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AN INTEGRAL OPERATOR SOLUTION TO THE MATRIX TODA EQUATIONS

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ABSTRACT. In previous work the author found solutions to the Toda equations that were expressed in terms of determinants of integral operators. Here it is observed that a simple variant yields solutions to the matrix Toda equations. As an application another derivation is given of a differential equation of Sato, Miwa and Jimbo for a particular Fredholm determinant.

During the last 20 years, beginning with [**2**], many connections have been established between determinants of integral operators and solutions of differential equations. A result proved in [**2**] can be shown to be equivalent to one concerning the integral operator K on $L^2(\mathbf{R}^+)$ with kernel

$$\frac{e^{-t(u+u^{-1}+v+v^{-1})/4}}{u+v}.$$

It is that the function $\tau := \log\det(I-\lambda^2K^2)$ has the representation

$$\tau = -\frac{1}{2}\int_t^\infty s\left(\left(\frac{d\varphi}{ds}\right)^2 - \sinh^2\varphi\right)ds, \tag{1}$$

where $\varphi = \varphi(t;\lambda)$ satisfies the differential equation

$$\frac{d^2\varphi}{dt^2} + \frac{1}{t}\frac{d\varphi}{dt} = \frac{1}{2}\sinh 2\varphi \tag{2}$$

with boundary condition

$$\varphi(t;\lambda) \sim 2\lambda K_0(t) \quad \text{as } t \longrightarrow \infty.$$

(Here K_0 is the usual modified Bessel function.) The differential equation for φ, the cylindrical sinh-Gordon equation, is reducible to a special case of the Painlevé III equation. The result of [**2**] was the

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first of several in which special integral operators were shown to have determinants expressible in terms of Painlevé functions.

The proof in [**2**] was combinatorial in nature and quite difficult. Simpler proofs of a somewhat stronger result have been obtained since then. Note that differentiating (1) twice and using the equation (2) gives the equivalent relation

$$\frac{d^2\tau}{dt^2} + \frac{1}{t}\frac{d\tau}{dt} = -\sinh^2\varphi. \tag{3}$$

It follows from results in [**1**], see also [**4**], that if we define $\tau^\pm := \log\det(I \pm \lambda K)$, then

$$\frac{d^2\tau^\pm}{dt^2} + \frac{1}{t}\frac{d\tau^\pm}{dt} = \frac{1-e^{\pm 2\varphi}}{4},$$

where φ solves (2). Adding the two equations give (3).

Subtracting the two equations and comparing with (2) shows that

$$\varphi = \log\det(I+\lambda K) - \log\det(I-\lambda K)$$

solves (2). Another proof of this fact was given in [**5**]. Here families of operators G_k (with $k \in \mathbf{Z}$) depending on parameters x and y were produced such that the functions $q_k := \log\det(I - G_{k+1}) - \log\det(I - G_k)$ satisfy the Toda equations

$$\frac{\partial^2 q_k}{\partial x \partial y} = e^{q_k - q_{k-1}} - e^{q_{k+1}-q_k}, \quad k \in \mathbf{Z}.$$

In a special case $\det(I - G_k)$ was a function of the product xy and $G_k(t/4, t/4)$ was equal to $(-1)^k\lambda K$ with K as given above. Equation (2) followed from these facts and the observation that $q_0 = \varphi$, $q_{-1} = q_1 = -\varphi$. Notice that these solutions of the Toda equations are 2-periodic in the sense that $q_{k+2} = q_k$.

The purpose of this note is to give a "Toda" proof of a generalization of the first-mentioned result which was established in [**3**]. Here a parameter θ was introduced into the kernel of K, so that it equals

$$\left(\frac{u}{v}\right)^{\theta/2} \frac{e^{-t(u+u^{-1}+v+v^{-1})/4}}{u+v}.$$

It was shown that, if we define

$$\tau := \log\det(I - \lambda^2 KK'),$$

$'$ is the transpose, then (3) holds, where φ now satisfies

$$\frac{d^2\varphi}{dt^2} + \frac{1}{t}\frac{d\varphi}{dt} = \frac{1}{2}\sinh 2\varphi + \frac{\theta^2}{t^2}\tanh\varphi\,\mathrm{sech}^2\varphi \tag{4}$$

with boundary condition,

$$\varphi(t;\lambda) \sim 2\lambda K_\theta(t) \quad \text{as } t \longrightarrow \infty.$$

This can also be reduced to a special case of the Painlevé III equation.

Since the determinant of $I - \lambda^2 KK'$ is equal to the determinant of the operator matrix

$$\begin{pmatrix} I & \lambda K \\ \lambda K' & I \end{pmatrix},$$

it is not surprising that this fact can be proved by extending the results of [**5**] to obtain solutions of the 2-periodic *matrix* Toda equations by means of operators with matrix-valued kernels. Notice that in the scalar case described above, if we set $Q_k := e^{q_k}$ then the Toda equations become

$$\frac{\partial}{\partial y}\left(\frac{\partial Q_k}{\partial x}\Big/ Q_k\right) = \frac{Q_k}{Q_{k-1}} - \frac{Q_{k+1}}{Q_k}. \tag{5}$$

The matrix Toda equations are the generalizations of this given by

$$\frac{\partial}{\partial y}\left(\frac{\partial Q_k}{\partial x}Q_k^{-1}\right) = Q_k Q_{k-1}^{-1} - Q_{k+1}Q_k^{-1}, \tag{6}$$

where the Q_k are now matrix functions of x and y.

We shall now be more explicit about the relevant result of [**5**] and its matrix extension. Define $E(u) := e^{-(xu+yu^{-1})}$ and let $p(u)$ be a suitable function on $\mathbf{R}^+$. (It is only required that the operators which occur are trace class.) Define G to be the integral operator on $L^2(\mathbf{R}^+)$ with kernel

$$G(u,v) = \frac{p(u)E(u)p(v)E(v)}{u+v}, \tag{7}$$

set $G_k := (-1)^k G$ and assume that the operators $I - G_k$ are invertible. Then a (clearly 2-periodic) solution of the Toda system (5) is given by

$$Q_k = \frac{\det(I - G_{k+1})}{\det(I - G_k)}. \tag{8}$$

Moreover, we also have

$$Q_k = 1 + (-1)^k (pE_0, (I - G_k)^{-1} pE_{-1}),$$

where we define $E_i(u) := u^i E(u)$.

An examination of the derivation of this reveals that, with only trivial changes, one can establish the following matrix version: In the formula (7) replace $p(u)$ and $p(v)$ by matrix functions $p(u)$ and $q(v)$, respectively. Then a solution to (6) is given by

$$Q_k = I + (-1)^k (qE_0, (I - G_k)^{-1} pE_{-1}), \tag{9}$$

where the inner product is interpreted as matrix multiplication (in the order indicated) followed by integration. We also have

$$\det Q_k = \frac{\det(I - G_{k+1})}{\det(I - G_k)}, \tag{10}$$

which is the replacement of (8).

Next we state a fact about these solutions which could easily have been derived in [**5**] but was not. This is that for the (scalar) solutions of (5) we have

$$-\frac{\partial^2}{\partial x \partial y} \log \det(I - G_k) = \frac{Q_k}{Q_{k-1}} - 1,$$

and more generally for the (matrix) solutions of (6) we have

$$-\frac{\partial^2}{\partial x \partial y} \log \det(I - G_k) = \operatorname{tr}(Q_k Q_{k-1}^{-1} - I). \tag{11}$$

At the end of this note, we shall explain how this is proved.

We consider the special case where

$$p(u) = \begin{pmatrix} f(u) & 0 \\ 0 & g(u) \end{pmatrix}, \qquad q(u) = \begin{pmatrix} 0 & g(u) \\ f(u) & 0 \end{pmatrix}.$$

For the present f and g are general although eventually they will be the functions $u^{\pm\theta/2}$. We shall take $k = 0$ and write Q for Q_0. The kernel of G is

$$G(u,v) = \begin{pmatrix} 0 & f(u)E(u)g(v)E(v)/(u+v) \\ g(u)E(u)f(v)E(v)/(u+v) & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix},$$

say. Since

$$I \pm G = \begin{pmatrix} I & \pm A \\ \pm B & I \end{pmatrix}, \tag{12}$$

we have

$$\det(I \pm G) = \det(I - AB), \tag{13}$$

so (10) gives

$$\det Q_k = 1. \tag{14}$$

From (12), the form of the matrices p and q and (9), we easily see that the diagonal elements of Q_1 are equal to those of $Q = Q_0$ while the off-diagonal elements are the negatives of each other. Similarly, interchanging f and g has the same effect on $I - G$ as left and right-multiplying by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and from this it follows that the two diagonal entries of Q, as well as the two off-diagonal entries, are obtained from each other by interchanging the roles of f and g. Denoting the effect of this interchange by a tilde, we see that we may write our matrices as

$$Q = \begin{pmatrix} 1+b & a \\ \tilde{a} & 1+\tilde{b} \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 1+b & -a \\ -\tilde{a} & 1+\tilde{b} \end{pmatrix}.$$

Observe that (14), which gives the identity

$$b + \tilde{b} + b\tilde{b} = a\tilde{a}, \tag{15}$$

also gives

$$Q^{-1} = \begin{pmatrix} 1+\tilde{b} & -a \\ -\tilde{a} & 1+b \end{pmatrix}, \qquad Q_1^{-1} = \begin{pmatrix} 1+\tilde{b} & a \\ \tilde{a} & 1+b \end{pmatrix}.$$

And from these and (11) with $k = 0$, we obtain

$$-\frac{\partial^2}{\partial x \partial y} \log\det(I - G) = 4a\tilde{a}. \tag{16}$$

Let us see what the matrix Toda equations (6) give. When $k = 0$, the equation is

$$\frac{\partial^2 Q}{\partial x \partial y} Q^{-1} + \frac{\partial Q}{\partial x}\frac{\partial Q^{-1}}{\partial y} = QQ_1^{-1} - Q_1 Q^{-1}.$$

Comparing the entries of these matrices gives the four equations (we use subscript notation now for partial derivatives)

(i) $b_{xy}(1+\tilde{b}) - a_{xy}\tilde{a} + b_x\tilde{b}_y - a_x\tilde{a}_y = 0$,

(ii) $\tilde{b}_{xy}(1+b) - \tilde{a}_{xy}a + \tilde{b}_x b_y - \tilde{a}_x a_y = 0$,

(iii) $a_{xy}(1+b) - ab_{xy} + a_x b_y - b_x a_y = 4a(1+b)$,

(iv) $\tilde{a}_{xy}(1+\tilde{b}) - \tilde{a}\tilde{b}_{xy} + \tilde{a}_x\tilde{b}_y - \tilde{b}_x\tilde{a}_y = 4\tilde{a}(1+\tilde{b})$.

Equations (i) and (ii) may be written

$$(b_x(1+\tilde{b}) - a_x\tilde{a}))_y = 0, \quad (\tilde{b}_x(1+b) - \tilde{a}_x a))_y = 0,$$

and since all our functions vanish as $y \to +\infty$, we deduce

$$b_x(1+\tilde{b}) = a_x\tilde{a}, \qquad \tilde{b}_x(1+b) = \tilde{a}_x a. \tag{17}$$

We derive analogous identities for y-derivatives as follows. Denote by T the unitary operator defined by $Th(u) = u^{-1}h(u^{-1})$, and denote by a carat the effect of the replacements $f(u) \to f(u^{-1})$, $g(u) \to g(u^{-1})$. Then (we now display the dependence of everything on the parameters

x and y) we find that $TG(x,y)T = \hat{G}(y,x)$, $T(qE_0(x,y)) = \hat{q}\hat{E}_{-1}(y,x)$, $T(pE_{-1}(x,y)) = \hat{p}\hat{E}_0(y,x)$. Thus, if we set

$$U := (qE_0, (I-G)^{-1}pE_{-1}), \qquad V := (qE_{-1}, (I-G)^{-1}pE_0),$$

then $U(x,y) = \hat{V}(y,x)$. On the other hand, the symmetry of G (the fact that its kernel satisfies $G(u,v)' = G(v,u)$) implies that $V' = (p'E_0, (I-G)^{-1}q'E_{-1})$. We have, using the same tilde notation as before and setting

$$S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$p' = \tilde{q}S, \qquad q' = S\tilde{p}, \qquad SGS = \tilde{G},$$

and from this we deduce that $V' = \tilde{U}$. Combining this with the already established $U(x,y) = \hat{V}(y,x)$, we deduce $\tilde{U}(x,y) = \hat{U}'(y,x)$, in other words,

$$a(x,y) = \hat{a}(y,x), \qquad \tilde{a}(x,y) = \hat{\tilde{a}}(y,x)$$
$$b(x,y) = \hat{\tilde{b}}(y,x), \qquad \tilde{b}(x,y) = \hat{b}(y,x).$$

Combining these with (17) for the operator $\hat{G}$, we obtain

$$\tilde{b}_y(1+b) = a_y\tilde{a}, \qquad b_y(1+\tilde{b}) = \tilde{a}_y a.$$

Eliminating b_{xy} and $\tilde{b}_{xy}$ from equations (i) and (iii), and (ii) and (iv), respectively, and using our formulas for the derivatives of b and $\tilde{b}$ as well as (15), we find the equations

$$\begin{aligned} a_{xy} &= \frac{\tilde{a}}{1+a\tilde{a}} a_x a_y + 4a(1+a\tilde{a}), \\ \tilde{a}_{xy} &= \frac{a}{1+a\tilde{a}} \tilde{a}_x \tilde{a}_y + 4\tilde{a}(1+a\tilde{a}). \end{aligned} \tag{18}$$

These equations hold whatever the functions f and g. We now use them to obtain the cited result of [**3**]. By (13) we see that the determinant in question is equal to $\det(I-G)$ evaluated at $x = y = t/4$ in the case where

$$f(u) = \sqrt{\lambda}u^{\theta/2}, \qquad g(u) = \sqrt{\lambda}u^{-\theta/2}.$$

Observe first that $\hat{a} = \tilde{a}$ in this case, so that $\tilde{a}(x, y) = a(y, x)$. We now show that

$$a(x,y) = (x/y)^{\theta/2} a(\sqrt{xy}, \sqrt{xy}),$$
$$\tilde{a}(x,y) = (y/x)^{\theta/2} \tilde{a}(\sqrt{xy}, \sqrt{xy}). \tag{19}$$

For this, we take any $r > 0$ and use the unitary operator T now defined by $Th(u) = r^{1/2} h(ru)$. Denote now by a carat the result of the replacement $(x, y) \to (rx, y/r)$. Since $TGT^{-1} = \hat{G}$ and

$$T(qE_0) = r^{1/2} \begin{pmatrix} r^{-\theta/2} & 0 \\ 0 & r^{\theta/2} \end{pmatrix} q\hat{E}_0,$$
$$T(pE_{-1}) = r^{-1/2} p\hat{E}_{-1} \begin{pmatrix} r^{\theta/2} & 0 \\ 0 & r^{-\theta/2} \end{pmatrix},$$

we deduce

$$Q = \begin{pmatrix} r^{-\theta/2} & 0 \\ 0 & r^{\theta/2} \end{pmatrix} \hat{Q} \begin{pmatrix} r^{\theta/2} & 0 \\ 0 & r^{-\theta/2} \end{pmatrix},$$

which gives the asserted identities upon setting $r = \sqrt{y/x}$.

We also deduce from $TGT^{-1} = \hat{G}$ in the same way that $\det(I - G)$ is a function of xy, and we shall eventually set $x = y = t/4$. Since, for a function of $t = 4\sqrt{xy}$,

$$\frac{\partial^2}{\partial x \partial y} = 4\left(\frac{d^2}{dt^2} + t^{-1}\frac{d}{dt}\right),$$

the left side of (3) equals 1/4 times the left side of (16) evaluated at $x = y = t/4$. Thus, if we set $c(t) := a(t/4, t/4) = \tilde{a}(t/4, t/4)$ and define φ by $\sinh\varphi = c$, then (3) holds and it remains to verify (4). Using (19) we find that either equation in (18) becomes at $x = y = t/4$,

$$\frac{d^2c}{dt^2} + \frac{1}{t}\frac{dc}{dt} = \frac{c}{1+c^2}\left(\frac{dc}{dt}\right)^2 + c(1+c^2) + \frac{\theta^2}{t^2}\left(c - \frac{c^3}{1+c^2}\right),$$

and (4) follows upon substituting $c = \sinh\varphi$.

Remark. In [**1**], differential identities were found, by different methods, for the quantities we called $a, \tilde{a}, b, \tilde{b}$. These identities do not seem to give our equations (18). A general result was also stated there which would imply in particular that (2) holds rather than (4) for the operator kernel with general θ. The authors are aware of the error in their paper and plan to publish an erratum.

Appendix

We derive (11) here. Taking the logarithmic derivative of (10) with respect to x gives

$$\operatorname{tr}\left(\frac{\partial Q_k}{\partial x}Q_k^{-1}\right) = \frac{\partial}{\partial x}\log\det(I - G_{k+1}) - \frac{\partial}{\partial x}\log\det(I - G_k),$$

and so taking traces in (6) gives

$$\frac{\partial^2}{\partial x\partial y}\log\det(I - G_{k+1}) - \frac{\partial^2}{\partial x\partial y}\log\det(I - G_k) = \operatorname{tr}(Q_k Q_{k-1}^{-1} - Q_{k+1}Q_k^{-1}).$$

Suppose it were true (which it certainly is not) that $G_k \to 0$ in trace norm and $Q_k \to I$ as $k \to +\infty$. Then replacing k successively by $k, k+1, \ldots$ in the above relation and adding would give (11).

In order to make this argument work, we use a family of operator solutions to (5), depending on parameter ω, these also being special cases of those derived in [**5**]. We assume that ω belongs to

$$\Omega := \{\omega \in \mathbf{C}\backslash\mathbf{R}^+ : \Re\omega < 1, \Re\omega^{-1} < 1\},$$

set $E(\omega, u) := e^{-[(1-\omega^{-1})xu + (1-\omega)yu^{-1}]/2}$, define G to be the operator on $L_2(\mathbf{R}^+)$ with kernel

$$\frac{p(u)E(\omega,u)p(v)E(\omega,v)}{u - \omega v},$$

and set $G_k := \omega^k G$. Then

$$Q_k = 1 + \omega^k(pE_0, (I - G_k)^{-1}pE_{-1})$$

(where we now define $E_i(u) := u^i E(\omega, u)$) satisfies (5) and (8) whenever these make sense, i.e., when the operators $I - G_k$ that appear in the expressions are invertible. In the matrix version, the factors $p(u)$ and $p(v)$ are replaced by the matrix functions $p(u)$ and $q(v)$, the constant 1 in the definition of Q_k is replaced by I, and (6) and (10) hold. Notice that we are interested in the case $\omega = -1$.

Let W be any open set whose closure is a compact subset of $\{\omega \in \Omega : |\omega| < 1\}$. Then for some k' all the operators G_k with $k \geq k'$ will have norm less than 1 when $\omega \in W$ and so the $I - G_k$ will be invertible. (We think of x and y as lying in fixed intervals bounded away from 0.) Now let k_0 be arbitrary. For fixed x and y, removing a finite set from W will ensure that all $I - G_k$ with $k \geq k_0$ are invertible. If x and y are confined to sufficiently small intervals, there will still be a nonempty open subset W_0 of W such that all $I - G_k$ with $k \geq k_0$ and $\omega \in W_0$ are invertible. Moreover, since $|\omega| < 1$ in W_0, it is clear that $G_k \to 0$ in trace norm and $Q_k \to I$ as $k \to +\infty$, so the argument given above shows that (11) holds in this case for all $k \geq k_0$. But both sides of the identity are analytic functions of $\omega \in \Omega$ and, taking a suitable path in Ω running from a point in W_0 to $\omega = -1$, we deduce (11) for $\omega = -1$, in other words, for our given operator.

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