimal. Indeed, to reduce the amount of work for the quadratures applied during the computation of the stiffness matrix, wavelets with smaller supports but with the same moment conditions seem to be preferable. Thus, in Section 3.1, we consider the piecewise linear univariate biorthogonal wavelets used by Petersdorff, Schwab and Rathsfeld [**32, 36**]. These wavelets have the smallest support among all the piecewise linear wavelets with two vanishing moments. By reflection techniques we define boundary wavelets and get a stable wavelet system (Riesz basis) over the interval. Applying well-known tensor product techniques in Section 3.2, we introduce a wavelet basis over the square, and by using the parameterization mappings, we end up with continuous wavelet functions over the boundary manifold. For these wavelets, we will prove the Riesz basis property and the usual decay property for the coefficients of a smooth bilinear function. If the support of the wavelet does not intersect the lines belonging to the common boundary of two subsurfaces defined by different parameter representations, then the wavelets have two vanishing moments. Note that the techniques for the proof of these properties are well known from the works of, e.g., Cohen, Daubechies, Feauveau, Dahmen, Kunoth and Schneider [**6, 16, 11, 39**]. Therefore, some parts of the proof are only sketched.

Following the ideas of Harten and Yad-Shalom [**22**], we define a wavelet basis for the space of test functionals in Section 3.3. In Section 3.4 we describe the wavelet algorithm which is based on the just introduced bases in the test and trial spaces. Analogously to the results by Dahmen, Prößdorf, Schneider, Petersdorff and Schwab [**14, 39, 31**], we will show that the $n \times n$ stiffness matrix corresponding to the wavelet bases admits a compression up to a matrix with no more than $O(n[\log n]^4)$ nonzero entries and that, replacing the full stiffness matrix by the compressed matrix, we get the same asymptotic convergence rate $O(n^{-1})$ as for the conventional collocation solution. For this estimate, the second order moment condition for the wavelets along the common boundary of two subsurfaces defined by different parameter representations is not necessary. Note that the logarithmic factor $[\log n]^4$ could be slightly improved if the factor j in the compression criterion (3.66) of Theorem 3.1 is replaced by a power of j with exponent less than one. Essential improvements are possible if wavelets with more vanishing moments are used and if the compression is extended to matrix entries corresponding to wavelets with overlapping supports, cf. the compression of the Galerkin matrix due to Schneider [**39**]. However, the complete removal of this factor similar to the compression of the Galerkin matrix seems not to be possible since the basis transform corresponding to the test wavelets is not bounded, cf. Lemma 3.4.

Clearly, using the compressed matrix, the iterative solution, e.g., by a cascadic GMRes algorithm, of the collocation system requires no more than $O(n[\log n]^4)$ arithmetic operations. In Section 4 we will introduce a quadrature algorithm for the computation of the compressed stiffness matrix with no more than $O(n^{4/3}[\log n]^{4/3})$ operations. The corresponding error of the discretized collocation solution is less than $O(n^{-1}\log n)$. Note that this quadrature algorithm is more or less an adoption of the Johnson-Scott algorithm [**23**], cf. also the references in [**23**], for the computation of conventional stiffness matrices to the case of wavelet transformed stiffness matrices. The complexity result is true if each of the parametrization mappings is analytic in a neighborhood of the parameter domain and if the kernel function of the singular integral operator admits a representation, cf. (4.3), which is typically fulfilled for boundary integral operators. Moreover, in contrast to the estimates for the Galerkin method by Petersdorff, Schwab and Schneider [**31, 39**], we even do not need the global analyticity of the parametrizations. Local analyticity is sufficient. More exactly, if the thrice continuously differentiable surface is given by certain grid points and if this surface is replaced by a suitable interpolation, then we may suppose that the parametrizations are twice continuously differentiable and piecewise polynomial. For this situation, the complexity estimate $O(n^{4/3}[\log n]^{4/3})$ remains true. Finally, we indicate how an algorithm of complexity $O(n)$ times a certain power of $\log n$ can be obtained.

For a numerical experiment with the method of the present paper, we refer to the paper [**35**]. In that article we considered a singular integral equation corresponding to an oblique derivative boundary value problem of Laplace's equation with application in geodesy, cf. Moritz [**28**], Klees and Engels [**24**]. To this we applied a slightly modified version of the wavelet and quadrature algorithm defined in Sections 3.4 and 4.2. The underlying manifold was a part of the earth's surface which is not smooth and which was approximated by Overhauser interpolation over the uniform grid of a square shaped parameter domain. Thus a global parametrization mapping was applied for the numerical computations. Using this we could replace the singularity subtraction technique of Section 4.2 by a global singularity technique. Furthermore, to reduce the computing time, we used test functionals with one vanishing moment, only. Though these test wavelets lead to asymptotically slower methods, we expect them to be faster for linear systems of size less than 10,000. Due to the lower compression rates the refinement step from $\{\Gamma_i^j\}$ to $\{\Gamma_{i'}^{\xi'}\}$ for the quadrature partition, cf. Section 4.2, turns out to be redundant. Implementing our wavelet algorithm including the three modifications mentioned above, we observed that the stiffness matrix of dimension $n = 9025$ can be compressed to 5.1 percent such that the additional relative compression error is still less than 10^{-5}. The wavelet algorithm reduces the computing time on a DEC 3000 AXP 400 α-processor workstation from 10,500s for a conventional algorithm to 890s. For more details and results, see [**35**].

2. The collocation method for the singular integral equation.

2.1. *The singular integral equation.* Now we consider a smooth two-dimensional surface Γ in the three-dimensional Euclidean space $\mathbf{R}^3$. This surface is supposed to be the union of the closed bounded surface pieces Γ_m, $m = 1, \dots, m_\Gamma$ such that, for every m, there exists an infinitely differentiable coordinate mapping κ_m from the reference domain $\mathcal{S} := [0,1] \times [0,1]$ to Γ_m. Moreover, we suppose that this mapping extends to a mapping over a small neighborhood of $\mathcal{S}$ and that the intersection of two subsurfaces Γ_m and $\Gamma_{m'}$ is either empty or consists of a common corner point or is equal to a common side of Γ_m and $\Gamma_{m'}$, respectively. In case the intersection $\Gamma_m \cap \Gamma_{m'}$ is a side, we suppose that the parametrizations κ_m and $\kappa_{m'}$ restricted to this common side coincide. The singular integral equation over Γ takes the form

$$Au(x) := a(x)u(x) + \int_\Gamma K_A(x,y)u(y)\, d_y\Gamma = v(x), \quad x \in \Gamma, \tag{2.1}$$

where a is a smooth function and $K_A(x,y)$ is the singular kernel function of operator A. We suppose that K_A is infinitely differentiable over $\Gamma\times\Gamma\backslash\{(x,x) : x \in \Gamma\}$ and that the derivatives satisfy the Calderón-Zygmund estimate

$$|\partial_x^\alpha \partial_y^\beta K_A(x,y)| \le C(\alpha,\beta,A,\Gamma)|x-y|^{-(2+|\alpha|+|\beta|)} \tag{2.2}$$

for any multi-indices α and β. The integral on the lefthand side of (2.1) is to be understood in the sense of a principle value, cf. [**26**]. Operator A is supposed to be a classical pseudodifferential operator of order zero and maps the Sobolev space $H^s(\Gamma)$ of order s into $H^s(\Gamma)$. In local coordinates, (2.1) takes the form

$$\begin{aligned} a(\kappa_k(t))u(\kappa_k(t)) + \sum_{m=1}^{m_\Gamma} \int_{\mathcal{S}} K_A(\kappa_k(t),\kappa_m(s))u(\kappa_m(s))|\kappa_m'(s)|\, ds \\ = v(\kappa_k(t)), \\ t \in \mathcal{S}, \quad k = 1, \dots, m_\Gamma, \end{aligned} \tag{2.3}$$

where $|\kappa_m'(s)|$ denotes the density of the surface integral, i.e., the norm of the vector product $\partial_{s_1}\kappa_m(s) \times \partial_{s_2}\kappa_m(s)$.

For the stability of the numerical methods, the concept of strong ellipticity plays a crucial role. We call A strongly elliptic if A satisfies the so-called Gårding inequality, i.e.,

$$\operatorname{Re}\langle Au, u\rangle_{L^2(\Gamma)} \ge \gamma\|u\|_{L^2(\Gamma)} - |\langle Tu, u\rangle_{L^2(\Gamma)}| \tag{2.4}$$

for any $u \in L^2(\Gamma)$. In (2.4) the operator $T \in \mathcal{L}(L^2(\Gamma))$ is supposed to be compact and γ stands for a positive constant independent of u. Note that the classical pseudodifferential operator A is strongly elliptic if and only if the real part of its main symbol is greater than a positive constant.

Finally, we remark that the smoothness assumptions can be relaxed. This will be indicated in Section 4.1.

2.2. *The bilinear trial functions and the collocation.* We will seek an approximate solution for u of (2.1) in the space of bilinear functions over Γ. To define these functions, we first introduce functions over the square $\mathcal{S}$. We set $N := N_j := 3 \cdot 2^j$ and $h := h_j := 1/N$ and consider the grid $\Delta_j^{\mathcal{S}} := \{\tau_{i,k} : i,k = 0,\dots,N\}$, where $\tau_{i,k} := (ih, kh)$. The space of piecewise bilinear functions $S_j^{\mathcal{S}} := \operatorname{span}\{\varphi_\tau^{\mathcal{S}} : \tau \in \Delta_j^{\mathcal{S}}\}$ over the grid $\Delta_j^{\mathcal{S}}$ is defined by the basis functions $\varphi_\tau^{\mathcal{S}}(t) := N\varphi^T(N \cdot [t-\tau])$, where $\varphi^T((t_1,t_2)) := \varphi(t_1)\varphi(t_2)$ is the tensor product of the univariate hat function

$$\varphi(s) := \begin{cases} 1 - |s| & \text{if } |s| \le 1 \\ 0 & \text{else.} \end{cases} \tag{2.5}$$

Using the parametrizations κ_m, we define the grid $\Delta_j := \{\xi_{i,k}^m : m = 1,\dots,m_\Gamma, i,k = 0,\dots,N\}$ over Γ by $\xi_{i,k}^m := \kappa_m(\tau_{i,k})$ and the space of trial functions $S_j := \operatorname{span}\{\varphi_\xi : \xi \in \Delta_j\}$ by $\varphi_{\xi_{i,k}^m}(\kappa_m(t)) := \varphi_{i,k}^m(\kappa_m(t)) := \varphi_{\tau_{i,k}}^{\mathcal{S}}(t)$. Note that, if $\xi \in \Delta_j$ belongs to more than one subsurface Γ_m, then it admits several representations of the form $\xi = \xi_{i,k}^m$. Nevertheless, we consider ξ as one point. The corresponding basis function φ_ξ is the sum of the functions $\kappa_m(t) \mapsto \varphi_{i,k}^m(\kappa_m(t)) := \varphi_{\tau_{i,k}}^{\mathcal{S}}(t)$ defined over the different Γ_m. Clearly, the functions of S_j are bilinear with respect to the parametrization and $\varphi_\xi(\xi') = N\delta_{\xi,\xi'}$ holds for any $\xi, \xi' \in \Delta_j$.

With the collocation method, we seek an approximate solution $u_j \in S_j$ to u by solving the collocation equations

$$(Au_j)(\xi) = v(\xi), \quad \xi \in \Delta_j. \tag{2.6}$$

We introduce the interpolation projection P_j onto S_j by

$$P_j f \in S_j, \quad P_j f(\xi) = f(\xi), \quad \xi \in \Delta_j. \tag{2.7}$$

Clearly the collocation system (2.6) is equivalent to $P_j A u_j = P_j v$. The collocation is called stable if, for sufficiently large j, the collocation operators $\mathcal{A}_j := P_j A|_{S_j} \in \mathcal{L}(S_j)$ are invertible and the L^2-norms of the inverse operators are uniformly bounded.

Theorem 2.1. i) [**34**] *Suppose that Γ is homeomorphic to the two-dimensional torus and that $m_\Gamma := 1$, i.e., $\kappa := \kappa_1 : \mathcal{S} \to \Gamma$ is a global parametrization. Moreover, we assume A to be strongly elliptic. Then the collocation method is stable in H^s for $0 \le s < 3/2$. The collocation solution u_j defined by* (2.6) *converges in H^s to the exact solution u of $Au = v$ for any $v \in H^s$ with $s > 1$, and the collocation error satisfies*

$$\|u_j - u\|_{H^s} \le C 2^{-j(t-s)} \|u\|_{H^t}$$

for $0 \le s \le t \le 2$, $s < 3/2$, $1 < t$.

(ii) [**33**] *Suppose that $\Gamma = \mathcal{S}$, that S_j and Δ_j are modified such that Δ_j contains only the interior grid points and that S_j is spanned by the basis functions vanishing at the boundary of $\Gamma = \mathcal{S}$. Moreover, we assume A to be strongly elliptic. Then the collocation method is stable in L^2. The collocation solution u_j defined by* (2.6) *converges in L^2 to the exact solution u of $Au = v$ for any $v \in L^2$ such that $\|P_j v - v\|_{L^2} \to 0$. If u is in H^2 and vanishes over the boundary of $\mathcal{S}$, then*

$$\|u_j - u\|_{L^2} \le C 2^{-2j} \|u\|_{H^2}.$$

Unfortunately, we do not know stability results for the collocation method in the general case. Nevertheless, we suppose in the following that the collocation method is stable. Then the error estimates of the last theorem remain valid.

Choosing the conventional finite element basis $\{\varphi_\xi\}_{\xi \in \Delta_j}$, the collocation equation (2.6) is equivalent to the system

$$\sum_{\xi \in \Delta_j} h(A\varphi_\xi)(\xi') w_\xi = hv(\xi'), \quad \xi' \in \Delta_j \tag{2.8}$$

for the coefficients w_ξ of $u_j := \sum_{\xi \in \Delta_j} w_\xi \varphi_\xi$. Thus, the stiffness matrix of the collocation is $A_j := (h(A\varphi_\xi)(\xi'))_{\xi', \xi \in \Delta_j}$.

3. The wavelet algorithms.

3.1 *Univariate wavelet functions.* Using the parametrizations, it will be sufficient to define the wavelet basis functions over the square $\mathcal{S}$. Since these wavelets can be defined by tensor product techniques, we begin with the definition of univariate wavelets. To introduce wavelets over the real axis $\mathbf{R}$, we consider the uniform grids $\Delta_j^{\mathbf{R}} := \{ih_j : i \in \mathbf{Z}\}$ and the difference grids $\nabla_l^{\mathbf{R}} := \Delta_{l+1}^{\mathbf{R}} \backslash \Delta_l^{\mathbf{R}}$ for $l \geq 0$ and $\nabla_{-1}^{\mathbf{R}} := \nabla_0^{\mathbf{R}}$. Clearly, $\Delta_j^{\mathbf{R}} = \cup_{l=-1}^{j-1} \nabla_l^{\mathbf{R}}$ and the space of piecewise linear functions $S_j^{\mathbf{R}}$ over the grid $\Delta_j^{\mathbf{R}}$ is spanned by the finite element basis $\{\varphi_{j,\sigma}^{\mathbf{R}} : \sigma \in \Delta_j^{\mathbf{R}}\}$ given by $\varphi_{j,\sigma}^{\mathbf{R}}(s) := \sqrt{N}_j \varphi(N_j \cdot [s-\sigma])$. It is easy to see that the finite element functions satisfy the refinement equations

$$\varphi_{l,ih_l}^{\mathbf{R}} = \frac{1}{2}\varphi_{l+1,[2i-1]h_{l+1}}^{\mathbf{R}} + \varphi_{l+1,[2i]h_{l+1}}^{\mathbf{R}} + \frac{1}{2}\varphi_{l+1,[2i+1]h_{l+1}}^{\mathbf{R}}. \tag{3.1}$$

Following the techniques for the construction of orthogonal wavelets, it is natural to define the wavelet shape function

$$\psi(s) := \frac{1}{2}\varphi(2s-1) - \varphi(2s) + \frac{1}{2}\varphi(2s+1) \tag{3.2}$$

and to introduce the wavelet basis functions by shifting the dilated shape function $s \mapsto \psi(N_l \cdot s)$ to the points of the reference grid $\nabla_l^{\mathbf{R}}$. More exactly, we set $\psi_\sigma^{\mathbf{R}}(s) := \varphi_{0,\sigma}^{\mathbf{R}}(s)$ for $\sigma \in \nabla_{-1}^{\mathbf{R}}$ as well as $\psi_\sigma^{\mathbf{R}}(s) := \sqrt{N}_l \psi(N_l \cdot [s-\sigma])$ for $\sigma \in \nabla_l^{\mathbf{R}}$ with $l \geq 0$. We arrive at the hierarchical basis $\{\psi_\sigma^{\mathbf{R}} : \sigma \in \nabla_l^{\mathbf{R}}, l = -1, \dots, j-1\}$ of the finite element space $S_j^{\mathbf{R}}$ and at the multiscale decomposition $S_j^{\mathbf{R}} = \sum_{-1}^{j-1} W_l^{\mathbf{R}}$, where the wavelet space $W_l^{\mathbf{R}}$ is spanned by $\{\psi_\sigma^{\mathbf{R}} : \sigma \in \nabla_l^{\mathbf{R}}\}$.

We remark that these basis functions are not wavelets in the sense of [**16, 4**]. The $\psi_\sigma^{\mathbf{R}}$ are biorthogonal wavelets in the sense of [**6**], where the dual scaling function does not have a finite support but decays exponentially. From Proposition 4.8 of [**6**] with $L = 2$ and $k = 2$, we infer that the dual scaling function belongs even to the Sobolev space $H^{1/2+\varepsilon}(\mathbf{R})$ for a certain small positive ε. For a few more details, we refer the reader to the proof of Lemma 3.5 in [**36**]. The wavelet functions $\psi_\sigma^{\mathbf{R}}$, $\sigma \in \nabla_l^{\mathbf{R}}$ of level $l \geq 0$ have two vanishing moments, i.e., they are orthogonal to constant and linear functions. Moreover, among all the basis functions with two vanishing moments, the $\psi_\sigma^{\mathbf{R}}$ have the smallest support.

Now we define wavelet functions over the interval $\mathcal{I} := [0,1]$. We consider the uniform grids $\Delta_j^{\mathcal{I}} := \{ih_j : i = 0, \dots, N_j\}$ and the difference grids $\nabla_l^{\mathcal{I}} := \Delta_{l+1}^{\mathcal{I}} \backslash \Delta_l^{\mathcal{I}}$ for $l \geq 0$ and $\nabla_{-1}^{\mathcal{I}} := \Delta_0^{\mathcal{I}}$. Clearly the space of piecewise linear functions $S_j^{\mathcal{I}}$ over the grid $\Delta_j^{\mathcal{I}}$ is spanned by the finite element basis $\{\varphi_{j,\sigma}^{\mathcal{I}} : \sigma \in \Delta_j^{\mathcal{I}}\}$ given by $\varphi_{j,\sigma}^{\mathcal{I}} := \varphi_{j,\sigma}^{\mathbf{R}}|_{\mathcal{I}}$. Similarly, the wavelet functions could be defined as the restrictions to $\mathcal{I}$ of the corresponding functions over $\mathbf{R}$. However, we will change those basis functions which do not vanish at the end points of the interval. To this end, we consider the space of "even" functions over $\mathbf{R}$, i.e., the functions satisfying $f(s) = f(-s) = f(2-s)$ for $s \in [0,1]$. The correct basis functions for this space are the functions

$$s \longmapsto \psi_\sigma(s) + \psi_\sigma(-s) + \psi_\sigma(2-s) = \psi_\sigma(s) + \psi_{-\sigma}(s) + \psi_{2-\sigma}(s).$$

If we restrict these to $\mathcal{I}$, we arrive at the wavelet basis $\{\psi_\sigma^{\text{even}} : \sigma \in \Delta_j^{\mathcal{I}}\}$ defined by

$$\psi_\sigma^{\text{even}} := \begin{cases} \varphi_{0,\sigma}^{\mathbf{R}}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_{-1}^{\mathcal{I}} \\ \psi_\sigma^{\mathbf{R}}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } 0,1 \notin \operatorname{supp} \psi_\sigma^{\mathbf{R}} \\ \{\psi_{h_{l+1}}^{\mathbf{R}} + \psi_{-h_{l+1}}^{\mathbf{R}}\}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } \sigma = h_{l+1} \\ \{\psi_{1-h_{l+1}}^{\mathbf{R}} + \psi_{1+h_{l+1}}^{\mathbf{R}}\}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } \sigma = 1 - h_{l+1}. \end{cases} \tag{3.3}$$

We denote the corresponding wavelet spaces span $\{\psi_\sigma^{\text{even}} : \sigma \in \nabla_l^{\mathcal{I}}\}$ by $W_l^{\mathcal{I}}$ and obtain $W_l^{\mathcal{I}} = W_l^{\mathbf{R}}|_{\mathcal{I}}$ and $S_j^{\mathcal{I}} = \sum_{-1}^{j-1} W_l^{\mathcal{I}}$. Clearly only those wavelets of level $l \geq 0$ have two vanishing moments for which the support is contained in the interior of $\mathcal{I}$. The wavelets of level $l \geq 0$ with support intersecting the boundary $\{0,1\}$ have one vanishing moment, only. Instead of the orthogonality of the wavelet basis, we get

Lemma 3.1. i) *There exists a constant $C > 0$ such that, for any j and any sequence $(u_\sigma)_{\sigma\in\Delta_j^{\mathcal{I}}}$, we get*

$$\begin{aligned}\frac{1}{C}\sqrt{\sum_{\sigma\in\Delta_j^{\mathcal{I}}}|u_\sigma|^2} &\le \Big\|\sum_{\sigma\in\Delta_j^{\mathcal{I}}}u_\sigma\psi_\sigma^{\text{even}}\Big\|_{L^2(\mathcal{I})}\\ &\le C\sqrt{\sum_{\sigma\in\Delta_j^{\mathcal{I}}}|u_\sigma|^2}.\end{aligned}\tag{3.4}$$

ii) *There exist constants $C > 0$ and $0 < q < 1$ such that, for any $l < l'$, $u^l \in S_l^{\mathcal{I}}$ and $u^{l'} \in W_{l'}^{\mathcal{I}}$, we get*

$$|\langle u^l, u^{l'}\rangle_{L^2(\mathcal{I})}| \le Cq^{l'-l}\|u^l\|_{L^2(\mathcal{I})}\|u^{l'}\|_{L^2(\mathcal{I})}.\tag{3.5}$$

Proof. Now and in the following, we denote by C a generic constant the value of which varies from instant to instant. We note that the corresponding assertions hold for the wavelets over the real axis. Indeed, the analogue of i) is proved in Theorem 3.8 of [**6**]. For the proof of ii), we consider the projection $Q_j^{\mathbf{R}}$ onto $S_j^{\mathbf{R}}$ parallel to the closure of $\cup_{l=j}^{\infty}W_l^{\mathbf{R}}$. This projection $Q_j^{\mathbf{R}} \in \mathcal{L}(l^2(\mathbf{R}))$ is uniformly bounded with respect to j, cf. (3.4). We observe that the vanishing moment condition for ψ implies that the constant function is contained in the span of the dual scaling function, i.e., in $\text{im}\,[Q_j^{\mathbf{R}}]^*$. From this fact and the exponential decay of the dual scaling functions, it is not hard to derive the usual L^2 convergence order $O(\sqrt{h_j})$ for the approximation of an $H^{1/2}$ function f by $[Q_j^{\mathbf{R}}]^*f$. By duality arguments, we can approximate an L^2 function f by $Q_j^{\mathbf{R}}f$ with an $H^{-1/2}$ error of $O(\sqrt{h_j})$. This and the well-known inverse property for piecewise linear functions yields, cf., e.g., the proof of Lemma 6.3 in [**39**],

$$\begin{aligned}|\langle u^l, u^{l'}\rangle_{L^2(\mathbf{R})}| &\le \|u^l\|_{H^{1/2}(\mathbf{R})}\|u^{l'}\|_{H^{-1/2}(\mathbf{R})}\\ &\le \|u^l\|_{H^{1/2}(\mathbf{R})}\|(I-Q_{l'-1}^{\mathbf{R}})u^{l'}\|_{H^{-1/2}(\mathbf{R})}\\ &\le C2^{l/2}\|u^l\|_{L^2(\mathbf{R})}C2^{-l'/2}\|u^{l'}\|_{L^2(\mathbf{R})}\\ &\le C\left(\frac{1}{\sqrt{2}}\right)^{l'-l}\|u^l\|_{L^2(\mathbf{R})}\|u^{l'}\|_{L^2(\mathbf{R})},\end{aligned}\tag{3.6}$$

and ii) for the case of the real axis is proved.

Now we consider $\mathcal{I}$. The second inequality in (3.4) follows easily from the corresponding estimate over the axis. To see the first, we choose a sufficiently large integer M and extend $u_j = \sum_{\sigma\in\Delta_j^{\mathcal{I}}} u_\sigma \psi_\sigma^{\text{even}}$ to the real axis by setting

$$u_j := \sum_{\sigma\in\Delta_j^{\mathbf{R}}} u_\sigma \psi_\sigma^{\mathbf{R}},$$

$$u_\sigma := \begin{cases} u_{\sigma-2m} & \text{if } 2m \le \sigma \le 2m+1 \\ & \text{and } m=-M,\dots,M, \\ u_{-\sigma+2m} & \text{if } 2m-1 \le \sigma \le 2m \\ & \text{and } m=-M,\dots,M, \\ 0 & \text{else.} \end{cases} \tag{3.7}$$

This function satisfies $u_j(s) = u_j(s-2m)$ for $0 < s < 1$ and $m = -M,\dots,M-1$, $u_j(s) = u(-s-2m)$ for $0 < s < 1$ and $m = -M+1,\dots,M$, and $u_j(s) = 0$ if $|s| \ge 2M+1$. The assertion i) for the real axis leads to

$$\begin{aligned} 4M\|u_j\|_{L^2(\mathcal{I})} + 2C\sqrt{\sum_{\sigma\in\Delta_j^{\mathcal{I}}} |u_\sigma|^2} &\ge \|u_j\|_{L^2(\mathbf{R})} \\ &\ge \frac{1}{C}2M\sqrt{\sum_{\sigma\in\Delta_j^{\mathcal{I}}} |u_\sigma|^2} \\ \|u_j\|_{L^2(\mathcal{I})} \ge \left\{\frac{1}{2C} - \frac{C}{2M}\right\}\sqrt{\sum_{\sigma\in\Delta_j^{\mathcal{I}}} |u_\sigma|^2}, \end{aligned} \tag{3.8}$$

which proves the first inequality of (3.4), i.e., the assertion i). Assertion ii) follows by similar arguments from the corresponding result over the axis. □

Similarly, we can define a wavelet basis in the subspace $S_{0,j}^{\mathcal{I}}$ of those functions of $S_j^{\mathcal{I}}$ which vanish at the end points 0 and 1. To this end we consider the space of "odd" functions over $\mathbf{R}$, i.e., the functions satisfying $f(s) = -f(-s) = -f(2-s)$ for $s \in [0,1]$. The correct basis functions for this space are the functions $s \mapsto \psi_\sigma(s) - \psi_{-\sigma}(s) - \psi_{2-\sigma}(s)$. If we restrict these to $\mathcal{I}$, we arrive at the wavelet basis $\{\psi_\sigma^{\mathrm{odd}} : \sigma \in \Delta_j^{\mathcal{I}} \backslash \{0,1\}\}$ defined by

$$(3.9) \qquad \psi_\sigma^{\mathrm{odd}} := \begin{cases} \varphi_{0,\sigma}^{\mathbf{R}}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_{-1}^{\mathcal{I}} \backslash \{0,1\} \\ \psi_\sigma^{\mathbf{R}}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } 0,1 \notin \operatorname{supp} \psi_\sigma^{\mathbf{R}}, \\ \{\psi_{h_{l+1}}^{\mathbf{R}} - \psi_{-h_{l+1}}^{\mathbf{R}}\}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } \sigma = h_{l+1}, \\ \{\psi_{1-h_{l+1}}^{\mathbf{R}} - \psi_{1+h_{l+1}}^{\mathbf{R}}\}|_{\mathcal{I}} & \text{if } \sigma \in \nabla_l^{\mathcal{I}},\ l \geq 0, \\ & \text{and } \sigma = 1 - h_{l+1}. \end{cases}$$

We denote the corresponding wavelet spaces span $\{\psi_\sigma^{\mathrm{odd}} : \sigma \in \nabla_l^{\mathcal{I}}\}$ by $W_{0,l}^{\mathcal{I}}$ and obtain $S_{0,j}^{\mathcal{I}} = \sum_{-1}^{j-1} W_{0,l}^{\mathcal{I}}$. Again, only those wavelets of level $l \geq 0$ have two vanishing moments for which the support is contained in the interior of $\mathcal{I}$. The wavelets of level $l \geq 0$ with support intersecting the boundary $\{0,1\}$ have no vanishing moment. The assertions of Lemma 3.1 hold also for the basis $\{\psi_\sigma^{\mathrm{odd}}\}$ and for the spaces $W_{0,l}^{\mathcal{I}}$.

We conclude this section with some results on the dual wavelet functions. For definiteness, we restrict our consideration to the dual wavelets of the wavelets $\psi_\sigma^{\mathcal{I}} := \psi_\sigma^{\mathrm{even}}$. From [**6**], cf. also [**36**, Lemma 3.5], we infer the existence of a dual scaling function $\tilde{\varphi}$ and a dual mother wavelet $\tilde{\psi}$. These functions $\tilde{\psi}$ and $\tilde{\varphi}$ belong to $H^{1/2+\varepsilon}$ for a certain $\varepsilon > 0$ and decay exponentially. Setting $\tilde{\varphi}_{l,\sigma}^{\mathbf{R}}(s) := \sqrt{N_l}\tilde{\varphi}(N_l \cdot [s-\sigma])$, $\sigma \in \Delta_l^{\mathbf{R}}, \tilde{\psi}_\sigma^{\mathbf{R}}(s) := \tilde{\varphi}_{0,\sigma}^{\mathbf{R}}$, $\sigma \in \nabla_{-1}^{\mathbf{R}}$, and $\tilde{\psi}_\sigma^{\mathbf{R}}(s) := \sqrt{N_l}\tilde{\psi}(N_l \cdot [s-\sigma])$, $\sigma \in \nabla_l^{\mathbf{R}}$, $l \geq 0$, we get the duality relations $\langle \psi_\sigma^{\mathbf{R}}, \tilde{\psi}_{\sigma'}^{\mathbf{R}} \rangle = \delta_{\sigma,\sigma'}$ and $\langle \varphi_{j,\sigma}^{\mathbf{R}}, \tilde{\varphi}_{j,\sigma'}^{\mathbf{R}} \rangle = \delta_{\sigma,\sigma'}$ for any $\sigma, \sigma' \in \Delta_j^{\mathbf{R}}$. Clearly, the projection $Q_j^{\mathbf{R}}$ onto $S_j^{\mathbf{R}}$ parallel to the closure of $\sum_{l=j}^{\infty} W_l^{\mathbf{R}}$ can be represented as

$$(3.10) \qquad Q_j^{\mathbf{R}} f(s) = \sum_{\sigma \in \Delta_j^{\mathbf{R}}} \langle \tilde{\varphi}_{j,\sigma}^{\mathbf{R}}, f \rangle \varphi_{j,\sigma}^{\mathbf{R}}(s) = \sum_{\sigma \in \Delta_j^{\mathbf{R}}} \langle \tilde{\psi}_\sigma^{\mathbf{R}}, f \rangle \psi_\sigma^{\mathbf{R}}(s).$$

These projections are uniformly bounded in $L^2(\mathbf{R})$ since $\{\psi_\sigma^{\mathbf{R}}\}$ is a Riesz basis. For the construction of dual wavelets over $\mathcal{I}$, we introduce the restriction operator $R : L^2_{\mathrm{loc}}(\mathbf{R}) \to L^2(\mathcal{I})$ by $Rf := f|_{\mathcal{I}}$, the prolongation operator $K : L^2(\mathcal{I}) \to L^2_{\mathrm{loc}}(\mathbf{R})$ and the L^2 adjoint operators R^*, K^* by

$$Kf(s) := \begin{cases} f(s-2m) & \text{if } 2m \le s \le 2m+1 \text{ and } m \in \mathbf{Z}, \\ f(-s+2m) & \text{if } 2m-1 \le s \le 2m \text{ and } m \in \mathbf{Z} \end{cases}$$

$$K^*g(s) := \sum_{m\in\mathbf{Z}} \{g(2m-s) + g(s+2m)\}, \tag{3.11}$$

$$R^*g(s) := \begin{cases} f(s) & \text{if } s \in \mathcal{I}, \\ 0 & \text{else.} \end{cases}$$

Now we define the dual elements over $\mathcal{I}$ by $\tilde{\varphi}^{\mathcal{I}}_{j,\sigma} := K^*\tilde{\varphi}^{\mathbf{R}}_{j,\sigma}$ and the dual wavelets by $\tilde{\psi}^{\mathcal{I}}_{\sigma} := K^*\tilde{\psi}^{\mathbf{R}}_{\sigma}$. It is not hard to obtain that $\langle\tilde{\psi}^{\mathcal{I}}_{\sigma}, \psi^{\mathcal{I}}_{\sigma'}\rangle = \langle\tilde{\psi}^{\mathbf{R}}_{\sigma}, K\psi^{\mathcal{I}}_{\sigma'}\rangle = \delta_{\sigma,\sigma'}$ and that $\langle\tilde{\varphi}^{\mathcal{I}}_{j,\sigma}, \varphi^{\mathcal{I}}_{j,\sigma'}\rangle = \delta_{\sigma,\sigma'}$ for any $\sigma, \sigma' \in \Delta^{\mathcal{I}}_j$. Moreover, the projection $Q^{\mathcal{I}}_j$ onto $S^{\mathcal{I}}_j$ parallel to the closure of $\sum_{l=j}^{\infty} W^{\mathcal{I}}_l$ can be represented as

$$Q^{\mathcal{I}}_j f(s) = \sum_{\tau\in\Delta^{\mathcal{I}}_j} \langle\tilde{\psi}^{\mathcal{I}}_{\tau}, f\rangle\psi^{\mathcal{I}}_{\tau}(s), \qquad Q^{\mathcal{I}}_j = RQ^{\mathbf{R}}_j K. \tag{3.12}$$

Analogously to Lemma 3.1, we get

Lemma 3.2. i) *There exists a constant $C > 0$ such that, for any j and any sequence $(u_\sigma)_{\sigma\in\Delta^{\mathcal{I}}_j}$, we get*

$$\begin{aligned} \frac{1}{C}\sqrt{\sum_{\sigma\in\Delta^{\mathcal{I}}_j} |u_\sigma|^2} &\le \Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} u_\sigma \tilde{\psi}^{\mathcal{I}}_{\sigma} \Big\|_{L^2(\mathcal{I})} \\ &\le C\sqrt{\sum_{\sigma\in\Delta^{\mathcal{I}}_j} |u_\sigma|^2}. \end{aligned} \tag{3.13}$$

ii) *There exist constants $C > 0$ and $0 < q < 1$ such that, for any $l < l'$, $u^l \in \operatorname{span}\{\tilde{\varphi}^{\mathcal{I}}_{l,\tau} : \tau \in \Delta^{\mathcal{I}}_l\}$ and $u^{l'} \in \operatorname{span}\{\tilde{\psi}^{\mathcal{I}}_{\tau} : \tau \in \nabla^{\mathcal{I}}_{l'}\}$, we get*

$$|\langle u^l, u^{l'}\rangle_{L^2(\mathcal{I})}| \le Cq^{l'-l}\|u^l\|_{L^2(\mathcal{I})}\|u^{l'}\|_{L^2(\mathcal{I})}. \tag{3.14}$$

Proof. Assertion i) is a simple consequence of a duality argument, of the duality relations between the basis $\{\tilde{\psi}^{\mathcal{I}}_{\sigma}\}$ and $\{\psi^{\mathcal{I}}_{\sigma}\}$ and of

Lemma 3.1 i). For assertion ii), we remark that it suffices to prove the inverse property and the approximation property for the space $\operatorname{span}\{\tilde{\varphi}^{\mathcal{I}}_{j,\tau} : \tau \in \Delta^{\mathcal{I}}_j\} = \operatorname{im}[Q^{\mathcal{I}}_j]^*$, compare (3.6). However, the estimate for the approximation error $f - [Q^{\mathcal{I}}_j]^* f$ in $H^{-\varepsilon}(\mathcal{I})$, $0 < \varepsilon < 1/2$ with f from $L^2(\mathcal{I})$ is equivalent to the well-known $L^2(\mathcal{I})$ estimate for $f - Q^{\mathcal{I}}_j f$ with f from $H^{\varepsilon}(\mathcal{I})$. Thus the approximation property is clear.

For the inverse property estimating the $H^{\varepsilon}(\mathcal{I})$ norm of $u_j \in \operatorname{im}[Q^{\mathcal{I}}_j]^*$ by $Ch_j^{-\varepsilon}$ times the L^2 norm of u_j, we consider $u_j = \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \tilde{\varphi}^{\mathcal{I}}_{j,\sigma}$ and set $\xi_{-\sigma} := \xi_\sigma$ as well as $\Delta^{[-1,1]}_j := \Delta^{\mathcal{I}}_j \cup -\Delta^{\mathcal{I}}_j$. We obtain

(3.15)
$$\sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \tilde{\varphi}^{\mathcal{I}}_{j,\sigma} = \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \{\tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(2m-s) + \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(s+2m)\},$$
$$\begin{aligned}\Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \tilde{\varphi}^{\mathcal{I}}_{j,\sigma} \Big\|_{H^{\varepsilon}(\mathcal{I})} &\le \Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(2m-s) \Big\|_{H^{\varepsilon}(\mathcal{I})} \\ &\quad + \Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(s+2m) \Big\|_{H^{\varepsilon}(\mathcal{I})} \\ &\le \Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,-\sigma}(2m+s) \Big\|_{H^{\varepsilon}([-1,0])} \\ &\quad + \Big\| \sum_{\sigma\in\Delta^{\mathcal{I}}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(s+2m) \Big\|_{H^{\varepsilon}([0,1])} \\ &\le \Big\| \sum_{\sigma\in\Delta^{[-1,1]}_j} \xi_\sigma \sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(s+2m) \Big\|_{H^{\varepsilon}([-1,1])}.\end{aligned}$$

The last norm can be estimated by standard techniques. Indeed, the H^{ε} norm of a function f over the periodic interval $[-1,1]$ is equivalent to $\{\sum_{k\in\mathbf{Z}} \max[|k|,1]^{2\varepsilon}|f_k|^2\}^{1/2}$, where the kth Fourier coefficient f_k of a function f is given by $f_k := (1/2)\int_{-1}^{1} f(s)e^{-i\pi sk}\,ds$. Using the norm equivalence, the formula

$$\Big[\sum_{m\in\mathbf{Z}} \tilde{\varphi}^{\mathbf{R}}_{j,\sigma}(2m-\cdot)\Big]_k = \frac{1}{2\sqrt{N_j}} e^{i\pi\sigma k}[\mathcal{F}\tilde{\varphi}]\Big(\frac{k}{2N_j}\Big),$$
$$[\mathcal{F}\tilde{\varphi}](s) := \int_{-\infty}^{\infty} \tilde{\varphi}(t)e^{i2\pi st}\,dt, \tag{3.16}$$

and the estimate, which follows from [**6**, Proposition 4.8] by choosing $L = 2$ and $k = 2$,

$$|\mathcal{F}\tilde{\varphi}(s)| \leq C \min\{1, |s|^{-1}\}, \tag{3.17}$$

it is not hard to obtain

$$\begin{aligned}
\Big\| \sum_{\sigma \in \Delta_j^{\mathcal{I}}} \xi_\sigma \tilde{\varphi}_{j,\sigma}^{\mathcal{I}} \Big\|_{H^\varepsilon(\mathcal{I})} &\leq C N_j^\varepsilon \sqrt{\sum_{\sigma \in \Delta_j^{[-1,1]}} |\xi_\sigma|^2} \\
&\leq C N_j^\varepsilon \sqrt{\sum_{\sigma \in \Delta_j^{\mathcal{I}}} |\xi_\sigma|^2} \\
&\leq C N_j^\varepsilon \Big\| \sum_{\sigma \in \Delta_j^{\mathcal{I}}} \xi_\sigma \tilde{\varphi}_{j,\sigma}^{\mathcal{I}} \Big\|_{L^2(\mathcal{I})},
\end{aligned} \tag{3.18}$$

where the last inequality follows analogously to (3.13). Thus, the inverse property is proved, too. □

3.2. *Wavelet functions over the square S and over Γ*. Our aim is to introduce wavelets over the surface Γ. These wavelets will be tensor products of the wavelets and scaling functions in the space $S_j^{\mathcal{I}}$ and $S_{0,j}^{\mathcal{I}}$, respectively. In the first step, we define wavelets as tensor products of functions from $S_j^{\mathcal{I}}$ and then, using the parametrization κ_1, we define functions over Γ_1. These functions are extended by a simple extension procedure to piecewise bilinear functions on Γ vanishing at the grid points of the other subdomains. For the basis over the neighbor Γ_2 of Γ_1, however, the linear functions on the common edge already belong to the span of basis functions of the first step. Thus we need a basis of functions vanishing at the common edge. In general, for any Γ_m to be considered in the further steps, we are given a certain set of edges on which the linear functions belong already to the span of wavelets of the previous steps, and we have to define basis functions vanishing over these edges. This will be realized by taking appropriate tensor products of functions from $S_j^{\mathcal{I}}$ and $S_{0,j}^{\mathcal{I}}$, respectively.

Now we turn to $\mathcal{S}$ and seek a basis of bilinear functions vanishing at the set of edges $\mathcal{E}$. Here $\mathcal{E}$ is an arbitrary but fixed subset of $\{e_j : j = 1, \dots, 4\}$ with $e_1 := [0,1] \times \{0\}$, $e_2 := [0,1] \times \{1\}$,

$e_3 := \{0\} \times [0,1]$ and $e_4 := \{1\} \times [0,1]$. We set

$$\psi^y_\sigma := \begin{cases} \psi^{\text{even}}_\sigma & \text{if } \sigma \le 1/2 \text{ and } e_1 \notin \mathcal{E} \\ \psi^{\text{odd}}_\sigma & \text{if } \sigma \le 1/2 \text{ and } e_1 \in \mathcal{E} \\ \psi^{\text{even}}_\sigma & \text{if } \sigma > 1/2 \text{ and } e_2 \notin \mathcal{E} \\ \psi^{\text{odd}}_\sigma & \text{if } \sigma > 1/2 \text{ and } e_2 \in \mathcal{E} \end{cases}$$
$$\psi^x_\sigma := \begin{cases} \psi^{\text{even}}_\sigma & \text{if } \sigma \le 1/2 \text{ and } e_3 \notin \mathcal{E} \\ \psi^{\text{odd}}_\sigma & \text{if } \sigma \le 1/2 \text{ and } e_3 \in \mathcal{E} \\ \psi^{\text{even}}_\sigma & \text{if } \sigma > 1/2 \text{ and } e_4 \notin \mathcal{E} \\ \psi^{\text{odd}}_\sigma & \text{if } \sigma > 1/2 \text{ and } e_4 \in \mathcal{E}. \end{cases} \tag{3.19}$$

Setting $\Delta^{\mathcal{S}}_l := \Delta^{\mathcal{I}}_l \times \Delta^{\mathcal{I}}_l$, $\Delta^{\mathcal{S},\mathcal{E}}_l := \Delta^{\mathcal{S}}_l \setminus \cup\, \mathcal{E}$, $\nabla^{\mathcal{S},\mathcal{E}}_{-1} := \Delta^{\mathcal{S},\mathcal{E}}_0$ as well as $\nabla^{\mathcal{S},\mathcal{E}}_l := \nabla^{\mathcal{S},\mathcal{E}}_{l+1} \setminus \Delta^{\mathcal{S},\mathcal{E}}_l$ if $l \ge 0$, we get

$$\begin{aligned} \nabla^{\mathcal{S},\mathcal{E}}_l &= \bigcup_{t=1}^{3} \nabla^{\mathcal{S},\mathcal{E}}_{t,l}, \\ \nabla^{\mathcal{S},\mathcal{E}}_{1,l} &:= \nabla^{\mathcal{I}}_l \times \Delta^{\mathcal{I}}_l \setminus \cup\, \mathcal{E}, \\ \nabla^{\mathcal{S},\mathcal{E}}_{2,l} &:= \Delta^{\mathcal{I}}_l \times \nabla^{\mathcal{I}}_l \setminus \cup\, \mathcal{E}, \\ \nabla^{\mathcal{S},\mathcal{E}}_{3,l} &:= \nabla^{\mathcal{I}}_l \times \nabla^{\mathcal{I}}_l \setminus \cup\, \mathcal{E}, \end{aligned} \tag{3.20}$$

for $l \ge 0$. The basis functions over $\mathcal{S}$ are defined as

$$\psi^{\mathcal{S}}_\tau(t_1,t_2) := \begin{cases} \varphi^{\mathcal{I}}_{0,\tau_1}(t_1)\varphi^{\mathcal{I}}_{0,\tau_2}(t_2) & \text{if } \tau = (\tau_1,\tau_2) \in \nabla^{\mathcal{S},\mathcal{E}}_{-1} \\ \psi^x_{\tau_1}(t_1)\varphi^{\mathcal{I}}_{l,\tau_2}(t_2) & \text{if } l \ge 0 \text{ and } \tau = (\tau_1,\tau_2) \in \nabla^{\mathcal{S},\mathcal{E}}_{1,l} \\ \varphi^{\mathcal{I}}_{l,\tau_1}(t_1)\psi^y_{\tau_2}(t_2) & \text{if } l \ge 0 \text{ and } \tau = (\tau_1,\tau_2) \in \nabla^{\mathcal{S},\mathcal{E}}_{2,l} \\ \psi^x_{\tau_1}(t_1)\psi^y_{\tau_2}(t_2) & \text{if } l \ge 0 \text{ and } \tau = (\tau_1,\tau_2) \in \nabla^{\mathcal{S},\mathcal{E}}_{3,l}. \end{cases} \tag{3.21}$$

Clearly, the functions $\{\psi^{\mathcal{S}}_\tau : \tau \in \Delta^{\mathcal{S},\mathcal{E}}_j\}$ span the space $S^{\mathcal{S},\mathcal{E}}_j$ of all bilinear functions of $S^{\mathcal{S}}_j$ which vanish over the edge points of $\cup\mathcal{E}$. We get $S^{\mathcal{S},\mathcal{E}}_j = \sum_{l=-1}^{j-1} W^{\mathcal{S},\mathcal{E}}_l$ where $W^{\mathcal{S},\mathcal{E}}_l := \operatorname{span}\{\psi^{\mathcal{S}}_\tau : \tau \in \nabla^{\mathcal{S},\mathcal{E}}_l\}$.

Besides these basis functions we also need the simple extension procedure mentioned in the beginning of this section. We retain the definition of the finite element functions $\varphi_\tau^{\mathcal{S}}$ from Section 2.2. For a moment, however, we write $\varphi_{j,\tau}^{\mathcal{S}} := \varphi_\tau^{\mathcal{S}}$ in order to indicate the dependence on the level j. The trace of a bilinear function of $S_j^{\mathcal{S}}$ on the edge is a linear function. If the bilinear function belongs to $W_l^{\mathcal{S},\mathcal{E}}$, then the trace on the edge is a piecewise linear function over the restriction of $\Delta_{l+1}^{\mathcal{S}}$ to the edge. Thus, suppose we are given a function f over the union of the edges in $\mathcal{E}$ which is piecewise linear over the uniform grid $\Delta_{l+1}^{\mathcal{S}}|_{\cup\mathcal{E}}$. Then we denote by $\mathcal{P}_l f$ the function

$$\mathcal{P}_l f(t) := \sum_{\tau \in \Delta_{l+1}^{\mathcal{S}} \cap \cup\mathcal{E}} \frac{f(\tau)}{\varphi_{l+1,\tau}^{\mathcal{S}}(\tau)} \varphi_{l+1,\tau}^{\mathcal{S}}(t), \tag{3.22}$$

i.e., the unique piecewise bilinear prolongation of f to a function in $S_{l+1}^{\mathcal{S}}$ which vanishes over the grid points of $\Delta_{l+1}^{\mathcal{S},\mathcal{E}}$.

Now we turn to Γ. We suppose that the Γ_m, $m = 1, \dots, m_\Gamma$ are given in such an order that, for any $2 \le m \le m_\Gamma$, each vertex of the subdomain Γ_m belongs to an edge common with $\cup_{m'=1}^{m-1}\Gamma_{m'}$ or does not belong to $\cup_{m'=1}^{m-1}\Gamma_{m'}$. To each m with $1 \le m \le m_\Gamma$ there belongs a possibly empty set $\mathcal{E}_m \subseteq \{e_j : j = 1, \dots, 4\}$ such that $\{\kappa_m(e) : e \in \mathcal{E}_m\}$ are just the edges which are contained in $\cup_{m'=1}^{m-1}\Gamma_{m'}$. Obviously, we have $\Delta_j = \cup_{m=1}^{m_\Gamma}\kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m})$. To define the wavelet basis over Γ we first set

$$\hat{\psi}_\xi(x) := \begin{cases} \psi_\tau^{\mathcal{S}}(t) & \text{if } \xi = \kappa_m(\tau) \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m}) \\ & \text{and if } x = \kappa_m(t) \\ 0 & \text{else.} \end{cases} \tag{3.23}$$

For $m' > m$ and $\xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m}) \cap \kappa_{m'}(\cup\mathcal{E}_{m'})$, however, the function $\hat{\psi}_\xi$ vanishes over the interior of $\Gamma_{m'}$ and does not vanish over the common edge $\Gamma_m \cap \Gamma_{m'}$. The same kind of discontinuity along an edge occurs also for wavelet functions $\hat{\psi}_\xi$ with ξ in the interior of Γ_m but close to the common edge, i.e., if $\xi = \kappa_m(\tau) \in \kappa_m(\nabla_l^{\mathcal{S},\mathcal{E}_m})$, if $\kappa_m(e) = \Gamma_m \cap \Gamma_{m'}$, and if the distance of τ to e is equal to h_{l+1}. To get a continuous function from S_j, we extend the traces from the edge to a bilinear function over

$\Gamma_{m'}$. Finally, we arrive at

(3.24)
$$\psi_\xi(x) := \begin{cases} \hat{\psi}_\xi(x) & \text{if } \xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m}) \\ & \text{and if } x \in \Gamma_m, \\ [\mathcal{P}_l(\hat{\psi}_\xi \circ \kappa_{m'}|_{\cup\mathcal{E}_{m'}})](t) & \text{if } \xi = \kappa_m(\tau) \in \kappa_m(\nabla_l^{\mathcal{S},\mathcal{E}_m}), \\ & x \in \Gamma_{m'}, \\ & \text{and } \Gamma_m \cap \Gamma_{m'} = \kappa_m(e), \\ & \operatorname{dist}(\tau, e) \leq h_{l+1} \\ 0 & \text{else.} \end{cases}$$

Clearly the functions $\{\psi_\xi : \xi \in \Delta_j\}$ span the space of all bilinear functions of S_j. The functions ψ_ξ have two vanishing moments whenever $\xi \in \Delta_j \backslash \Delta_0$ and the support $\operatorname{supp} \psi_\xi$ is contained in the interior of Γ_m. Note that two vanishing moments mean that the ψ_ξ are orthogonal to "polynomials" of degree less than two, i.e., $\langle \psi_\xi, f \rangle = 0$ for any bilinear polynomial $f \circ \kappa_m$ over $\mathcal{S}$. The scalar product $\langle \cdot, \cdot \rangle$ is defined by

$$\langle f, g \rangle := \sum_{m=1}^{m_\Gamma} \int_{\mathcal{S}} f(\kappa_m(t)) \overline{g(\kappa_m(t))} \, dt.$$

Furthermore, the ψ_ξ satisfy the following properties:

Lemma 3.3. i) *There exists a constant $C > 0$ such that, for any j and any sequence $(u_\xi)_{\xi \in \Delta_j}$, we get*

$$\frac{1}{C} \sqrt{\sum_{\xi \in \Delta_j} |u_\xi|^2} \leq \Big\| \sum_{\xi \in \Delta_j} u_\xi \psi_\xi \Big\|_{L^2(\Gamma)} \leq C \sqrt{\sum_{\xi \in \Delta_j} |u_\xi|^2}. \tag{3.25}$$

ii) *There exists a constant $C > 0$ such that the coefficients f_ξ of the piecewise bilinear interpolant $P_j f = \sum_{\xi \in \Delta_j} f_\xi \psi_\xi$ to an arbitrary function f from the Sobolev space $H^2(\Gamma)$ satisfy*

$$\sqrt{\sum_{\xi \in \Delta_j} 2^{4l} |f_\xi|^2} \leq C \sqrt{j} \|f\|_{H^2(\Gamma)}, \tag{3.26}$$

where $l = l(\xi)$ denotes the level of ξ, i.e., $\xi \in \nabla_l := \Delta_{l+1} \backslash \Delta_l$.

Proof. i) First we consider the square $\mathcal{S}$ and the space $S_j^{\mathcal{S},\mathcal{E}}$. For these, we will show

$$\begin{aligned}\frac{1}{C}\sqrt{\sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}}|u_\tau|^2} &\leq \Big\|\sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}} u_\tau\psi_\tau^{\mathcal{S}}\Big\|_{L_2(s)} \\ &\leq C\sqrt{\sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}}|u_\tau|^2}.\end{aligned} \tag{3.27}$$

We set $u^l := \sum_{\tau\in\nabla_l^{\mathcal{S},\mathcal{E}}} u_\tau\psi_\tau^{\mathcal{S}}$ and prove

$$|\langle u^l, u^{l'}\rangle_{L^2(\mathcal{S})}| \leq Cq^{|l'-l|}\|u^l\|_{L^2(\mathcal{S})}\|u^{l'}\|_{L^2(\mathcal{S})}, \tag{3.28}$$

where q is a fixed constant less than one. To simplify the formulae, we assume that $l < l'$, that $u^l := \sum_{\tau\in\nabla_{1,l}^{\mathcal{S},\mathcal{E}}} u_\tau\psi_\tau^{\mathcal{S}}$, and that $u^{l'} := \sum_{\tau\in\nabla_{3,l}^{\mathcal{S},\mathcal{E}}} u_\tau\psi_\tau^{\mathcal{S}}$. From Lemma 3.1 ii) and i) we conclude

(3.29)

$$\begin{aligned}\langle u^l, u^{l'}\rangle &= \int_0^1 \sum_{\tau_1,\tau_1'} \psi_{\tau_1}^x(t_1)\psi_{\tau_1'}^x(t_1)\int_0^1\Big[\sum_{\tau_2} u_{(\tau_1,\tau_2)}\varphi_{l,\tau_2}^{\mathcal{I}}(t_2)\Big] \\ &\qquad\cdot\Big[\sum_{\tau_2'} u_{(\tau_1',\tau_2')}\psi_{\tau_2'}^y(t_2)\Big]\,dt_2\,dt_1 \\ |\langle u^l, u^{l'}\rangle| &\leq Cq^{l'-l}\int_0^1\sum_{\tau_1,\tau_1'}\Big\|\sum_{\tau_2} u_{(\tau_1,\tau_2)}\varphi_{l,\tau_2}^{\mathcal{I}}\Big\| \\ &\qquad\cdot|\psi_{\tau_1}^x(t_1)|\Big\|\sum_{\tau_2'} u_{(\tau_1',\tau_2')}\psi_{\tau_2'}^y\Big\||\psi_{\tau_1'}^x(t_1)|\,dt_1 \\ &\leq Cq^{l'-l}\int_0^1\sum_{\tau_1,\tau_1'}\sqrt{\sum_{\tau_2}|u_{(\tau_1,\tau_2)}|^2}|\psi_{\tau_1}^x(t_1)| \\ &\qquad\cdot\sqrt{\sum_{\tau_2'}|u_{(\tau_1',\tau_2')}|^2}|\psi_{\tau_1'}^x(t_1)|\,dt_1.\end{aligned}$$

We observe that (3.4) holds also if the $\psi_\sigma^{\text{even}}$ are replaced by $|\psi_\sigma^x|$, by $|\psi_\sigma^y|$ or by $|\varphi_{l,\sigma}^{\mathcal{I}}|$ if the summation runs over functions of a fixed level.

Using this, we arrive at

$$|\langle u^l, u^{l'}\rangle| \le C q^{l'-l} \sqrt{\sum_{(\tau_1,\tau_2)} |u_{(\tau_1,\tau_2)}|^2} \sqrt{\sum_{(\tau_1',\tau_2')} |u_{(\tau_1',\tau_2')}|^2}. \tag{3.30}$$

On the other hand, Lemma 3.1 i) and the well-known analogue for the finite element functions imply

$$\begin{aligned}\int_{\mathcal{S}} |u^l(t)|^2\, dt &= \int_0^1 \int_0^1 \Big| \sum_{(\tau_1,\tau_2)} u_{(\tau_1,\tau_2)} \psi^x_{\tau_1}(t_1) \varphi^{\mathcal{I}}_{l,\tau_2}(t_2)\Big|^2 dt_1\, dt_2 \\ &= \int_0^1 \int_0^1 \Big| \sum_{\tau_1} \Big[\sum_{\tau_2} u_{(\tau_1,\tau_2)} \varphi^{\mathcal{I}}_{l,\tau_2}(t_2)\Big] \psi^x_{\tau_1}(t_1)\Big|^2 dt_1\, dt_2 \\ &\sim \int_0^1 \sum_{\tau_1} \Big| \sum_{\tau_2} u_{(\tau_1,\tau_2)} \varphi^{\mathcal{I}}_{l,\tau_2}(t_2)\Big|^2 dt_2 \\ &= \sum_{\tau_1} \int_0^1 \Big| \sum_{\tau_2} u_{(\tau_1,\tau_2)} \varphi^{\mathcal{I}}_{l,\tau_2}(t_2)\Big|^2 dt_2 \\ &\sim \sum_{(\tau_1,\tau_2)} |u_{(\tau_1,\tau_2)}|^2.\end{aligned} \tag{3.31}$$

Here the symbol $\sim$ means that the lefthand side is less than constant times the righthand side and vice versa. Relation (3.31), the analogues result for $u^{l'}$, and (3.30) prove (3.28). The estimates (3.28) and (3.31), however, imply

$$\begin{aligned}\Big\| \sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}} u_\tau \psi_\tau^{\mathcal{S},\mathcal{E}} \Big\|^2 &= \Big\langle \sum_{l=-1}^{j-1} u^l, \sum_{l'=-1}^{j-1} u^{l'} \Big\rangle \\ &= \sum_{l,l'=-1}^{j-1} \langle u^l, u^{l'}\rangle \\ \Big\| \sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}} u_\tau \psi_\tau^{\mathcal{S},\mathcal{E}} \Big\|^2 &\le C \sum_{l,l'=-1}^{j-1} q^{|l-l'|} \|u^l\|\, \|u^{l'}\| \\ &\le C \sum_{l=-1}^{j-1} \|u^l\|^2 \le C \sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}} |u_\tau|^2\end{aligned} \tag{3.32}$$

which proves the upper estimate in (3.27).

To get the lower estimate, we consider the dual wavelets

$$\tilde{\psi}_\tau^{\mathcal{S}}(t_1,t_2) := \begin{cases} \tilde{\varphi}_{0,\tau_1}^{\mathcal{I}}(t_1)\tilde{\varphi}_{0,\tau_2}^{\mathcal{I}}(t_2) & \text{if } \tau=(\tau_1,\tau_2)\in\nabla_{-1}^{\mathcal{S},\mathcal{E}} \\ \tilde{\psi}_{\tau_1}^{x}(t_1)\tilde{\varphi}_{l,\tau_2}^{\mathcal{I}}(t_2) & \text{if } l\geq 0 \\ & \text{and } \tau=(\tau_1,\tau_2)\in\nabla_{1,l}^{\mathcal{S},\mathcal{E}}, \\ \tilde{\varphi}_{l,\tau_1}^{\mathcal{I}}(t_1)\tilde{\psi}_{\tau_2}^{y}(t_2) & \text{if } l\geq 0 \\ & \text{and } \tau=(\tau_1,\tau_2)\in\nabla_{2,l}^{\mathcal{S},\mathcal{E}} \\ \tilde{\psi}_{\tau_1}^{x}(t_1)\tilde{\psi}_{\tau_2}^{y}(t_2) & \text{if } l\geq 0 \\ & \text{and } \tau=(\tau_1,\tau_2)\in\nabla_{3,l}^{\mathcal{S},\mathcal{E}}, \end{cases} \tag{3.33}$$

where the $\tilde{\varphi}_{l,\sigma}^{\mathcal{I}}$, $\tilde{\psi}_\sigma^x$ and the $\tilde{\psi}_\sigma^y$ are the univariate dual functions to the functions $\varphi_{l,\sigma}^{\mathcal{I}}$, ψ_σ^x and ψ_σ^y, respectively, cf., the end of Section 3.1. The univariate duality relations $\langle\tilde{\psi}_\sigma^x,\psi_{\sigma'}^x\rangle=\delta_{\sigma,\sigma'}$, $\langle\tilde{\psi}_\sigma^y,\psi_{\sigma'}^y\rangle=\delta_{\sigma,\sigma'}$ and $\langle\tilde{\varphi}_{j,\sigma}^{\mathcal{I}},\varphi_{j,\sigma'}^{\mathcal{I}}\rangle=\delta_{\sigma,\sigma'}$ imply the duality relations $\langle\tilde{\psi}_\tau^{\mathcal{S}},\psi_\tau^{\mathcal{S}}\rangle=\delta_{\tau,\tau'}$ over $\mathcal{S}$. Applying the arguments leading to the upper estimate of (3.27) to the dual system, we get

$$\Big\|\sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}} v_\tau\tilde{\psi}_\tau^{\mathcal{S}}\Big\|_{L^2(\mathcal{S})} \leq C\sqrt{\sum_{\tau\in\Delta_j^{\mathcal{S},\mathcal{E}}}|v_\tau|^2}. \tag{3.34}$$

Consequently,

(3.35)

$$\begin{aligned}\Big\|\sum_{\tau\in\Delta_j^{\mathcal{S}}} u_\tau\psi_\tau^{\mathcal{S}}\Big\|_{L^2(\mathcal{S})} &\geq \sup_{\|\sum_{\tau\in\Delta_j^{\mathcal{S}}} v_\tau\tilde{\psi}_\tau^{\mathcal{S}}\|\leq 1}\Big\langle \sum_{\tau\in\Delta_j^{\mathcal{S}}} u_\tau\psi_\tau^{\mathcal{S}},\sum_{\tau\in\Delta_j^{\mathcal{S}}} v_\tau\tilde{\psi}_\tau^{\mathcal{S}}\Big\rangle \\ &\geq \sup_{\sqrt{\sum_{\tau\in\Delta_j^{\mathcal{S}}}|v_\tau|^2}\leq C^{-1}}\Big|\sum_{\tau\in\Delta_j^{\mathcal{S}}} u_\tau\overline{v_\tau}\Big| \\ &\geq C^{-1}\sqrt{\sum_{\tau\in\Delta_j^{\mathcal{S}}}|u_\tau|^2},\end{aligned}$$

and (3.27) is proved.

For the proof of (3.25), we observe that the piecewise bilinear prolongation $\mathcal{P}_l f$ of a univariate function f of level l defined over an edge is the tensor product of this f times the finite element $\varphi^{\mathbf{R}}_{l+1,0}|_{\mathcal{I}}$ or $\varphi^{\mathbf{R}}_{l+1,1}|_{\mathcal{I}}$. Using

$$|\langle \varphi^{\mathbf{R}}_{l+1,0}|_{\mathcal{I}}, \varphi^{\mathbf{R}}_{l'+1,0}|_{\mathcal{I}}\rangle| \le C 2^{-|l-l'|/2} \tag{3.36}$$

and (3.4) and repeating the arguments leading to (3.32), we arrive at

$$\Big\| \sum_{\xi \in \kappa_{m'}(\Delta_j^{\mathcal{S},\mathcal{E}_{m'}})} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_m)} \le C \sqrt{\sum_{\xi \in \kappa_{m'}(\Delta_j^{\mathcal{S},\mathcal{E}_{m'}})} |u_\xi|^2}. \tag{3.37}$$

From this and (3.27), the upper estimate in (3.25) follows easily. To get the lower estimate, we conclude from (3.27) that

$$\begin{aligned} \sqrt{\sum_{\xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m})} |u_\xi|^2} &\le C \Big\| \sum_{\xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m})} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_m)} \\ &\le C \Big\| \sum_{\xi \in \Delta_j} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_m)} \\ &\quad + C \sum_{m'=1}^{m-1} \Big\| \sum_{\xi \in \kappa_{m'}(\Delta_j^{\mathcal{S},\mathcal{E}_{m'}})} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_m)}. \end{aligned} \tag{3.38}$$

Using the just proved upper bound (3.37), we continue

$$\begin{aligned} \sqrt{\sum_{\xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m})} |u_\xi|^2} &\le C \Big\| \sum_{\xi \in \Delta_j} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_m)} \\ &\quad + C \sum_{m'=1}^{m-1} \sqrt{\sum_{\xi \in \kappa_{m'}(\Delta_j^{\mathcal{S},\mathcal{E}_{m'}})} |u_\xi|^2}. \end{aligned} \tag{3.39}$$

Now we substitute (3.39) with $m = 1$ into the righthand side of (3.39) with $m = 2$, substitute the resulting inequality into the righthand side of (3.39) with $m = 3$, substitute the obtained inequality into the righthand side of (3.39) with $m = 4$, and so on. For $m = 1, \dots, m_\Gamma$, we arrive at

$$\sqrt{\sum_{\xi \in \kappa_m(\Delta_j^{\mathcal{S},\mathcal{E}_m})} |u_\xi|^2} \le C \sum_{m'=1}^{m} \Big\| \sum_{\xi \in \Delta_j} u_\xi \psi_\xi \Big\|_{L^2(\Gamma_{m'})}. \tag{3.40}$$

Summing up over all m, we obtain the lower estimate of (3.25).

ii) First we recall the well-known estimate

$$\|f - P_j f\|_{L^2(\Gamma)} \le C h_j^2 \|f\|_{H^2(\Gamma)} \tag{3.41}$$

for the interpolation projection P_j unto the piecewise bilinear functions. Here the norm $\|\cdot\|_{H^2(\Gamma)}$ is the sum of the H^2 Sobolev norms over the subsurfaces Γ_m, $m = 1, \ldots, m_\Gamma$. Now we consider the complementary space $S_j^{\text{compl}} := \operatorname{cl}\operatorname{span}\{\psi_\xi : \xi \in \Delta_{j'} \backslash \Delta_j, j' > j\}$ of S_j and denote the projection of $L^2(\Gamma)$ onto S_j with null space S_j^{compl} by Q_j. From i) we conclude that Q_j is uniformly bounded with respect to j. In view of (3.41), we get

$$\begin{aligned} \|f - Q_j f\|_{L^2(\Gamma)} &\le C h_j^2 \|f\|_{H^2(\Gamma)}, \\ \|(Q_l - Q_{l-1}) f\|_{L^2(\Gamma)} &\le C h_l^2 \|f\|_{H^2(\Gamma)}. \end{aligned} \tag{3.42}$$

We set $Q_j f = \sum \tilde{f}_\xi \psi_\xi$. Together with (3.25) we arrive at

$$\begin{aligned} \sqrt{\sum_{\xi \in \Delta_j : l(\xi) = l} |\tilde{f}_\xi|^2} &\le C 2^{-2l} \|f\|_{H^2(\Gamma)}, \\ \sqrt{\sum_{\xi \in \Delta_j} 2^{4l(\xi)} |\tilde{f}_\xi|^2} &\le C \sqrt{j} \|f\|_{H^2(\Gamma)}. \end{aligned} \tag{3.43}$$

In order to derive (3.26), with the help of (3.25), (3.41) and (3.42), we conclude that

$$\begin{aligned} \sqrt{\sum_{\xi \in \Delta_j} |\tilde{f}_\xi - f_\xi|^2} &\le \|Q_j f - P_j f\|_{L^2(\Gamma)} \\ &\le C h_j^2 \|f\|_{H^2(\Gamma)} \\ &\le C 2^{-2j} \|f\|_{H^2(\Gamma)}. \end{aligned} \tag{3.44}$$

Together with inequality (3.43) we arrive at

$$
\begin{aligned}
\sqrt{\sum_{\xi\in\Delta_j} 2^{4l(\xi)}|f_\xi|^2} &\leq \sqrt{\sum_{\xi\in\Delta_j} 2^{4l(\xi)}|\tilde{f}_\xi - f_\xi|^2} \\
&\quad + \sqrt{\sum_{\xi\in\Delta_j} 2^{4l(\xi)}|\tilde{f}_\xi|^2} \\
&\leq 2^{2j}\sqrt{\sum_{\xi\in\Delta_j} |\tilde{f}_\xi - f_\xi|^2} + C\sqrt{j}\|f\|_{H^2(\Gamma)} \\
&\leq C\sqrt{j}\|f\|_{H^2(\Gamma)}. \qquad \square
\end{aligned} \tag{3.45}
$$

Note that, if $\varphi_{j,\xi} := \varphi_\xi$ denotes the finite element function of Section 2.2, then there holds

$$
\frac{1}{C}\sqrt{\sum_{\xi\in\Delta_j} |v_\xi|^2} \leq \Big\| \sum_{\xi\in\Delta_j} v_\xi \varphi_{j,\xi} \Big\|_{L^2(\Gamma)} \leq C\sqrt{\sum_{\xi\in\Delta_j} |v_\xi|^2}. \tag{3.46}
$$

By E_j we denote the wavelet transform, i.e., the basis transform mapping the vector $(v_\xi)_{\xi\in\Delta_j}$ of coefficients v_ξ of a function $u_j \in S_j$ with respect to the basis $\{\varphi_{j,\xi}\}$ to the vector $(u_\xi)_{\xi\in\Delta_j}$ of coefficients u_ξ with respect to the basis $\{\psi_\xi\}$. Then Lemma 3.3 i) implies that E_j is invertible and that the l^2 operator norms of E_j and E_j^{-1} are uniformly bounded with respect to j. Finally, we remark that the application of E_j and E_j^{-1} can be realized by fast pyramid algorithms, cf. [**16, 4**]. For one application of E_j or E_j^{-1}, no more than $O(N_j^2)$ arithmetic operations are required.

3.3 *The wavelet test functionals.* Similarly to the new wavelet basis ψ_ξ in the trial space S_j, we can introduce a "wavelet" basis for the space of test functionals. Note that, in view of (2.6), the space of test functionals is spanned by the Dirac delta functionals δ_ξ, $\xi \in \Delta_j$, where $\delta_\xi(f) := f(\xi)$. The wavelet functionals will be linear combinations of the delta functionals. To introduce wavelet functionals, we first consider the square $\mathcal{S}$. Analogously to (3.20), we set $\nabla^{\mathcal{S}}_{-1} := \Delta^{\mathcal{S}}_0$ and

$$\begin{aligned} \nabla^{\mathcal{S}}_l &= \bigcup_{t=1}^{3} \nabla^{\mathcal{S}}_{t,l}, \\ \nabla^{\mathcal{S}}_{1,l} &:= \nabla^{\mathcal{I}}_l \times \Delta^{\mathcal{I}}_l, \\ \nabla^{\mathcal{S}}_{2,l} &:= \Delta^{\mathcal{I}}_l \times \nabla^{\mathcal{I}}_l, \\ \nabla^{\mathcal{S}}_{3,l} &:= \nabla^{\mathcal{I}}_l \times \nabla^{\mathcal{I}}_l, \end{aligned} \tag{3.47}$$

for $l \geq 0$. The basis functionals $\vartheta^{\mathcal{S}}_\tau$, $\tau = (\tau_1, \tau_2) \in \Delta^{\mathcal{S}}_j$ over $\mathcal{S}$ are defined by

$$\vartheta^{\mathcal{S}}_\tau(f) := \begin{cases} f(\tau)/N_0 & \text{if } \tau \in \nabla^{\mathcal{S}}_{-1}, \\ (f(\tau) - (1/2)\{f(\tau_1 - h_{l+1}, \tau_2) & \\ +f(\tau_1 + h_{l+1}, \tau_2)\})/N_l & \text{if } \tau \in \nabla^{\mathcal{S}}_{1,l} \\ & \text{and } l \geq 0, \\ (f(\tau) - (1/2)\{f(\tau_1, \tau_2 - h_{l+1}) & \\ +f(\tau_1, \tau_2 + h_{l+1})\})/N_l & \text{if } \tau \in \nabla^{\mathcal{S}}_{2,l} \cup \nabla^{\mathcal{S}}_{3,l} \\ & \text{and } l \geq 0. \end{cases} \tag{3.48}$$

Since the points $(\tau_1 \pm h_{l+1}, \tau_2)$ belong to $\Delta^{\mathcal{S}}_l$ for $\tau \in \nabla^{\mathcal{S}}_{1,l}$, we easily get that the span of $\{\vartheta^{\mathcal{S}}_\tau : \tau \in \nabla^{\mathcal{S}}_{1,l}\} \cup \{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_l\}$ is equal to the span of $\{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_l \cup \nabla^{\mathcal{S}}_{1,l}\}$. Similarly, for $\tau \in \nabla^{\mathcal{S}}_{2,l} \cup \nabla^{\mathcal{S}}_{3,l}$, the points $(\tau_1, \tau_2 \pm h_{l+1})$ belong to $\Delta^{\mathcal{S}}_l \cup \nabla^{\mathcal{S}}_{1,l}$, and the span of $\{\vartheta^{\mathcal{S}}_\tau : \tau \in \nabla^{\mathcal{S}}_{2,l} \cup \nabla^{\mathcal{S}}_{3,l}\} \cup \{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_l \cup \nabla^{\mathcal{S}}_{1,l}\}$ is equal to the span of $\{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_l \cup \nabla^{\mathcal{S}}_l\}$. Thus, the span of $\{\vartheta^{\mathcal{S}}_\tau : \tau \in \nabla^{\mathcal{S}}_l\} \cup \{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_l\}$ is equal to the span of $\{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_{l+1}\}$ and we have $\operatorname{span}\{\delta_\tau : \tau \in \Delta^{\mathcal{S}}_j\} = \operatorname{span}\{\vartheta^{\mathcal{S}}_\tau : \tau \in \Delta^{\mathcal{S}}_j\}$. Now the functionals ϑ_ξ, $\xi \in \Delta_j$ over Γ are defined by $\vartheta_\xi(f) := \vartheta^{\mathcal{S}}_\tau(f \circ \kappa_m)$ where $\xi = \kappa_m(\tau)$ and $\tau \in \Delta^{\mathcal{S}}_j$. Clearly, $\operatorname{span}\{\delta_\xi : \xi \in \Delta_j\} = \operatorname{span}\{\vartheta_\xi : \xi \in \Delta_j\}$.

To prepare the analysis of the corresponding wavelet transform, we introduce the dual wavelet basis which is some sort of hierarchical basis. We write $t = (t_1, t_2)$ and $\tau = (\tau_1, \tau_2)$, retain the notation of $\varphi^{\mathcal{I}}_{l,\sigma}$ from Section 3.1 and set

$$\chi^{\mathcal{S}}_\tau(t) := \begin{cases} \varphi^{\mathcal{I}}_{0,\tau_1}(t_1)\varphi^{\mathcal{I}}_{0,\tau_2}(t_2) & \text{if } \tau \in \nabla^{\mathcal{S}}_{-1} \\ \varphi^{\mathcal{I}}_{l+1,\tau_1}(t_1)\varphi^{\mathcal{I}}_{l,\tau_2}(t_2) & \text{if } \tau \in \nabla^{\mathcal{S}}_{1,l} \text{ and } l \geq 0 \\ \varphi^{\mathcal{I}}_{l+1,\tau_1}(t_1)\varphi^{\mathcal{I}}_{l+1,\tau_2}(t_2) & \text{if } \tau \in \nabla^{\mathcal{S}}_{2,l} \cup \nabla^{\mathcal{S}}_{3,l} \\ & \text{and } l \geq 0. \end{cases} \tag{3.49}$$

These functions satisfy $\vartheta_\tau^{\mathcal{S}}(\chi_{\tau'}^{\mathcal{S}}) = \delta_{\tau,\tau'}$. Now the dual functions χ_ξ, $\xi \in \Delta_j$ over Γ are defined by $\chi_\xi(\kappa_m(t)) := \chi_\tau^{\mathcal{S}}(t)$ where $\xi = \kappa_m(\tau)$, $\tau \in \Delta_j^{\mathcal{S}}$ and $t \in \mathcal{S}$. Clearly we get $\vartheta_\xi(\chi_{\xi'}) = \delta_{\xi,\xi'}$ for any $\xi, \xi' \in \Delta_j$, and the interpolation projection P_j of (2.7) admits the representation

$$P_j f = \sum_{\xi \in \Delta_j} h_j f(\xi) \varphi_{j,\xi} = \sum_{\xi \in \Delta_j} \vartheta_\xi(f) \chi_\xi. \tag{3.50}$$

Now we introduce the "wavelet" transform R_j mapping a vector of functional values $(\vartheta_\xi(f))_{\xi \in \Delta_j}$ into the vector of function values $(h_j f(\xi))_{\xi \in \Delta_j}$. This is nothing else than the basis transform mapping the vector $(u_\xi)_{\xi \in \Delta}$ of coefficients u_ξ of a function $u_j \in S_j$ with respect to the basis $\{\chi_\xi\}$ to the vector $(v_\xi)_{\xi \in \Delta}$ of coefficients v_ξ with respect to the basis $\{\varphi_{j,\xi}\}$. Though we have the norm equivalence (3.46) for the functions $\varphi_{j,\xi}$, the estimate (3.25) with ψ_ξ replaced by χ_ξ is not true and the l^2 operator norms of R_j and R_j^{-1}, respectively, are not uniformly bounded anymore. Instead of (3.25) we have the following result.

Lemma 3.4. *There exists a constant $C > 0$ such that, for any j, we get*

$$\begin{aligned} C^{-1}\sqrt{j} \le \|R_j\|_{\mathcal{L}(l^2(\Delta_j))} \le C\sqrt{j}, \\ C^{-1}2^j \le \|R_j^{-1}\|_{\mathcal{L}(l^2(\Delta_j))} \le C2^j. \end{aligned} \tag{3.51}$$

Proof. Setting $u_j = \sum_{\xi \in \Delta_j} v_\xi \varphi_{j,\xi} = \sum_{\xi \in \Delta_j} u_\xi \chi_\xi$ as well as $u := (u_\xi)_{\xi \in \Delta_j}$, $v := (v_\xi)_{\xi \in \Delta_j}$, we get $R_j u = v$. From (3.50), we infer

$$v_\xi = h u_j(\xi) = \sum_{\xi' \in \Delta_j} u_{\xi'} h \chi_{\xi'}(\xi). \tag{3.52}$$

The last sum contains no more than $C \cdot j$ terms different from zero and each term can be estimated by

$$|u_{\xi'}| \cdot h \cdot \sup_x |\chi_{\xi'}(x)| \le C |u_{\xi'}| 2^{l(\xi') - j}. \tag{3.53}$$

By the Cauchy-Schwarz inequality, we conclude

$$(3.54)\qquad \begin{aligned} |v_\xi|^2 &\le Cj \sum_{\xi'\in\Delta_j:\chi_{\xi'}(\xi)\neq 0} 2^{2(l(\xi')-j)}|u_{\xi'}|^2,\\ \sum_{\xi\in\Delta_j}|v_\xi|^2 &\le Cj\sum_{\xi'\in\Delta_j}2^{2(l(\xi')-j)}|u_{\xi'}|^2 \sum_{\xi\in\Delta_j:\chi_{\xi'}(\xi)\neq 0} 1. \end{aligned}$$

Taking into account that the support of $\chi_{\xi'}$ contains no more than $C2^{2(j-l(\xi'))}$ grid points ξ, we continue

$$(3.55)\qquad \sum_{\xi\in\Delta_j}|v_\xi|^2 \le Cj\sum_{\xi'\in\Delta_j}|u_{\xi'}|^2.$$

This proves $\|R_h\| \le C\sqrt{j}$. For the converse estimate, we choose $u_{\xi'} := 2^{-l(\xi')}$. A simple calculation yields $\|u\| \le C\sqrt{j}$ and $\|v\| \sim \|u_j\|_{L^2} \ge \|u_j\|_{L^1} \ge Cj$. Hence, we conclude $\|R_j\| \ge C\sqrt{j}$.

Now we turn to R_j^{-1}. Analogously to (3.52), we arrive at

$$(3.56)\qquad u_{\xi'} = \sum_{\xi\in\Delta_j} v_\xi \vartheta_{\xi'}(\varphi_{j,\xi}).$$

In this sum the number of terms different from zero is bounded by a constant. Each term can be estimated by $|v_\xi|2^{(j-l(\xi'))}$, and the Cauchy-Schwarz inequality yields

$$(3.57)\qquad \begin{aligned} |u_{\xi'}|^2 &\le C2^{2(j-l(\xi'))}\sum_{\xi\in\Delta_j:\vartheta_{\xi'}(\varphi_{j,\xi})\neq 0}|v_\xi|^2,\\ \sum_{\xi'\in\Delta_j}|u_{\xi'}|^2 &\le C\sum_{\xi\in\Delta_j}|v_\xi|^2\sum_{\xi'\in\Delta_j:\vartheta_{\xi'}(\varphi_{j,\xi})\neq 0}2^{2(j-l(\xi'))}. \end{aligned}$$

For fixed $\xi \in \Delta_j$ and fixed l, $-1 \le l \le j-1$, the number of $\xi' \in \nabla_l$ with $\vartheta_{\xi'}(\varphi_{j,\xi}) \neq 0$ is bounded by a constant. Consequently, we obtain

$$(3.58)\qquad \begin{aligned} \sum_{\xi'\in\Delta_j}|u_{\xi'}|^2 &\le C\sum_{\xi\in\Delta_j}|v_\xi|^2\sum_{l=-1}^{j-1}2^{2(j-l)},\\ \|u\|_{l^2} &\le C2^j\|v\|_{l^2} \end{aligned}$$

and $\|R_j^{-1}\| \leq C2^j$. On the other hand, choosing $v_\xi := 2^{-j}$ for one point $\xi = \xi'' \in \nabla_{-1}$ and $v_\xi := 0$ otherwise, we arrive at $\|v\| \leq C2^{-j}$ and $|u_{\xi''}| \geq C$. In other words, $\|u\| \geq C$ and $\|R_j^{-1}\| \geq C2^{-j}$. □

Remark 3.1. Suppose that $\mathbf{s}$ is a fixed number between 1 and 3/2. Then there exists a constant $C > 0$ such that, for any j and any sequence $(u_\xi)_{\xi\in\Delta_j}$, we get

$$\frac{1}{C}\sqrt{\sum_{\xi\in\Delta_j} 2^{2\mathbf{s}}|u_\xi|^2} \leq \Big\| \sum_{\xi\in\Delta_j} u_\xi \chi_\xi \Big\|_{H^{\mathbf{s}}(\Gamma)} \leq C\sqrt{\sum_{\xi\in\Delta_j} 2^{2\mathbf{s}}|u_\xi|^2}. \tag{3.59}$$

This result can be proved analogously to [**39**].

Finally, we remark that the application of R_j can be realized by fast pyramid algorithms, too. The matrix R_j^{-1} contains no more than three nonzero entries in each row. Consequently, for one application of R_j or R_j^{-1}, no more than $O(N_j^2)$ arithmetic operations are required.

3.4. *The wavelet algorithm.* Using the new wavelet bases from Sections 3.2 and 3.3, the collocation equation (2.6) is equivalent to

$$\vartheta_{\xi'}(Au_j) = \vartheta_{\xi'}(v), \quad \xi' \in \Delta_j, \qquad u_j = \sum_{\xi\in\Delta_j} u_\xi \psi_\xi. \tag{3.60}$$

The matrix equation $A_j(w_\xi)_{\xi\in\Delta_j} = (hv(\xi'))_{\xi'\in\Delta_j}$ can be replaced by the equivalent equation $B_j(u_\xi)_{\xi\in\Delta_j} = (\vartheta_{\xi'}(v))_{\xi'\in\Delta_j}$, where the matrix B_j is defined as $(\vartheta_{\xi'}(A\psi_\xi))_{\xi',\xi\in\Delta_j}$. This B_j is called the wavelet transform of A_j, and we get $A_j = R_jB_jE_j$. Note that we will identify the operators in $\mathcal{L}(S_j)$ with their matrices corresponding to the basis $\{\varphi_{j,\xi}\}$. In particular, we get $A_j = \mathcal{A}_j \in \mathcal{L}(S_j)$.

Now the wavelet algorithm looks as follows. We solve the matrix equation $A_j(w_\xi)_{\xi\in\Delta_j} = (hv(\xi'))_{\xi'\in\Delta_j}$ iteratively, e.g., by GMRes. The main part of the computation is spent for the multiplication of iterative solutions $z := (z_\xi)_{\xi\in\Delta_j}$ or residual vectors z by the matrix A_j. In the wavelet algorithm, this step is done by first multiplying z by E_j, then by B_j and finally by R_j. As has been mentioned near the ends of Sections 3.2 and 3.3, the basis transforms $z \mapsto E_jz$ and $[B_jE_jz] \mapsto R_j[B_jE_jz]$

can be realized via fast pyramid type algorithms. For the multiplication by B_j, we will prove that, due to the moment conditions and the smallness of the supports of the bases $\{\vartheta_{\xi'}, \xi' \in \Delta_j\}$ and $\{\psi_\xi, \xi \in \Delta_j\}$, the majority of entries in B_j is very small, cf. Lemma 3.5. Thus, setting these entries equal to zero, we end up with a compressed matrix C_j and the multiplication by B_j can be replaced by the multiplication with C_j. The additional error due to compression will be less than the discretization error of the conventional collocation, cf. Theorem 3.1. Since the matrix C_j is sparse, the multiplication by C_j is fast. In fact, cf. Theorem 3.1, no more than $O(N_j^2[\log N_j]^4)$ arithmetic operations are necessary for the multiplication by the $O(N_j^2) \times O(N_j^2)$ matrix C_j. Hence, if the matrix C_j is already given and if the equation $[R_j C_j E_j](w_\xi)_{\xi\in\Delta_j}(hv(\xi'))_{\xi'\in\Delta_j}$ is solved by an iterative algorithm, e.g., by a cascadic GMRes algorithm, then an approximate solution $u_j = \sum_{j\in\Delta_j} w_\xi \varphi_{j,\xi}$ with an error less than Ch_j^2 can be computed with no more than $Ch_j^{-2}[\log h_j^{-1}]^4$ arithmetic operations.

In any case, the main part of the computing time for boundary element methods is spent for the calculation of the stiffness matrix. For the wavelet algorithm, we do not need the whole matrices A_j or B_j but only the compressed matrix C_j which saves a lot of computing time. However, this reduction in computing time is not so easy to achieve as it might seem at first glance. In fact, a sophisticated algorithm of quadrature is needed to guarantee small quadrature errors and to reduce the amount of work. We will discuss this issue in Section 4.

Remark 3.2. It is possible to solve $B_j(u_\xi)_{\xi\in\Delta_j} = (\vartheta_{\xi'}(v))_{\xi'\in\Delta_j}$ directly. For details we refer to the papers by Dahmen, Kunoth, Prößdorf and Schneider [**11, 14**]. In the situation considered in the present paper, however, the condition number of the original matrix A_j is uniformly bounded, and we expect the actual value of the condition number of the wavelet transform B_j to be much worse even if it is uniformly bounded.

Now we describe the compression algorithm. The results and proofs are analogous to those given by Dahmen, Prößdorf, Schneider, Petersdorff and Schwab [**14, 31**]. Hence, we present the results and only those parts of the proofs which are new. We begin with the estimate for the entries of B_j.

Lemma 3.5. *Suppose $\xi \in \Delta_j$ is equal to $\xi = \kappa_m(\tau)$ for $1 \le m \le m_\Gamma$ and $\tau \in \Delta_j^{\mathcal{S},\mathcal{E}_m}$ such that the support of ψ_ξ is contained in the interior of Γ_m. Then for this ξ and for $\xi' \in \Delta_j$, the entry $b_{\xi',\xi} := \vartheta_{\xi'}(A\psi_\xi)$ of the wavelet transform B_j can be estimated as*

$$|b_{\xi',\xi}| \le 2^{-3l(\xi)-3l(\xi')}[\mathrm{dist}\,(\mathrm{supp}\,\psi_\xi, \mathrm{conv}\,\vartheta_{\xi'})]^{-6}, \tag{3.61}$$

where $\mathrm{supp}\,\psi_\xi$ denotes the support of the function ψ_ξ and $\mathrm{conv}\,\vartheta_{\xi'}$ stands for the convex hull, in the parameter domain, of the support of the functional $\vartheta_{\xi'}$. By $\mathrm{dist}\,(\mathrm{supp}\,\psi_\xi, \mathrm{conv}\,\vartheta_{\xi'})$ we have denoted the distance between the sets $\mathrm{supp}\,\psi_\xi$ and $\mathrm{conv}\,\vartheta_{\xi'}$. The integer $l(\xi)$ denotes the level of ξ, i.e., $\xi \in \nabla_{l(\xi)} := \Delta_{l(\xi)+1} \backslash \Delta_{l(\xi)}$. For arbitrary $\xi, \xi' \in \Delta_j$, the entry $b_{\xi',\xi}$ can be estimated as

$$|b_{\xi',\xi}| \le 2^{-l(\xi)-3l(\xi')}[\mathrm{dist}\,(\mathrm{supp}\,\psi_\xi, \mathrm{conv}\,\vartheta_{\xi'})]^{-4}. \tag{3.62}$$

Proof. Instead of repeating the rigorous proof of [**14, 31, 39**], let us only explain where the different factors in (3.61) and (3.62) come from. For analogy reasons, it is sufficient to consider (3.61). One factor $2^{-l(\xi')}$ is from the scaling factor $N_{l(\xi)}^{-1}$ in the definition of (3.48). The second factor $2^{-2l(\xi')}$ is due to the third term in the Taylor series expansion of the kernel function at a point $x = \kappa_m(t)$ of $\mathrm{conv}\,\vartheta_{\xi'}$. Indeed, applying $\vartheta_{\xi'}$ to $f := A\psi_\xi$ and using that $\vartheta_{\xi'}$ vanishes over linear functions, we get

$$\begin{aligned} f(\kappa_m(s)) &= f(\kappa_m(t)) + \nabla f(\kappa_m(t)) \cdot (s-t) \\ &\quad + \frac{1}{2}\nabla^2 f(\kappa_m(t')) \cdot (s-t)^2, \end{aligned} \tag{3.63}$$

$$\begin{aligned} |N_{l(\xi')}\vartheta_{\xi'}(f)| &\le C \sup |\nabla^2 f(\kappa_m(t'))| \sup_{y \in \mathrm{conv}\,\vartheta_{\xi'}} |y-x|^2 \\ &\le C \sup |\nabla^2 f(x')| 2^{-2l(\xi')}. \end{aligned} \tag{3.64}$$

Similarly, writing $\vartheta_{\xi'}(A\psi_\xi) = \langle A\psi_\xi, \vartheta_{\xi'}\rangle = \langle \psi_\xi, A^*\vartheta_{\xi'}\rangle = \int f\psi_\xi$ with $f := A^*\vartheta_{\xi'}$, using the moment conditions of order two for the trial wavelet, and choosing $x \in \operatorname{supp} \psi_\xi$, we conclude, cf. (3.63),

$$\int f\psi_\xi = \int \frac{1}{2}\nabla^2 f(\kappa_m(t')) \cdot (s-t)^2 \psi_\xi(\kappa_m(s))\, ds,$$

$$\left|\int f\psi_\xi\right| \le C \sup |\nabla^2 f(x')| \int_{\operatorname{supp}\psi_\xi} |y-x|^2 |\psi_\xi(y)|\, dy \tag{3.65}$$

$$\le C \sup |\nabla^2 f(x')| 2^{-2l(\xi)} \int_{\operatorname{supp}\psi_\xi} |\psi_\xi(y)|\, dy.$$

Thus, a factor $2^{-2l(\xi)}$ in (3.61) is due to the second order moment conditions of the wavelet in the trial space and an additional $2^{-l(\xi)}$ arises from the scaling factor $N_{l(\xi)} \sim 2^{l(\xi)}$ in the definitions of Sections 3.1 and 3.2, cf., the factor $\sqrt{N_l}$ for the univariate wavelet $\psi_\sigma^{\mathbf{R}}$ and observe that the bivariate wavelets are tensor products of univariate wavelets, and from the measure meas $(\operatorname{supp}\psi_\xi) \sim 2^{-2l(\xi)}$. Applying these Taylor series arguments to the integrand in $\langle A\psi_\xi, \vartheta_{\xi'}\rangle$, it remains to estimate the fourth order derivatives of the kernel function $K_A(x,y)$ of the operator A for $x \in \operatorname{conv}\vartheta_{\xi'}$ and $y \in \operatorname{supp}\psi_\xi$. Applying (2.2), the estimate of the kernel function leads to the factor $[\operatorname{dist}(\operatorname{supp}\psi_\xi, \operatorname{conv}\vartheta_{\xi'})]^{-6}$ in (3.61). □

Theorem 3.1. *Suppose that the righthand side v of* (2.1) *belongs to the Sobolev space $H^2(\Gamma)$ and define the compressed matrix $C_j = (c_{\xi',\xi})_{\xi',\xi\in\Delta_j}$ by*

$$c_{\xi',\xi} := \begin{cases} b_{\xi',\xi} & \text{if } \operatorname{dist}(\operatorname{supp}\psi_\xi, \operatorname{conv}\vartheta_{\xi'}) \le (a2^j j)2^{-l(\xi')-l(\xi)} \\ 0 & \text{else,} \end{cases} \tag{3.66}$$

with a suitable constant $a > 1$. If a is large enough and if the collocation method (2.6) *is stable, cf. Theorem* 2.1, *then the operator $\tilde{A}_j := [R_j C_j E_j] \in \mathcal{L}(S_j)$ is stable, i.e., there is an $\tilde{h} > 0$ such that, for any $h_j < \tilde{h}$, the operator $\tilde{A}_j$ is invertible and its inverse $\tilde{A}_j^{-1}$ is uniformly bounded. Additionally, if $u_j \in S_j$ denotes the solution of $\tilde{A}_j u_j = P_j v$, then*

$$\|u - u_j\|_{L^2(\Gamma)} \le C h_j^2 \tag{3.67}$$

and the number of nonzero entries in the matrix C_j is less than $Ca^2 N_j^2 [\log N_j]^4 = Ca^2 h_j^{-2} [\log h_j^{-1}]^4$.

Proof. For some details of the proof we again refer to [**14, 31, 39**]. We only present those parts which are new. In particular, the bound for the number of nonzero entries can be derived analogously to [**14, 31**]. For the stability and for the convergence estimate, we have to prove

$$\|(A_j - \tilde{A}_j)\tilde{u}_j\|_{L^2(\Gamma)} \le Ca^{-2}h_j^{2-\mathbf{s}} \begin{cases} \|u\|_{H^2(\Gamma)} & \text{if } \mathbf{s}=2 \\ \|\tilde{u}_j\|_{L^2(\Gamma)} & \text{if } \mathbf{s}=0, \end{cases} \tag{3.68}$$

where $\tilde{u}_j$ is the interpolation $P_j u$ of the exact solution u to Equation (2.1).

To prove (3.68), we set $D_j := B_j - C_j = (d_{\xi',\xi})_{\xi',\xi\in\Delta_j}$ and get $A_j - \tilde{A}_j = R_j D_j E_j$. In view of the Lemmas 3.3 and 3.4 we have to estimate the matrix $D_j^{\mathbf{s}} := (d^{\mathbf{s}}_{\xi',\xi})_{\xi',\xi\in\Delta_j} \in \mathcal{L}(l^2(\Delta_j))$ with $d^{\mathbf{s}}_{\xi',\xi} := d_{\xi',\xi}2^{-\mathbf{s}l(\xi)}$. By Schur's lemma the norm can be bounded as follows

$$\|D_j^{\mathbf{s}}\|_{L(l^2(\Delta_j))} \le \sqrt{\sigma_1\sigma_2}, \tag{3.69}$$

$$\sigma_1 := \sup_{\xi'\in\Delta_j}\left[2^{l(\xi')}\sum_{\xi\in\Delta_j}|d^{\mathbf{s}}_{\xi',\xi}|2^{-l(\xi)}\right],$$

$$\sigma_2 := \sup_{\xi\in\Delta_j}\left[2^{l(\xi)}\sum_{\xi'\in\Delta_j}|d^{\mathbf{s}}_{\xi',\xi}|2^{-l(\xi')}\right].$$

Since the entries $d^{\mathbf{s}}_{\xi',\xi}$ with $\operatorname{supp}\psi_\xi$ contained in the interior of some Γ_m can be treated as in [**14, 31, 39**], we only estimate those parts σ_i^b of σ_i, $i=1,2$, where a ξ is involved such that $\operatorname{supp}\psi_\xi$ intersects the boundary of some Γ_m. We denote the set of these ξ by Δ_j^b and set $a_* := (a2^j j)2^{-l(\xi')-l(\xi)}$ as well as $\text{dist} := \text{dist}\,(\operatorname{supp}\psi_\xi, \operatorname{conv}\vartheta_{\xi'})$. Using (3.62) and (3.66), we get

$$\begin{aligned}\sigma_1^b &\le C\sup_{\xi'\in\Delta_j}\left[2^{l(\xi')}\sum_{\xi\in\Delta_j^b:\text{dist}>a_*}2^{-l(\xi)-3l(\xi')}\text{dist}^{-4}2^{-\mathbf{s}l(\xi)}2^{-l(\xi)}\right]\\ &\le C\sup_{\xi'\in\Delta_j}\left[2^{-2l(\xi')}\sum_{l=-1}^{j-1}2^{-l(1+\mathbf{s})}\sum_{\xi\in\Delta_j^b:\text{dist}>a_*,l(\xi)=l}\text{dist}^{-4}2^{-l(\xi)}\right].\end{aligned} \tag{3.70}$$

Applying

$$\sum_{\xi\in\Delta_j^b:\mathrm{dist}\,>a_*,l(\xi)=l} \mathrm{dist}^{\,-4}2^{-l(\xi)} \le C\int_{\{t\in\mathbf{R}:|t|>a_*\}} |t|^{-4}\,dt \le Ca_*^{-3}, \tag{3.71}$$

we arrive at

$$\begin{aligned}\sigma_1^b &\le C\sup_{\xi'\in\Delta_j}\left[2^{-2l(\xi')}\sum_{l=-1}^{j-1}2^{-l(1+\mathbf{s})}a_*^{-3}\right]\\ &\le C\sup_{\xi'\in\Delta_j}\left[2^{-2l(\xi')}\sum_{l=-1}^{j-1}2^{-l(1+\mathbf{s})}((a2^jj)2^{-l(\xi')-l})^{-3}\right]\\ &\le C\sup_{\xi'\in\Delta_j}\left[a^{-3}j^{-3}2^{-3j}2^{l(\xi')}\sum_{l=0}^{j-1}2^{l(2-\mathbf{s})}\right]\\ &\le Ca^{-3}j^{-2}2^{-\mathbf{s}j}.\end{aligned} \tag{3.72}$$

On the other hand, similarly to (3.71), we get

$$\sum_{\xi'\in\Delta_j:\mathrm{dist}\,>a_*,l(\xi)=l} \mathrm{dist}^{\,-4}2^{-2l(\xi)} \le C\int_{\{x\in\mathbf{R}^2:|x|>a_*\}} |x|^{-4}\,dx \le Ca_*^{-2}, \tag{3.73}$$

and, analogously to (3.72), we conclude
(3.74)

$$\begin{aligned}\sigma_2^b &\le C\sup_{\xi\in\Delta_j}\left[2^{l(\xi)}\sum_{\xi'\in\Delta_j:\mathrm{dist}\,>a_*}2^{-l(\xi)-3l(\xi')}\mathrm{dist}^{\,-4}2^{-\mathbf{s}l(\xi)}2^{-l(\xi')}\right]\\ &\le C\sup_{\xi\in\Delta_j}\left[2^{-\mathbf{s}l(\xi)}\sum_{l=-1}^{j-1}2^{-2l}\sum_{\xi'\in\Delta_j:\mathrm{dist}\,>a_*,l(\xi')=l}\mathrm{dist}^{\,-4}2^{-2l(\xi')}\right]\\ &\le C\sup_{\xi\in\Delta_j}\left[2^{-\mathbf{s}l(\xi)}\sum_{l=-1}^{j-1}2^{-2l}((a2^jj)2^{-l-l(\xi)})^{-2}\right]\\ &\le C\sup_{\xi\in\Delta_j}\left[a^{-2}j^{-2}2^{-2j}2^{(2-\mathbf{s})l(\xi)}\sum_{l=0}^{j-1}1\right]\\ &\le a^{-2}j^{-1}2^{-\mathbf{s}j}.\end{aligned}$$

The estimates (3.72) and (3.74), the analogous estimates for the entries $b_{\xi',\xi}$, $\xi \in \Delta_j \backslash \Delta_j^b$ and (3.69) yield that $\|D_j^{\mathbf{s}}\|_{\mathcal{L}(l^2(\Delta_j))}$ is less than $Ca^{-2}j^{-1}h_j^{\mathbf{s}}$. This, together with the Lemmas 3.3 and 3.4 implies (3.68). □

Remark 3.3. From the Lemmas 3.3 and 3.4, we get $\|C_j\| = \|R_j^{-1}\tilde{A}_j E_j^{-1}\| \sim 2^j$ and $\|R_j\| \sim \sqrt{j}$. Thus, the multiplication of a certain vector z by $R_j C_j E_j$ can lead to an additional error of $O(2^j\sqrt{j})$ times the numerical error of z.

4. The error and complexity of the quadrature algorithm.

4.1. *Assumptions on the parametrization and the kernel function.* Clearly, the assumptions on the parametrization and the kernel function in Section 2.1 are not necessary for the results of the previous sections. Indeed, for the kernel $K_A(x,y)$ and $x \neq y$, the existence of continuous derivatives up to the order four (two derivatives with respect to each variable x and y) is sufficient. For the parametrization, a differentiability up to order three is sufficient. If differentiability is guaranteed only up to orders less than four and three, then a different wavelet algorithm is possible. More precisely, for appropriate real numbers $\alpha \geq 1$, $\beta \geq 1$ and $\gamma > 0$ the compressed matrix C_j can be defined by

$$(4.1) \qquad c_{\xi',\xi} := \begin{cases} b_{\xi',\xi} & \text{if dist}\,(\text{supp}\,\psi_\xi, \text{conv}\,\vartheta_{\xi'}) \\ & \leq \max\{2^{-l(\xi)}, 2^{-l(\xi')}, (a2^j j^\gamma)2^{-\alpha l(\xi')-\beta l(\xi)}\} \\ 0 & \text{else.} \end{cases}$$

The error $\|u-u_j\|_{L^2(\Gamma)}$ for the solution of the corresponding discretized equation $\tilde{A}_j u_j = P_j v$ will be of order $O(h^\delta)$, $0 < \delta \leq 2$, which should be the best possible under the weaker differentiability assumptions. The number of nonzero entries will be of order $N_j^{\delta'}$, $2 < \delta' \leq 4$. Thus, this wavelet method is suboptimal since it reduces the number of arithmetic operations from N_j^4 for a conventional finite element algorithm to $N_j^{\delta'} > N_j^2$.

Now we will define our quadrature algorithm for the following situation:

i) Suppose the surface is three times continuously differentiable.

ii) Suppose that the surface is given by a finite number of grid points only, i.e., that the κ_m are given over the grid $\Delta_j^{\mathcal{S}}$.

iii) We replace the true surface by a piecewise polynomial interpolant. This is given by the parametrizations κ_m which interpolate the given values $\{\kappa_m(\xi) : \xi \in \Delta_j^{\mathcal{S}}\}$.

iv) Suppose that κ_m is twice continuously differentiable over $\mathcal{S}$ and polynomial over each patch $\{(t_1, t_2) : (k-1)h_j \leq t_1 \leq kh_j, (i-1)h_j \leq t_2 \leq ih_j\}$. Furthermore, suppose that there exists a constant independent of m and the patch such that

$$\sup_{t \in \mathcal{S}} |\partial^\alpha \kappa_m(t)| \leq C \tag{4.2}$$

for any nonnegative multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| := \alpha_1 + \alpha_2 \leq 3$.

v) To ensure the existence of the singular integrals in the principal value sense, we suppose that the approximating manifold is continuously differentiable also over the common boundary of two subsurfaces defined by different parameter representations.

vi) For the kernel function $K_A(x, y)$, we require the representation, cf., e.g., [**31**],

$$K_A(x, y) = \sum_{\mathbf{k} \leq |\alpha|} s_\alpha(x, y, n_y)(x - y)^\alpha |x - y|^{-2-\mathbf{k}}, \tag{4.3}$$

where $\mathbf{k}$ is an odd integer, n_y is the unit normal to Γ at y, and the sum is taken over a finite number of multi-indices α.

vii) Suppose that, for any $m = 1, \ldots, m_\Gamma$, the functions $s_\alpha : \Gamma_m \times \Gamma_m \times S^2 \to \mathbf{R}$ admit continuous extensions to the sets

$$\begin{aligned} &\Gamma_m \times \{t \in \mathbf{C}^3 : \operatorname{dist}(t, \Gamma_m) \leq \varepsilon_A\} \times S^2, \\ &\Gamma_m \times \Gamma_m \times \{t \in \mathbf{C}^3 : \operatorname{dist}(t, S^2) \leq \varepsilon_A\}, \end{aligned} \tag{4.4}$$

such that s_α is a complex analytic function with respect to the second and third variable, respectively.

Clearly, the replacement of the true surface by the approximating piecewise polynomial surface leads to additional errors. Though these effects require an extra analysis, we will not discuss this issue. If the interpolation of the thrice differentiable surface is defined, e.g., by tensor product Overhauser interpolation, cf. [**29**], and by straightforward modifications at the lines $\Gamma_m \cap \Gamma_{m'}$, then the global continuous differentiability of the new surface can be guaranteed. Moreover, the piecewise second derivatives of the approximating surface are close to those of the true surface. Therefore, we conjecture that the compression results of Section 3 and the results of the present chapter remain true for the Overhauser interpolation of a three times continuously differentiable surface.

4.2. *The quadrature algorithm.* In this section we define the quadrature rules for the computation of the matrix entries $c_{\xi',\xi}$ of the compressed wavelet transform C_j. From (3.48) we conclude that, for each $\xi' \in \nabla_l$, there exist three points ξ_ι of Δ_{l+1} and three real coefficients λ_ι such that $\vartheta_{\xi'}(f) = \sum_{\iota=1}^3 \lambda_\iota f(\xi_\iota)$. Clearly for $\xi' \in \nabla_{-1}$, we have $\lambda_2 = \lambda_3 = 0$. If the entry $c_{\xi',\xi}$ is not zero, then it is equal to

$$\begin{aligned} c_{\xi',\xi} &= \sum_{\iota=1}^{3} \lambda_\iota A\psi_\xi(\xi_\iota) \\ &= \sum_{\iota=1}^{3} \lambda_\iota \left\{ a(\xi_\iota)\psi_\xi(\xi_\iota) + \int_\Gamma K_A(\xi_\iota, y)\psi_\xi(y)\, d_y\Gamma \right\}. \end{aligned} \tag{4.5}$$

Depending on $\vartheta_{\xi'}$, we will split Γ into the union of subdomains $\Gamma^{\xi'}_{i'}$, $i' \in \mathcal{N}$. Over this partition we will define a composite quadrature rule

$$\begin{aligned} \int_{\Gamma^{\xi'}_{i'}} f(y)\, d_y\Gamma &\sim \sum_{\mu \in \mathcal{M}_{i'}} f(x_\mu)\omega_\mu, \\ \int_\Gamma f(y)\, d_y\Gamma &\sim \sum_{i' \in \mathcal{N}} \sum_{\mu \in \mathcal{M}_{i'}} f(x_\mu)\omega_\mu, \\ &=: \sum_{\mu \in \mathcal{M}} f(x_\mu)\omega_\mu, \\ \mathcal{M} &:= \bigcup_{i' \in \mathcal{N}} \mathcal{M}_{i'}, \end{aligned} \tag{4.6}$$

which depends also on $\xi_\iota \in \operatorname{supp} \vartheta_{\xi'}$. However, before we apply such a quadrature rule to the computation of the integrals in (4.5), we have to perform a singularity subtraction step over some of the domains $\Gamma_{i'}^{\xi'}$, i.e., for i' in a certain subset $\mathcal{N}' = \mathcal{N}'(\xi', \xi_\iota) \subseteq \mathcal{N}$. Singularity subtraction means the following. We will introduce a main part $K_M(x, y)$ of the kernel function $K_A(x, y)$ which has the same singularity behavior for $y \to x$. In other words, $K_A(x, y) - K_M(x, y)$ will have a weak singularity only. Moreover, the function $K_M(x, y)$ will be chosen such that its integration can be performed by an analytic formula. Using this, we write

(4.7)

$$
\begin{aligned}
c_{\xi',\xi} = \sum_{\iota=1}^{3} \lambda_\iota \Bigg\{ a(\xi_\iota)\psi_\xi(\xi_\iota) \\
&+ \sum_{i' \in \mathcal{N}'} \Bigg[\int_{\Gamma_{i'}^{\xi'}} [K_A(\xi_\iota, y)\psi_\xi(y) - K_M(\xi_\iota, y)\psi_\xi(\xi_{i'}^{\xi', \xi_i})] \, d_y\Gamma \\
&\qquad + \int_{\Gamma_{i'}^{\xi'}} K_M(\xi_\iota, y) \, d_y\Gamma \psi_\xi(\xi_{i'}^{\xi', \xi_\iota}) \Bigg] \\
&+ \sum_{i' \in \mathcal{N} \setminus \mathcal{N}'} \int_{\Gamma_{i'}^{\xi'}} K_A(\xi_\iota, y)\psi_\xi(y) \, d_y\Gamma \Bigg\},
\end{aligned}
$$

where the point $\xi_{i'}^{\xi', \xi_\iota}$ is chosen to be equal to ξ_ι if $\xi_\iota \in \Gamma_{i'}^{\xi'}$ and where $\xi_{i'}^{\xi', \xi_\iota}$ is an arbitrary but fixed point $\xi_{i'}^{\xi'} \in \Gamma_{i'}^{\xi'}$ not depending on ξ_ι if $\xi_\iota \notin \Gamma_{i'}^{\xi'}$. The integrands $y \mapsto [K_A(\xi_\iota, y)\psi_\xi(y) - K_M(\xi_\iota, y)\psi_\xi(\xi_{i'}^{\xi', \xi_\iota})]$ in (4.7) have milder singularities at $y = \xi_\iota$ than the corresponding integrands $y \mapsto K_A(\xi_\iota, y)\psi_\xi(y)$ in (4.5). Applying the rules (4.6) to (4.7), we arrive at the final formula,

$$
\begin{aligned}
(4.8) \quad c_{\xi',\xi} \sim c'_{\xi',\xi} := \sum_{\iota=1}^{3} \lambda_\iota \Bigg\{ a(\xi_\iota)\psi_\xi(\xi_\iota) + \sum_{\mu \in \mathcal{M}} K_A(\xi_\iota, x_\mu)\psi_\xi(x_\mu)\omega_\mu \\
+ \sum_{i' \in \mathcal{N}' : \Gamma_{i'}^{\xi'} \cap \operatorname{supp} \psi_\xi \neq \varnothing} \Bigg[\int_{\Gamma_{i'}^{\xi'}} K_M(\xi_\iota, y) \, d_y\Gamma \\
- \sum_{\mu \in \mathcal{M}_{i'}} K_M(\xi_\iota, x_\mu)\omega_\mu \Bigg] \psi_\xi(\xi_{i'}^{\xi', \xi_\iota}) \Bigg\}
\end{aligned}
$$

It remains to introduce the $\Gamma_{i'}^{\xi'}$, the rule (4.6), the set $\mathcal{N}'$, and the main part K_M of the kernel.

First we fix a $\xi' \in \Delta_j$ and we introduce the underlying partition for the quadrature. Since the quadrature rules are accurate for polynomial integrands but not for piecewise polynomials, we have to choose the partition such that all the functions ψ_ξ are polynomials over the subdomains. We consider the uniform partitions

$$(4.9)\qquad \begin{aligned} \Gamma &= \bigcup_{m=1}^{m_\Gamma} \bigcup_{k,k'=1}^{N_l} D^{m,l,k,k'}, \\ D^{m,l,k,k'} &:= \kappa_m([(k-1)h_l, kh_l] \times [(k'-1)h_l, k'h_l]) \end{aligned}$$

of step size h_l with $l = 0, 1, \dots, j$. For the subdomains of these partitions, we call a function f "polynomial" over $D^{m,l,k,k'}$ if $f \circ \kappa_m$ is a polynomial over $[(k-1)h_l, kh_l] \times [(k'-1)h_l, k'h_l]$. By $\Gamma = \cup_{i=1}^{M^j} \Gamma_i^j$ we denote the coarsest partition into subdomains from the partitions (4.9) such that the restriction to these subdomains of the functions ψ_ξ, for which $c_{\xi',\xi} \neq 0$, is a "bilinear polynomial". More exactly, we define $\Gamma = \cup_{i=1}^{M^j} \Gamma_i^j$ recursively. First we set $\Gamma = \cup_{i=1}^{M^0} \Gamma_i^0$ equal to the partition (4.9) with $l = 0$. We define $\Gamma = \cup_{i=1}^{M^1} \Gamma_i^1$ as the refinement of $\Gamma = \cup_{i=1}^{M^0} \Gamma_i^0$, where a $\Gamma_i^0 = D^{m,0,k,k'}$ remains unchanged if the functions ψ_ξ for which $c_{\xi',\xi} \neq 0$ are "polynomials" over Γ_j^0 and where all the other $\Gamma_i^0 = D^{m,0,k,k'}$ are divided into the four subdomains $D^{m,1,2k-1,2k'-1}$, $D^{m,1,2k,2k'-1}$, $D^{m,1,2k-1,2k'}$ and $D^{m,1,2k,2k'}$. Next $\Gamma = \cup_{i=1}^{M^2} \Gamma_i^2$ is the refinement of $\Gamma = \cup_{i=1}^{M_1} \Gamma_i^1$, where every subdomain remains unchanged except those $\Gamma_i^1 = D^{m,1,k,k'}$ for which there exists a ξ such that $c_{\xi',\xi} \neq 0$ and ψ_ξ is not a "polynomial" over Γ_i^1. These Γ_i^1 are divided into the four subdomains $D^{m,2,2k-1,2k'-1}$, $D^{m,2,2k,2k'-1}$, $D^{m,2,2k-1,2k'}$ and $D^{m,2,2k,2k'}$. Proceeding in the same manner, we finally get the partition $\Gamma = \cup_{i=1}^{M^j} \Gamma_i^j$.

Unfortunately, this partition is still not sufficiently fine. Indeed, applying the one point quadrature rule over each Γ_i^j, $i = 1, \dots, M^j$, leads to large quadrature errors due to the singularity of the kernel $K_A(\xi_\iota, y)$ for y close to ξ_ι. These errors cannot be improved by employing quadrature rules which are exact for higher order polynomials since the assumptions iii) and iv) of Section 4.1 admit low order estimates only. The only way to improve the quadrature errors is to work with smaller step size. Thus, to refine the partition $\Gamma = \cup_{i=1}^{M_j} \Gamma_i^j$ we consider a $\Gamma_i^j = D^{m,l,k,k'}$. Obviously, there exists an l'' such that

$$2^{-2l''} \leq \operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\} < 2^{-2(l''-1)}. \tag{4.10}$$

If $l'' < j - l$, then we replace $\Gamma_i^j = D^{m,l,k,k'}$ by the union of the $2^{2l''}$ subdomains $D^{m,l+l'',\tilde{k},\tilde{k}'}$ which are contained in Γ_i^j. For $l'' \geq j - l$, we replace $\Gamma_i^j = D^{m,l,k,k'}$ by the union of the $2^{2(j-l)}$ subdomains $D^{m,j,\tilde{k},\tilde{k}'}$ which are contained in Γ_i^j. We denote the final partition by $\Gamma = \cup_{i' \in \mathcal{N}} \Gamma_{i'}^{\xi'}$.

Now we define the quadrature rule (4.6) for $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ such that $\xi_\iota \notin \Gamma_{i'}^{\xi'}$. We write

$$\begin{aligned} \int_{\Gamma_{i'}^{\xi'}} f(y)\, d_y\Gamma &= \int_{(\tilde{k}-1)h_{l'}}^{\tilde{k}h_{l'}} \int_{(\tilde{k}'-1)h_{l'}}^{\tilde{k}'h_{l'}} f(\kappa_m(t_1,t_2))|\kappa_m'(t_1,t_2)|\, dt_2\, dt_1 \\ &\sim \sum_{\mu \in \mathcal{M}_{i'}} f(x_\mu)\omega_\mu, \end{aligned} \tag{4.11}$$

where the last quadrature rule is the tensor product of the univariate n_G-point Gauß rule. If $l' < j$, then the distance of $\Gamma_{i'}^{\xi'} = D^{m,l,\tilde{k},\tilde{k}'}$ to the singularity point ξ_ι of $y \mapsto K_A(\xi_\iota, y)$ is sufficiently large and the step size $h_{l'}$ sufficiently small such that the one point rule is sufficiently accurate. Hence, we set $n_G = 1$ for $l' < j$. If $l' = j$, then κ_m is polynomial over $\Gamma_{j'}^{\xi'}$ and higher order quadrature rules can be employed. Hence, for $l' = j$, we choose n_G to be the smallest integer such that, cf. [**23**, Section 2.3],

$$n_G \geq b \frac{j}{\max(1, \log_2[\operatorname{dist}\{\xi_\iota, \Gamma_{i'}^{\xi'}\}/h_j])} \tag{4.12}$$

where b is a fixed positive integer.

Next we turn to the definition of the set $\mathcal{N}'$ of indices $i' \in \mathcal{N}$ for which the singularity subtraction step, cf. (4.5)–(4.8) is necessary for the quadrature over $\Gamma_{i'}^{\xi'}$. If $\xi_\iota \in \Gamma_{i'}^{\xi'}$, then the integrand $y \mapsto K_A(\xi_\iota, y)$ is strongly singular and the quadratures do not converge without singularity subtraction. For $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$, we employ the low order one point rule. In this case the singularity subtraction is also necessary in order to improve the bounds of the derivatives of the integrand. Only if $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' = j$, then the higher order quadrature rules are so strong that the singularity subtraction is redundant. Thus, we introduce $\mathcal{N}'$ as the set of all $i' \in \mathcal{N}$ such that $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$ or such that $\xi_\iota \in \Gamma_{i'}^{\xi'}$.

For the definition of the main part kernel K_M, we observe that the transformed kernel function takes the form, cf. (4.3),

$$\begin{aligned} K_A(\kappa_m(t),\kappa_m(t'))|\kappa_M'(t')| \\ = \sum_{\mathbf{k}\le|\alpha|} s_\alpha(\kappa_m(t),\kappa_m(t'),n_{\kappa_m(t')}) \\ \cdot [\kappa_m(t)-\kappa_m(t')]^\alpha |\kappa_m(t)-\kappa_m(t')|^{-2-\mathbf{k}} |\kappa_m'(t')|. \end{aligned} \tag{4.13}$$

Hence, we define $K_M(x, y)$ by

$$\begin{aligned} K_M(\kappa_m(t),\kappa_m(t'))|\kappa_m'(t')| = \sum_{\mathbf{k}=|\alpha|} s_\alpha(\kappa_m(t),\kappa_m(t),n_{\kappa_m(t)}) \\ \cdot [D\kappa_m(t)\cdot(t-t')]^\alpha \\ \cdot |D\kappa_m(t)\cdot(t-t')|^{-2-\mathbf{k}}|\kappa_m'(t)|, \end{aligned} \tag{4.14}$$

where the surface density $|\kappa_m'(t)|$ is $|\partial_{t_1}\kappa_m(t) \times \partial_{t_2}\kappa_m(t)|$ and the Fréchet derivative $D\kappa_m(t)$ is the matrix $(\partial_{t_1}\kappa_m(t), \partial_{t_2}\kappa_m(t)) \in \mathbf{R}^{3\times 2}$.

Now it remains to introduce the quadrature over the $\Gamma_{i'}^{\xi'}$ with $\xi_\iota \in \Gamma_{i'}^{\xi'}$. For definiteness, we suppose $\xi_i = \kappa_m((\tilde{k}-1)h_j, (\tilde{k}'-1)h_j)$ and consider $\Gamma_{i'}^{\xi'} = D^{m,j,\tilde{k},\tilde{k}'}$. Cutting along the diagonal through ξ_ι, we divide $D^{m,j,\tilde{k},\tilde{k}'}$ into the two triangles $D_-^{m,j,\tilde{k},\tilde{k}'}$ and $D_+^{m,j,\tilde{k},\tilde{k}'}$ given by

$$\begin{aligned} D_+^{m,j,\tilde{k},\tilde{k}'} &:= \kappa_m(\{(t_1,t_2) : 0 \le [t_2-(\tilde{k}'-1)h_j] \\ &\qquad \le [t_1-(\tilde{k}-1)h_j] \le h_j\}), \\ D_-^{m,j,\tilde{k},\tilde{k}'} &:= \kappa_m(\{(t_1,t_2) : 0 \le [t_1-(\tilde{k}-1)h_j] \\ &\qquad \le [t_2-(\tilde{k}'-1)h_j] \le h_j\}). \end{aligned} \tag{4.15}$$

Over $D_+^{m,j,\tilde{k},\tilde{k}'}$ the integrand function takes the form, cf. (4.7),

$$\begin{aligned} g(t) &:= G(\kappa_m(t))|\kappa_m'(t)|, \\ G(y) &:= K_A(\xi_\iota,y)\psi_\xi(y) - K_M(\xi_\iota,y)\psi_\xi(\xi_\iota) \end{aligned} \tag{4.16}$$

and is known to have a weak singularity of the type

$$g\Big((\tilde{k}-1)h_j + t_1, (\tilde{k}'-1)h_j + t_2\Big) = \Phi\Big(t_1, \frac{t_2}{t_1}\Big)\frac{1}{t_1} + \cdots, \tag{4.17}$$

where $0 \le t_2 \le t_1 \le h_j$, where the function Φ is smooth, and where the dots stand for smoother terms. By Duffy's transformation $(t_1, t_2) = (t_1', t_1' t_2')$ such a singularity is transformed into a smooth function and we get

$$\int_0^{h_j} \int_0^{t_1} \Phi\Big(t_1, \frac{t_2}{t_1}\Big)\frac{1}{t_1}\, dt_2\, dt_1 = \int_0^{h_j} \int_0^{1} \Phi(t_1', t_2')\, dt_2'\, dt_1'. \tag{4.18}$$

Consequently, we set

$$\begin{aligned} \int_{D_+^{m,j,\tilde{k},\tilde{k}'}} G(y)\, d_y\Gamma &= \int_0^{h_j} \int_0^{1} g\Big((\tilde{k}-1)h_j + t_1', (\tilde{k}'-1)h_j + t_1' t_2'\Big) t_1'\, dt_2'\, dt_1' \\ &\sim \sum_{\mu \in \mathcal{M}_{i'} : x_\mu \in D_+^{m,j,\tilde{k},\tilde{k}'}} G(x_\mu)\omega_\mu, \end{aligned} \tag{4.19}$$

where the last quadrature rule is the tensor product of the n_G-point Gauß rule applied to the rectangle $[0, h_j] \times [0, 1]$. The order n_G of the univariate Gauß rules is chosen to be greater than or equal to bj with b the constant from (4.12). If we define the knots x_μ and the weights ω_μ in the same fashion for any $D^{m,j,\tilde{k},\tilde{k}'}$ with $\xi_\iota \in D^{m,j,\tilde{k},\tilde{k}'}$ and for any $D_+^{m,j,\tilde{k},\tilde{k}'}$ and $D_-^{m,j,\tilde{k},\tilde{k}'}$, then we arrive at the quadrature rule (4.6) for the remaining subdomains and the approximate values $c'_{\xi',\xi}$ for the nonzero values $c_{\xi',\xi}$ in (4.8) are completely defined.

Finally, for the computation of $\int_{\Gamma_{i'}^{\xi'}} K_M(\xi_\iota, y)\, d_y\Gamma$, cf. (4.8), in case of $|\alpha| = 1$ and $\mathbf{k} = 1$, cf. (4.3), we mention the formulae

$$\begin{aligned}\int_a^{a'}\int_b^{b'} & \frac{cx+dy}{\{ex^2+fxy+gy^2\}^{3/2}}\,dy\,dx \\ &= -\frac{2gc-fd}{\sqrt{g}[4eg-f^2]}\Big\{\operatorname{arsh}\frac{2gb'+fa'}{a'[4eg-f^2]} - \operatorname{arsh}\frac{2gb'+fa}{a[4eg-f^2]} \\ &\qquad - \operatorname{arsh}\frac{2gb+fa'}{a'[4eg-f^2]} + \operatorname{arsh}\frac{2gb+fa}{a[4eg-f^2]}\Big\} \\ &\quad - \frac{2ed-fc}{\sqrt{e}[4eg-f^2]}\Big\{\operatorname{arsh}\frac{2ea'+fb'}{b'[4eg-f^2]} - \operatorname{arsh}\frac{2ea'+fb}{b[4eg-f^2]} \\ &\qquad - \operatorname{arsh}\frac{2ea+fb'}{b'[4eg-f^2]} + \operatorname{arsh}\frac{2ea+fb}{b[4eg-f^2]}\Big\}, \\ &\quad 0<a<a', \quad 0<b<b', \quad f^2<4eg,\end{aligned}$$

$$\begin{aligned}\int_0^{a'}\int_b^{b'} & \frac{cx+dy}{\{ex^2+fxy+gy^2\}^{3/2}}\,dy\,dx \\ &= -\frac{2gc-fd}{\sqrt{g}[4eg-f^2]}\Big\{\operatorname{arsh}\frac{2gb'+fa'}{a'[4eg-f^2]} - \operatorname{arsh}\frac{2gb+fa'}{a'[4eg-f^2]} - \log\frac{b'}{b}\Big\} \\ &\quad - \frac{2ed-fc}{\sqrt{e}[4eg-f^2]}\Big\{\operatorname{arsh}\frac{2ea'+fb'}{b'[4eg-f^2]} - \operatorname{arsh}\frac{2ea'+fb}{b[4eg-f^2]}\Big\}, \\ &\quad 0=a<a', \quad 0<b<b', \quad f^2<4eg,\end{aligned}$$

$$\begin{aligned}\int_0^{h}\int_0^{h} & \frac{cx+dy}{\{ex^2+fxy+gy^2\}^{3/2}}\,dy\,dx \\ &= \text{p.f.}\lim_{\varepsilon\to 0}\int\int_{\{(x,y)\in[0,h]^2: ex^2+fxy+gy^2\ge\varepsilon^2\}}\cdots \\ &= -\frac{2gc-fd}{\sqrt{g}[4eg-f^2]}\Big\{1-\log\frac{h[e+f+g]}{\sqrt{g}} \\ &\qquad - \operatorname{arsh}\frac{2g+f}{[4eg-f^2]} + \operatorname{arsh}\frac{f}{[4eg-f^2]}\Big\} \\ &\quad - \frac{2ed-fc}{\sqrt{e}[4eg-f^2]}\Big\{1-\log\frac{h[e+f+g]}{\sqrt{e}} \\ &\qquad - \operatorname{arsh}\frac{2e+f}{[4eg-f^2]} + \operatorname{arsh}\frac{f}{[4eg-f^2]}\Big\}, \\ &\quad f^2<4eg.\end{aligned}$$

Note that the kernel of the singular integral equation corresponding to the oblique derivative boundary value problem, cf. [**27, 25, 28**], admits a representation (4.3) with $|\alpha| = \mathbf{k} = 1$. Further details of the algorithm for the assembling of the matrix are discussed in [**35**].

Remark 4.1. To reduce the number of quadrature knots for the computation of the singular integrals, i.e., for (4.19), it is possible to choose different Gauß orders $n_{G,1}$ for the t'_1 direction and $n_{G,2}$ for the t'_2 direction. It is sufficient to take $n_{G,1} \geq b$ and $n_{G,2} \geq bj$.

4.3. *The error of the quadrature.* We introduce the compressed and discretized matrix $C'_j := (c'_{\xi',\xi})_{\xi',\xi\in\Delta_j}$, where the nonzero entries $c'_{\xi',\xi}$ are given in (4.8). By A'_j we denote the operator in $\mathcal{L}(S_j)$ whose matrix with respect to the basis $\{\varphi_{j,\xi} : \xi \in \Delta_j\}$ is $R_jC'_jE_j$. Thus, the quadrature algorithm for the stiffness matrix A_j leads to the fully discretized equation $A'_ju_j = P_jv$.

Theorem 4.1. *Suppose that the righthand side v of* (2.1) *belongs to the Sobolev space $H^2(\Gamma)$ and that the compressed collocation method including the approximate operator $\tilde{A}_j$ is stable, cf. Theorem* 3.1. *If the compression parameter a, cf.* (3.66), *and the quadrature parameter b, cf.* (4.12), *are sufficiently large, then the operators $A'_j \in \mathcal{L}(S_j)$ are stable. Additionally, if the second order estimate of Theorem* 3.1 *is valid, and if $u_j \in S_j$ denotes the solution of $A'_ju_j = P_jv$, then*

$$\|u - u_j\|_{L^2(\Gamma)} \leq Ch_j^2 \log h_j. \tag{4.20}$$

The number of nonzero entries for the matrix C'_j is the same as that for C_j, i.e., it is less than $CN_j^2[\log N_j]^4$.

For the proof, we need the following two estimates of the quadrature error.

Lemma 4.1. i) [**18**] *Consider the square $[a,b] \times [c,d]$ of the size $h = b - a = d - c$. Suppose that f is twice continuously differentiable over $[a,b]\times[c,d]$ and that $GR(f)$ stands for the tensor product of the one point Gauß rule, i.e., the midpoint rule, applied to f over $[a,b] \times [c,d]$.*

Then it is not hard to see that

$$\left|\int_a^b \int_c^d f(t_1,t_2)\,dt_2\,dt_1 - GR(f)\right| \leq Ch^4 \sup_{\substack{\beta\in\{(2,0),(0,2)\} \\ a\leq t_1\leq b, c\leq t_2\leq d}} |\partial_t^\beta f(t_1,t_2)|, \tag{4.21}$$

where the constant C is independent of $[a,b]\times[c,d]$ and f.

ii) [**17, 31**]. *Now consider a rectangle $[a,b]\times[c,d]$, set $h := b-a$ and $h' := d-c$, and suppose that f is analytic over $[a,b]\times[c,d]$. Moreover, suppose that f admits complex analytic extensions to the sets*

$$\{(t_1,t_2)\in \mathbf{R}\times\mathbf{C} : a\leq t_1\leq b, |t_2-c|+|t_2-d|\leq(\varrho+\varrho^{-1})h'/2\},$$
$$\{(t_1,t_2)\in \mathbf{C}\times\mathbf{R} : c\leq t_2\leq d, |t_1-a|+|t_1-b|\leq(\varrho+\varrho^{-1})h/2\},$$

where $\varrho > 1$. We denote the ellipse $\{t_1\in\mathbf{C} : |t_1-a|+|t_1-b| = (\varrho+\varrho^{-1})h/2\}$ by $\mathcal{E}_\varrho(a,b)$, define $\mathcal{E}_\varrho(c,d)$ similarly and consider the tensor product of the univariate n_G-point Gauß rule $GR(f)$ applied to f over $[a,b]\times[c,d]$. Then, for a constant C independent of $[a,b]\times[c,d]$ and f, we get, cf. [**17**], *Equation* (*4.6.1.11*) *and* [**31**, *Proposition 4.3*]

$$\left|\int_a^b \int_c^d f(t_1,t_2)\,dt_2\,dt_1 - GR(f)\right| \leq Chh'\varrho^{-2n_G}\Big\{\max_{\substack{t_2\in\mathcal{E}_\varrho(c,d) \\ a\leq t_1\leq b}} |f(t_1,t_2)| + \max_{\substack{t_1\in\mathcal{E}_\varrho(a,b) \\ c\leq t_2\leq d}} |f(t_1,t_2)|\Big\}. \tag{4.22}$$

Proof of Theorem 4.1. i) First we suppose that the integrals over the subdomains $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' = j$ are computed exactly and consider the quadrature errors over the domains $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$. For any function $\tilde{u}_j = \sum_{\xi\in\Delta_j}\tilde{u}_\xi\psi_\xi \in S_j$, we introduce the functions $\tilde{u}_l := \sum_{\xi\in\Delta_l}\tilde{u}_\xi\psi_\xi = \sum_{l'=-1}^{l-1}\sum_{\xi\in\nabla_{l'}}\tilde{u}_\xi\psi_\xi$ and their coefficients $\tilde{w}_{l,\xi}$ defined by $\tilde{u}_l = \sum_{\xi\in\Delta_l}\tilde{w}_{l,\xi}\varphi_{l,\xi}$. We will represent $\tilde{A}_j - A'_j = R_j(C_j - C'_j)E_j \in \mathcal{L}(S_j)$ as

$$(\tilde{A}_j - A'_j)\tilde{u}_j = \sum_{\xi'\in\Delta_j}\Big\{\sum_{l=0}^{j}\sum_{\xi\in\Delta_l} e_{\xi',(l,\xi)\tilde{w}_{l,\xi}} + \sum_{\xi\in\Delta_j} e_{\xi',\xi}\tilde{u}_\xi\Big\}\chi_{\xi'}. \tag{4.23}$$

This representation will have similar properties as the matrix of the compression error, i.e., it permits the application of a Schur lemma argument. We will show the sparsity pattern of this representation and, later, we will derive a bound for $\tilde{A}_j - A'_j$ by estimating $e_{\xi',(l,\xi)}$ and $e_{\xi',\xi}$. To get (4.23), we suppose that ξ' is fixed. Then the coefficient of $\chi_{\xi'}$ in (4.23) is the sum of the quadrature errors over the domains $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'} \subseteq \Gamma_i^j = D^{m,l,k,k'}$ corresponding to the integrand functions

$$
\begin{aligned}
y \longmapsto \vartheta_{\xi'}(K_A(\cdot,y)\tilde{u}_j^{\xi'}(y) - K_M(\cdot,y)\tilde{u}_j^{\xi'}(\xi_{i'}^{\xi'})) \\
:= \sum_{\iota=1}^{3} \lambda_\iota (K_A(\xi_\iota,y)\tilde{u}_j^{\xi'}(y) - K_M(\xi_\iota,y)\tilde{u}_j^{\xi'}(\xi_{i'}^{\xi'})),
\end{aligned}
\tag{4.24}
$$

where

$$
\tilde{u}_j^{\xi'} := \sum_{\xi\in\Delta_j : c_{\xi',\xi}\neq 0} \tilde{u}_\xi \psi_\xi .
$$

We consider a fixed subdomain $\Gamma_i^j = D^{m,l_D,k,k'}$ containing sets of the form $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$. From the definition of the Γ_i^j we observe that there exists a $\psi_{\xi''}$ such that $c_{\xi',\xi''} \neq 0$, $l(\xi'') = l_D - 1$, and $\operatorname{supp}\psi_{\xi''} \cap \Gamma_i^j \neq \varnothing$ (otherwise the partition step leading to Γ_i^j would be redundant). In view of (3.66) we get

$$
\begin{aligned}
\operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \operatorname{supp}\psi_{\xi''}\} &\leq aj2^{j-l(\xi')-(l_D-1)}, \\
\operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\} &\leq C2^{-(l_D-1)} + aj2^{j-l(\xi')-(l_D-1)}.
\end{aligned}
\tag{4.25}
$$

Consequently, if j is sufficiently large, then, for any ψ_ξ with $l(\xi) < l_D - 1$ and $\operatorname{supp}\psi_\xi \cap \Gamma_i^j \neq \varnothing$, we arrive at

$$
\begin{aligned}
\operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \operatorname{supp}\psi_\xi\} &\leq C2^{-(l_D-1)} + aj2^{j-l(\xi')-(l_D-1)} \\
&\leq aj2^{j-l(\xi')-l(\xi)}.
\end{aligned}
\tag{4.26}
$$

This means $c_{\xi',\xi} \neq 0$. In other words, the restriction $\tilde{u}_j^{\xi'}|_{\Gamma_i^j}$ is equal to the $\tilde{u}_{l_D-1}$ plus some of the terms $\tilde{u}_\xi\psi_\xi$ with $\xi \in \nabla_{l_D-1}$. The quadrature error corresponding to (4.24) over Γ_i^j is equal to the quadrature error corresponding to the function

$$
\begin{aligned}
y \longmapsto \sum_{\xi\in\Delta_{l_{D-1}}} & \tilde{w}_{l_{D-1},\xi}\vartheta_{\xi'}\Big(K_A(\cdot,y)\varphi_{l_{D-1},\xi}(y) \\
& \qquad - K_M(\cdot,y)\varphi_{l_{D-1},\xi}(\xi^{\xi'}_{i'})\Big) \\
+ \sum_{\xi\in\nabla_{l_{D-1}}:c_{\xi',\xi}\neq 0} & \tilde{u}_\xi\vartheta_{\xi'}\Big(K_A(\cdot,y)\psi_\xi(y) \\
& \qquad - K_M(\cdot,y)\psi_\xi(\xi^{\xi'}_{j'})\Big).
\end{aligned}
\tag{4.27}
$$

The entry $e_{\xi',(l,\xi)}$ is now the sum over all quadrature errors for the integrand functions

$$
y \longmapsto \vartheta_{\xi'}(K_A(\cdot,y)\varphi_{l,\xi}(y) - K_M(\cdot,y)\varphi_{l,\xi}(\xi^{\xi'}_{i'}))
$$

taken over all subdomains $\Gamma^j_i = D^{m,l_D,k,k'}$ with $l_D - 1 = l$ and $\operatorname{supp}\varphi_{l,\xi}\cap\Gamma^j_i \neq \varnothing$. Similarly, for $c_{\xi',\xi}\neq 0$, the entry $e_{\xi',\xi}$ is defined as the sum over all quadrature errors of the functions

$$
y \longmapsto \vartheta_{\xi'}(K_A(\cdot,y)\psi_\xi(y) - K_M(\cdot,y)\psi_\xi(\xi^{\xi'}_{i'}))
$$

taken over all subdomains $\Gamma^j_i = D^{m,l_D,k,k'}$ with $l_D - 1 = l(\xi)$ and $\operatorname{supp}\psi_\xi\cap\Gamma^j_i\neq\varnothing$. For $c_{\xi',\xi}=0$, we set $e_{\xi',\xi}=0$.

Note that $e_{\xi',(l,\xi)}=0$ and $e_{\xi',\xi}=0$ is possible also if there is no Γ^j_i with $l_D-1=l$, $\operatorname{supp}\varphi_{l,\xi}\cap\Gamma^j_i\neq\varnothing$ and $l_D-1=l(\xi)$, $\operatorname{supp}\psi_\xi\cap\Gamma^j_i\neq\varnothing$, respectively. More precisely, $e_{\xi',(l,\xi)}\neq 0$ implies the existence of $\Gamma^j_i = D^{m,l_D,k,k'}$ such that $l_D-1=l$ and $\operatorname{supp}\varphi_{l,\xi}\cap\Gamma^j_i\neq\varnothing$. From the definition of Γ^j_i, we infer $c_{\xi',\xi'''}=0$, for all the $\psi_{\xi'''}$ such that $\psi_{\xi'''}|_{\Gamma^j_i}$ is not polynomial. Hence, for $\operatorname{supp}\psi_{\xi'''}\cap\Gamma^j_i\neq\varnothing$ and $l(\xi''')=l_D$, we get $c_{\xi',\xi'''}=0$. This implies, cf. (3.66),

$$
\operatorname{dist}(\operatorname{conv}\vartheta_{\xi'},\Gamma^j_i) \geq \min\operatorname{dist}(\operatorname{conv}\vartheta_{\xi'},\operatorname{supp}\psi_{\xi'''}) > aj2^{j-l(\xi')-l_D}.
$$

Consequently, $e_{\xi',(l,\xi)}\neq 0$ implies

$$
\operatorname{dist}(\operatorname{conv}\vartheta_{\xi'},\operatorname{supp}\varphi_{(l,\xi)}) > Caj2^{j-l(\xi')-l}.
\tag{4.28}
$$

Similarly, we get that $e_{\xi',\xi} \neq 0$ implies

$$\text{dist}\,(\text{conv}\,\vartheta_{\xi'}, \text{supp}\,\psi_\xi) > Caj2^{j-l(\xi')-l(\xi)}. \tag{4.29}$$

Having derived the sparsity pattern of representation (4.23), we turn to the estimate of its entries. From the definition of $\vartheta_{\xi'}$, we infer the existence of an $x' \in \text{conv}\,\vartheta_{\xi'}$ such that, cf. (3.64),

$$\begin{aligned}\vartheta_{\xi'}(K_A(\cdot,y)\varphi_{l,\xi}(y) - K_M(\cdot,y)\varphi_{l,\xi}(\xi^{\xi'_{i'}})) \\ = 2^{-3l(\xi')}\partial_x^\alpha[K_A(x',y)\varphi_{l,\xi}(y) - K_M(x',y)\varphi_{l,\xi}(\xi_{i'}^{\xi'})],\end{aligned} \tag{4.30}$$

where ∂_x^α denotes a certain second order derivative (directional derivative) with respect to x. Applying the composite tensor product one point Gauß rule GR to this integrand over the square $\Gamma_i^j = D^{m,l+1,k,k'}$ of side length $2^{-(l+1)}$ and using the second order convergence estimate (4.21), we conclude

$$\begin{aligned}|e_{\xi',(l,\xi)}| \leq C2^{-3l(\xi')}2^{-4l} \\ \cdot \sum_{\Gamma_i^j:\Gamma_i^j \subseteq \text{supp}\,\varphi_{l,\xi}} \sup_{\substack{\beta:|\beta|=2 \\ y\in D^{m,l+1,k,k'}}} |\partial_y^\beta \partial_x^\alpha [K_A(x',y)\varphi_{l,\xi}(y) \\ - K_M(x',y)\varphi_{l,\xi}(\xi_{i'}^{\xi'})]|.\end{aligned} \tag{4.31}$$

The scaling factor $N_l \sim 2^l$ in the definition of $\varphi_{l,\xi}$, an additional factor $N_l \sim 2^l$ for each derivative of $\varphi_{l,\xi}$, the estimate (2.2), and a similar estimate for the kernel K_M lead to

$$|e_{\xi',(l,\xi)}| \leq \sum_{k=0}^{2} C2^{-3l(\xi')-3l+kl}\text{dist}\,\{\text{conv}\,\vartheta_{\xi'}, \text{supp}\,\varphi_{l,\xi}\}^{-6+k}. \tag{4.32}$$

Analogously, we obtain

$$|e_{\xi',\xi}| \leq \sum_{k=0}^{2} C2^{-3l(\xi')-3l(\xi)+kl(\xi)}\text{dist}\,\{\text{conv}\,\vartheta_{\xi'}, \text{supp}\,\psi_\xi\}^{-6+k}. \tag{4.33}$$

The sparsity patterns (4.28) and (4.29) as well as the estimates (4.32) and (4.33) together with a Schur lemma argument similar to (3.69) imply that the $l^2(\cup\Delta_l \cup \Delta_j) \to l^2(\Delta_j)$ norm of the matrix with the entries $e_{\xi',(l,\xi)}$ and $e_{\xi',\xi}$ is less than $Ca^{-2}j^{-3/2}$. Using Lemmas 3.3 and 3.4, we get

$$\sqrt{\sum_{l=1}^{j}\sum_{\xi\in\Delta_l}|\tilde{w}_{l,\xi}|^2+\sum_{\xi\in\Delta_j}|\tilde{u}_\xi|^2} \le \sqrt{\sum_{\xi\in\Delta_j}(j+1-l(\xi))|\tilde{u}_\xi|^2} \le C\sqrt{j}\|\tilde{u}_j\|_{L^2(\Gamma)}, \tag{4.34}$$

and $\|\tilde{A}_j - A'_j\| \le Ca^{-2}j^{-1/2}$. Hence, for sufficiently large a or j, the operator A'_j is a small perturbation of $\tilde{A}_j$. Together with $\tilde{A}_j$, also A'_j has a uniformly bounded inverse.

Now we return to the error estimate (4.20). First we will show

$$\|(\tilde{A}_j - A'_j)\tilde{u}_j\|_{L^2(\Gamma)} \le Ch_j^2 \log h_j, \tag{4.35}$$

where $\tilde{u}_j = P_j u = P_j A^{-1} v$. From Lemma 3.4 we infer

$$\begin{aligned}\Big\|\sum_{\xi'\in\Delta_j} v_{\xi'}\chi_{\xi'}\Big\|_{L^2(\Gamma)} &\le C\sqrt{j}\sqrt{\sum_{\xi'\in\Delta_j}|v_{\xi'}|^2}\\ &\le C\sqrt{j}\sqrt{\sum_{\xi'\in\Delta_j}2^{-2l(\xi')}}\sup_{\xi'\in\Delta_j}|2^{l(\xi')}v_{\xi'}|\\ &\le Cj\sup_{\xi'\in\Delta_j}|2^{l(\xi')}v_{\xi'}|.\end{aligned} \tag{4.36}$$

Hence it suffices to estimate the quadrature errors of $2^{l(\xi')}\vartheta_{\xi'}(\tilde{A}_j\tilde{u}_j) = 2^{l(\xi')}\vartheta_{\xi'}(A_j\tilde{u}_j^{\xi'})$ for each ξ' separately. In order to apply (4.21) we have to estimate the second order derivatives with respect to y of the integrand function, cf. (4.24) and (4.30),

$$\begin{aligned} y \longmapsto\ & 2^{-3l(\xi')}\partial_x^\alpha[K_A(x',y)\tilde{u}_j^{\xi'}(y) - K_M(x',y)\tilde{u}_j^{\xi'}(\xi_{i'}^{\xi'})]\\ &= 2^{-3l(\xi')}\partial_x^\alpha[[K_A(x',y)-K_M(x',y)]\tilde{u}_j^{\xi'}(y)\\ &\qquad + K_M(x',y)[\tilde{u}_j^{\xi'}(y) - \tilde{u}_j^{\xi'}(\xi_{i'}^{\xi'})]] \quad . \end{aligned} \tag{4.37}$$

The kernel functions K_A and K_M, however, satisfy (2.2) and

$$|\partial_y^\beta\partial_x^\alpha[K_A(x,y) - K_M(x,y)]| \le C|x-y|^{-1-|\alpha|-|\beta|}. \tag{4.38}$$

Moreover, Lemma 3.3 ii) implies

$$\begin{aligned}\tilde{u}_j^{\xi'}(x) - \tilde{u}_j^{\xi'}(y) &= \sum u_\xi[\psi_\xi(x) - \psi_\xi(y)]\\ |\tilde{u}_j^{\xi'}(x) - \tilde{u}_j^{\xi'}(y)| &\le C \sum |u_\xi| 2^{2l(\xi)} |x-y|\\ &\le C \sqrt{\sum 2^{4l(\xi)} |u_\xi|^2} \sqrt{\sum_{\substack{\xi:\psi_\xi(x)\neq 0\\ \text{or } \psi_\xi(y)\neq 0}} 1}\ |x-y|\\ &\le Cj|x-y|.\end{aligned} \tag{4.39}$$

Similarly, we get $|\tilde{u}_j^{\xi'}(x)| \le C\sqrt{j}$ and $|\partial_x^\beta \tilde{u}_j^{\xi'}(x)| \le Cj$ where $|\beta| = 1$. Note that the higher derivatives with $\beta = (2,0)$ or $\beta = (0,2)$ vanish since $\tilde{u}_j^\xi$ is bilinear. Using these estimates and applying (4.21) to the quadrature error for the integration of (4.37) over $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'} \subseteq \Gamma_i^j = D^{m,l,k,k'}$, we arrive at the bound

$$C 2^{-4l'} 2^{-3l(\xi')} j \operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_{i'}^{\xi'}\}^{-5}. \tag{4.40}$$

In view of (4.10), we have $2^{-2l'} = 2^{-2(l+l'')} \le 2^{-2l}\operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\}$. Summing up (4.40) over all $\Gamma_{i'}^{\xi'} \subseteq \Gamma_i^j$, we get the bound

$$\begin{aligned}\sum_{\Gamma_{i'}^{\xi'}:\Gamma_{i'}^{\xi'}\subseteq\Gamma_i^j} & C 2^{-2l'} 2^{-2l} 2^{-3l(\xi')} j \operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\}^{-4}\\ &= C 2^{-4l} 2^{-3l(\xi')} j \operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\}^{-4}\end{aligned}$$

for the quadrature error over Γ_i^j. Hence, the quadrature error for $2^{l(\xi')}\vartheta_{\xi'}(\tilde{A}_j\tilde{u}_j)$ is less than

$$Cj 2^{-2l(\xi')} \sum_{l=0}^{j-1} 2^{-2l} \sum_{\Gamma_i^j:\Gamma_i^j = D^{m,l,k,k'}} \operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\}^{-4} 2^{-2l}. \tag{4.41}$$

We observe, from the definition of the Γ_i^j, that $c_{\xi',\xi} = 0$ for all ξ with $\operatorname{supp}\psi_\xi \cap \Gamma_i^j \neq \varnothing$ and $l(\xi) \ge l$ (otherwise Γ_i^j would have been divided in further steps). In view of (3.66) this means that

$$\operatorname{dist}\{\operatorname{conv}\vartheta_{\xi'}, \Gamma_i^j\} \ge aj 2^{j-l(\xi')-l}. \tag{4.42}$$

Using (3.73), we estimate (4.41) by

$$Cj2^{-2l(\xi')}\sum_{l=0}^{j-1}2^{-2l}(aj2^{j-l(\xi')-l})^{-2}\le Ca^{-2}2^{-2j}. \tag{4.43}$$

This together with (4.36) proves that the L^2 norm of the quadrature error is less than $Cj2^{-2j}$. The estimate (4.35) is proved. Now equation (4.20) follows easily from this estimate, the corresponding consistency estimate (3.68) and the boundedness of the inverses A'_j.

ii) Next we suppose that the integrals over the subdomains $\Gamma^{\xi'}_{i'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$ or with the singularity point ξ_ι in $\Gamma^{\xi'}_{i'}$ are computed exactly and consider the quadrature errors over the domains $\Gamma^{\xi'}_{i'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' = j$ and $\xi_i \notin \Gamma^{\xi'}_{i'}$. We fix a $\vartheta_{\xi'}$, a ψ_ξ and a $\Gamma^{\xi'}_{i'} = D^{m,j,\tilde{k},\tilde{k}'}$. For these, we estimate the quadrature error $d_{\xi',\xi} = c_{\xi',\xi} - c'_{\xi',\xi}$ over $\Gamma^{\xi'}_{i'}$ with the help of (4.22). Thus,

(4.44)

$$\begin{aligned} f(t) &= \sum_{\iota=1}^{3}\lambda_\iota K_A(\xi_\iota,\kappa_m(t))|\kappa'_m(t)|\psi_\xi(\kappa_m(t)),\\ K_A(\xi_\iota,\kappa_m(t)) &= \sum_{\mathbf{k}\le|\alpha|} s_\alpha\bigg(\xi_\iota,\kappa_m(t),\frac{\partial_{t_1}\kappa_m(t)\times\partial_{t_2}\kappa_m(t)}{|\partial_{t_1}\kappa_m(t)\times\partial_{t_2}\kappa_m(t)|}\bigg)\\ &\qquad\qquad\cdot(\xi_\iota-\kappa_m(t))^\alpha|\xi_\iota-\kappa_m(t)|^{-2-\mathbf{k}}. \end{aligned}$$

From the analyticity assumption on the s_α, cf. the analyticity domains (4.4), and the boundedness of the derivatives of the parametrization, cf. (4.2), we observe that the function $f|_{\kappa_m^{-1}(D^{m,j,\tilde{k},\tilde{k}'})}$ extends to a complex analytic function over a neighborhood $\{t : \operatorname{dist}\{t,\kappa_m^{-1}(D^{m,j,\tilde{k},\tilde{k}'})\}\le \varepsilon_B\}$. Here we have to require $\varepsilon_B \le \varepsilon_A/C$ for the analyticity of $t\mapsto s_\alpha(\xi_\iota,\kappa_m(t),\dots)$ and $\varepsilon_B \le \operatorname{dist}\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}/C$ for the analyticity of $t\mapsto|\xi_\iota-\kappa_m(t)|^{-2-k}$. Thus, the assumptions of Lemma 4.1 ii) are satisfied if we choose

$$\varrho := 1+\operatorname{dist}\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}/[C'h_j] \tag{4.45}$$

with a sufficiently large constant C'. To get a bound for f over $[a,b]\times\mathcal{E}_\varrho(c,d)$ and $\mathcal{E}_\varrho(a,b)\times[c,d]$, we observe that $|K_A(\xi_\iota,\kappa_m(t))|$ is less than

$C\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}^{-2}$, that $|\kappa'_m(t)|$ is bounded by a constant, and that the absolute value of the bilinear extension of $\psi_\xi(\kappa_m(\cdot))|_{\kappa_m^{-1}(D^{m,j,\tilde{k},\tilde{k}'})}$ is less than $C2^{l(\xi)}[2^{l(\xi)}\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}+1]^2$. Using these bounds, $\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\} \geq 2^{-j}$, and $|\lambda_\iota| \leq C2^{-l(\xi')}$, cf. (3.48), we get that the quadrature error for the integration of f over $D^{m,j,\tilde{k},\tilde{k}'}$ is less than

$$\sum_{\iota=1}^{3} C2^{-l(\xi')}2^{-2j}\varrho^{-2n_G}\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}^{-2} \cdot 2^{l(\xi)}[2^{l(\xi)}\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}+1]^2 \leq C2^{j}\varrho^{-2n_G}. \tag{4.46}$$

We have to sum up over all $\Gamma_{i'}^{\xi'} = D^{m,j,\tilde{k},\tilde{k}'} \subseteq \text{supp}\,\psi_\xi$. The number of subsquares $D^{m,j,\tilde{k},\tilde{k}'}$ is less than 2^{2j} and we arrive at

$$|d_{\xi',\xi}| \leq \sum_{\iota=1}^{3} C2^{3j}\varrho^{-2n_G}. \tag{4.47}$$

We will show that the l^2 norm of the matrix $(d_{\xi',\xi})_{\xi',\xi}$ is less than $C2^{-2j}$, where C is a constant. If this is done, then the norm of $\tilde{A}_j - A'_j$ is less than $C\sqrt{j}N_j^{-2}$, cf. (3.51), and the convergence rate (4.35) is proved. Moreover, since the operators A'_j and $|A'_j|^{-1}$ are small perturbations of the bounded operators $\tilde{A}_j$ and $[\tilde{A}_j]^{-1}$, respectively, they are uniformly bounded. The estimate (4.20) follows as in point i) of this proof.

Clearly to show the norm estimate for $(d_{\xi',\xi})_{\xi',\xi}$, it suffices to prove that the l^2 norm of the matrix entries (Frobenius norm) is less than the desired bound. Hence, we only have to show $|d_{\xi',\xi}| \leq C2^{-4j}$. In view of (4.47) and (4.45) it remains to prove the uniform boundedness of

$$C2^{3j}\varrho^{-2n_G}2^{4j} \leq C2^{7j}\varrho^{-2n_G} \leq C2^{8j-2\log_2\{1+\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}/[C'h_j]\}n_G}. \tag{4.48}$$

The last expression, however, is bounded if

$$\begin{aligned} n_G &\geq \frac{4j}{\log_2\{1+\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}/[C'h_j]\}} \\ &\geq C\frac{j}{\max\{1, \log_2[\text{dist}\,\{\xi_\iota, D^{m,j,\tilde{k},\tilde{k}'}\}/h_j]\}}. \end{aligned} \tag{4.49}$$

This is fulfilled if b is sufficiently large, cf. (4.12).

iii) Now we suppose that the integrals over the subdomains $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' < j$ or with the singularity point ξ_ι not in $\Gamma_{i'}^{\xi'}$ are computed exactly and consider the quadrature errors over the domains $\Gamma_{i'}^{\xi'} = D^{m,l',\tilde{k},\tilde{k}'}$ with $l' = j$ and $\xi_\iota \in \Gamma_{i'}^{\xi'}$. We proceed analogously to the step ii). For fixed $\vartheta_{\xi'}$, ψ_ξ, and $\Gamma_{i'}^{\xi'} = D^{m,j,\tilde{k},\tilde{k}'}$, we estimate the quadrature error $d_{\xi',\xi} = c_{\xi',\xi} - c'_{\xi',\xi}$ over $\Gamma_{i'}^{\xi'}$ with the help of (4.22). Thus, cf. (4.19),

$$\begin{aligned} f(t) &= f(t_1,t_2) \\ &= \lambda_\iota g((\tilde{k}-1)h_j + t_1, (\tilde{k}'-1)h_j + t_1 t_2)t_1, \\ &\qquad t \in [0,h_j] \times [0,1]. \end{aligned} \tag{4.50}$$

Due to the subtraction of singularity and due to Duffy's transformation, there is no singularity in the integrand anymore. From the analyticity assumption on the s_α, cf. the analyticity domains (4.4), and the boundedness of the derivatives of the parametrization, cf. (4.2), we observe that the function $f|_{[0,h_j]\times[0,1]}$ extends to a complex analytic function over the analyticity sets of Lemma 4.1 ii), if $\varrho h_j \le \varepsilon_A/C$ and $\varrho[1-0] \le \varepsilon_A/C$. Thus we choose $\varrho := 1/C'$ with a sufficiently large constant C'. To get a bound for f over $[0,h_j] \times \mathcal{E}_\varrho(0,1)$ and $\mathcal{E}_\varrho(0,h_j) \times [0,1]$, we observe that $|f(t)|$ is less than constant times $|\lambda_\iota|$ times the supremum norm of the extended polynomials $(t_1,t_2) \mapsto \psi_\xi(\kappa_m(t_1,t_1t_2))$ and of their first order derivatives. We get $|f(t)| \le C2^{-l(\xi')}2^{2j}$ as well as the bound

$$C2^{-j}\varrho^{-2n_G}2^{-l(\xi')}2^{2j} \le C2^j\varrho^{-2n_G}$$

for the quadrature error of f over $D^{m,j,\tilde{k},\tilde{k}'}$, cf. (4.22). We have to sum up over all $D^{m,j,\tilde{k},\tilde{k}'} \subseteq \operatorname{supp}\psi_\xi$ with $\xi_\iota \in D^{m,j,\tilde{k},\tilde{k}'}$, i.e., over no more than four sets for each Γ_m. Consequently, we arrive at

$$\begin{aligned} |d_{\xi',\xi}| &\le C2^j\varrho^{-2n_G} \\ 2^{4j}|d_{\xi',\xi}| &\le C2^{5j}\varrho^{-2n_G} \\ &\le C2^{6j-2\log_2\{1/C'\}n_G}. \end{aligned} \tag{4.51}$$

The last expression, however, is bounded if

$$n_G \ge \frac{3j}{\log_2\{1/C'\}}, \tag{4.52}$$

which is fulfilled for sufficiently large b. ◻

4.4. *The complexity.* Clearly the number of arithmetic operations for the computation of the stiffness matrix in the form of its discretized and compressed wavelet transform is bounded by a constant multiple of the number of quadrature knots.

Theorem 4.2. *The number of quadrature knots for the quadrature algorithm in Section* 4.2 *is less than* $CN_j^{8/3}[\log N_j]^{4/3}$.

Proof. First we fix a $\vartheta_{\xi'}$ and count the quadrature knots for the computation of $\vartheta_{\xi'}(A_j' u_j)$. To count the points contained in $\Gamma_i^j = D^{m,l,k,k'}$, we observe, cf. (4.10), (4.25) and (4.42),

$$\begin{aligned} 2^{-2l''} &\sim \operatorname{dist}\{\operatorname{supp}\vartheta_{\xi'}, \Gamma_i^j\} \\ &\sim aj2^{j-l(\xi')-l}, \\ l'' &\sim [l + l(\xi') - j - \log_2 j - C]/2. \end{aligned} \tag{4.53}$$

Thus $l'' < j - l$ holds if and only if $l < j - [l(\xi') - \log_2 j - C]/3$. For a fixed l with $l < j - [l(\xi') - \log_2 j - C]/3$, the subdomains $\Gamma_i^j = D^{m,l,k,k'}$ are contained in a domain of size $aj2^{j-l(\xi')-l}$, cf. (4.53), and are divided into square $\Gamma_{i'}^{\xi'}$ of size $2^{-l-l''} \sim 2^{-l-[l+l(\xi')-j-\log_2 j - C]/2}$. In each $\Gamma_{i'}^{\xi'}$ there is exactly one quadrature knot. Hence, the number of quadrature knots contained in all these Γ_i^j is equal to the number of subdomains $\Gamma_{i'}^{\xi'}$ in the union of the Γ_i^j, i.e., less than

$$C\left[\frac{aj2^{j-l(\xi')-l}}{2^{-l-[l+l(\xi')-j-\log_2 j - C]/2}}\right]^2 \le Cj2^{j-l(\xi')+l}. \tag{4.54}$$

On the other hand, all the subdomains $\Gamma_i^j = D^{m,l,k,k'}$ with $l \ge j - [l(\xi') - \log_2 j - C]/3$ are contained in a domain of size $aj2^{j-l(\xi')-\{j-[l(\xi')-\log_2 j - C]/3\}}$, cf. (4.53), and are divided into square $\Gamma_{i'}^{\xi'}$ of size 2^{-j}. Moreover, for the $O(n)$ subdomains $\Gamma_{i'}^{\xi'} = D^{m,j,\tilde{k},\tilde{k}'}$ which satisfy $\operatorname{dist}\{\xi_\iota, \Gamma_{i'}^{\xi'}\} \sim n2^{-j}$ and which are contained in the set of all these $\Gamma_i^j = D^{m,j,k,k'}$ with $l \ge j - [l(\xi') - \log_2 j - C]/3$, we get $n_G \sim Cj/(1 + \log n)$. The maximal number of such n is

$$n_{\max} = aj2^{j-l(\xi')-\{j-[l(\xi')-\log_2 j - C]/3\}}/2^{-j} \le Cj^{2/3}2^{j-2l(\xi')/3}. \tag{4.55}$$

Now the number of all quadrature knots in the union of all $\Gamma_i^j = D^{m,j,k,k'}$ with $l \geq j - [l(\xi') - \log_2 j - C]/3$ is bounded by, cf. [**23**, Section 6],

$$\sum_{n=1}^{n_{\max}} Cn \left[\frac{Cj}{1+\log n} \right]^2 \leq Cj^2 n_{\max}^2 \Big/ [\log n_{\max}]^2.$$

Using $\log n_{\max} \sim j$, we arrive at the bound $Cn_{\max}^2$. Consequently, the number of quadrature points for a fixed $\vartheta_{\xi'}$ is less than, cf. (4.55) and (4.54),

$$Cj^{4/3} 2^{2j-4l(\xi')/3} + \sum_{l=0}^{j-[l(\xi')-\log_2 j - C]/3} Cj 2^{j-l(\xi')+l} \leq Cj^{4/3} 2^{2j-4l(\xi')/3}. \tag{4.56}$$

Now we sum up the quadrature knots over all $\xi' \in \Delta_j$ and arrive at the bound

$$\sum_{l(\xi')=1}^{j-1} 2^{2l(\xi')} Cj^{4/3} 2^{2j-4l(\xi')/3} \leq Cj^{4/3} 2^{8j/3}. \qquad \square \tag{4.57}$$

Remark 4.2. Suppose that, in addition to the assumption iv) of Section 4.1, the parametrizations κ_m are thrice continuously differentiable over $\mathcal{S}$ and four times over the domains $\kappa_m^{-1}(D^{m,j,k,k'})$. Then the second term in the asymptotics of the kernel function K_A can be included into K_M such that, compare (4.38),

$$|\partial_y^\beta \partial_x^\alpha [K_A(x,y) - K_M(x,y)]| \leq C|x-y|^{-|\alpha|-|\beta|}. \tag{4.58}$$

Moreover, suppose that, for these K_M, the integrals $\int K_M(x,\cdot)\psi_\xi$ can be computed by analytic formulae. Then we set $\{\Gamma_{i'}^{\xi'} : i' \in \mathcal{N}\} := \{\Gamma_i^j : i = 1, \dots, M^j\}$, i.e., no further partition of the domains Γ_i^j is required, and define the quadrature rule over this partition analogously to Section 4.2. The discretized entries of the compressed stiffness matrix can be computed as

$$(4.59)\quad c_{\xi',\xi} \sim c'_{\xi',\xi} := \sum_{\iota=1}^{3} \lambda_\iota \Big\{ a(\xi_\iota)\psi_\xi(\xi_\iota) + \sum_{\mu\in\mathcal{M}} K_A(\xi_\iota, x_\mu)\psi_\xi(x_\mu)\omega_\mu + \sum_{i'\in\mathcal{N}':\Gamma^{\xi'}_{i'}\cap \operatorname{supp}\psi_\xi\neq\varnothing} \Big[\int_{\Gamma^{\xi'}_{i'}} K_M(\xi_\iota, y)\psi_\xi(y)\, d_y\Gamma - \sum_{\mu\in\mathcal{M}_{i'}} K_M(\xi_\iota, x_\mu)\psi_\xi(x_\mu)\omega_\mu \Big] \Big\}.$$

This algorithm leads to a stable and fully discretized method such that the assertion of Theorem 4.1 remains valid. The number of arithmetic operations is less than N_j^2 times a power of $\log N_j$. The proof for this almost optimal algorithm is analogous to those of Theorems 4.1 and 4.2.

Remark 4.3. Suppose that, in addition to the assumption iv) of Section 4.1, the parametrizations κ_m are bounded and analytic over small neighborhoods of $\mathcal{S}$. Then the singularity subtraction step is necessary only for the domains $D^{m,j,k,k'}$ containing the singularity points ξ_ι. Setting $\{\Gamma^{\xi'}_{i'} : i' \in \mathcal{N}\} := \{\Gamma^j_i : i = 1, \dots, M^j\}$ and defining the quadrature rule as the tensor product Gauß rule over this partition with the Gauß order n_G from (4.12), we again arrive at an algorithm such that the assertion of Theorem 4.1 remains valid and that the number of arithmetic operations is less than N_j^2 times a power of $\log N_j$.

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