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TOLERANT QUALOCATION—A QUALOCATION METHOD FOR BOUNDARY INTEGRAL EQUATIONS WITH REDUCED REGULARITY REQUIREMENT

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Dedicated to Professor Wolfgang L. Wendland on the occasion of his 60th birthday.

ABSTRACT. We study a modification of the qualocation method for boundary integral equations on smooth curves, which allows the same high-order convergence as the original qualocation method but with reduced smoothness assumptions on the exact solution. This 'tolerant qualocation' method differs from the original one only in that the exact inner product is used on the righthand side of the qualocation equation, whereas in the original method a specially designed approximate inner product is used on both sides. The modified method achieves exactly the same error estimates as the Petrov-Galerkin method, at greatly reduced cost.

1. Introduction. We present in this paper a modification of the qualocation method introduced in [**11, 15, 3**], which allows the same high-order convergence (in an appropriate negative norm), but with reduced smoothness assumptions on the exact solution. 'Tolerant qualocation' seems an appropriate name for this modification, as a reminder of its forgiving nature. As in [**3**], the problem studied is

$$Lu = f,$$

where L is a pseudodifferential operator, so that the equation represents a boundary integral equation on a smooth curve.

For a review of the qualocation method, we refer the reader to [**12**]. We briefly recall here that the qualocation method is a semi-discrete Petrov-Galerkin method in which the outer inner product is replaced by a specially chosen quadrature rule. The same quadrature rule is used in the inner product on the right side of the equation. That

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rule is designed so that the leading terms in the error vanish, thereby raising the order of convergence even up to that of the corresponding Petrov-Galerkin method.

The advantage of the qualocation method over the Petrov-Galerkin method is that it provides a simple alternative to the exact evaluation of the inner product and hence permits computation of the same order of difficulty as for the collocation method. (In the implementation of the qualocation method one has, as for the Galerkin and collocation methods, to compute the inner integral 'exactly.' Fully discrete versions of the qualocation method were discussed in [**7, 9, 14**]). The disadvantage is that an extra smoothness requirement is imposed on the exact solution. Suppose, for definiteness, that we deal with Symm's integral equation (in which L is an integral operator with logarithmic kernel), and that the trial space and the test space in the Petrov-Galerkin method are both taken to consist of piecewise-linear functions on a uniform mesh, with mesh size h. Then the Galerkin method yields a convergence rate of order $O(h^5)$ (in the $\|\cdot\|_{-3}$ norm) provided that $u \in H^2$, whereas the qualocation method with the so-called 3/7, 4/7 rule of [**11**] (see below) can also yield the same $O(h^5)$ convergence rate (in the same norm as above), but requires $u \in H^4$.

The increased smoothness requirement of the qualocation method is more damaging than might at first appear. Suppose, hypothetically, that our exact solution u of Symm's integral equation belongs to H^2 but not to $H^{2+\varepsilon}$ for any $\varepsilon > 0$. Then the best order of convergence that the qualocation method can give, with a smoothest spline trial space of *any* order and in any negative norm is $O(h^3)$, which is the order of convergence of simple piecewise-linear break-point collocation.

In the present work we make what is at first sight a very small modification to the qualocation method: namely, instead of using the approximate inner product on both sides of the equation, we now use it only on the left side, while retaining the exact inner product on the right (see (2.9) below). This seemingly small change has profound effects. In the first place, it turns out that it eliminates the extra smoothness requirement: we shall see that the smoothness requirement is now exactly the same as in the corresponding Petrov-Galerkin method. But we shall also see that this small change also necessitates a redesign of the qualocation method, and a fresh convergence analysis, even though the techniques, which were developed in [**10**] and [**1**], are traditional for the analysis of collocation and qualocation methods. (This is because the method is no longer *consistent*.) The main theoretical result is Theorem 4.1.

The idea for this modification came from earlier analyses [**9**] and [**7**] of certain fully discrete qualocation schemes, in which exact inner products were employed on the righthand side of the equation.

We admit that in practice the 'exact' integral on the righthand side cannot be done exactly. However, as in the implementation of the Petrov-Galerkin method a quadrature rule can be used to compute this integral to any desired accuracy, so that in practice the assumption of an exact integral on the righthand side is adequate.

As in earlier studies, the design process is carried out for an operator L_0 corresponding to the case of the circle and then extended to smooth curves by considering pseudodifferential operators of the form $L = L_0 + K$ with K a compact operator in an appropriate setting. The perturbation argument, which follows the idea of [**6**], is carried out in a way which does not require high smoothness of the operator K, in the hope of applying the method to a wider class of boundary value problems, e.g., problems for the Helmholtz equation. We also hope to consider the extension to polygons by extending the argument of [**4**]. The generalization of the result in this paper to more general operators, namely, elliptic operators with constant and nonconstant coefficients, is a topic of further study.

In Section 2 we will recall some known results for the qualocation method, which will be needed in our analysis. Section 3 treats the special case of the operator on the circle. A perturbation argument is then used in Section 4 to generalize the results to the case of operators on arbitrary smooth closed curves. Concrete quadrature rules are suggested in Section 5 for operators of various orders, with various orders of splines used as test and trial functions. Numerical results are presented in Section 6.

Throughout this paper, c denotes a constant which can take different values at different occurrences.

2. Preliminaries. Since a function on a smooth closed curve is equivalent to a periodic function, we shall without loss of generality consider in this paper spaces of complex-valued functions which are periodic with period 1. Let $\phi_n(x) := \exp(2\pi\iota n x)$. Each periodic function u has a Fourier expansion

$$u \sim \sum_{n\in\mathbf{Z}} \hat{u}(n)\phi_n,$$

where the Fourier coefficients are given by

$$\hat{u}(n) = \int_0^1 u(x)\overline{\phi_n(x)}\,dx,$$

provided u is in $L^1(0,1)$. For $s \in \mathbf{R}$, the norm $\|\cdot\|_s$ is defined by

$$\|u\|_s^2 = |\hat{u}(0)|^2 + \sum_{n\in\mathbf{Z}^*} |n|^{2s}|\hat{u}(n)|^2,$$

where $\mathbf{Z}^* = \mathbf{Z}\backslash\{0\}$. The Sobolev space H^s consists of all periodic distributions u for which the norm $\|u\|_s$ is finite.

As in [**3**], we are concerned with pseudodifferential operators of the form

$$L = L_0 + K, \tag{2.1}$$

where the principal part L_0 of the operator L is given by

$$L_0 u := \sum_{n\in\mathbf{Z}} [n]_\beta \hat{u}(n)\phi_n, \tag{2.2}$$

with $\beta \in \mathbf{R}$ and with $[n]_\beta$ defined either by

$$[n]_\beta := \begin{cases} 1 & \text{if } n = 0, \\ |n|^\beta & \text{if } n \neq 0 \end{cases}$$

(in which case L_0 is an even operator of order β), or by

$$[n]_\beta := \begin{cases} 1 & \text{if } n = 0, \\ (\operatorname{sign} n)|n|^\beta & \text{if } n \neq 0 \end{cases}$$

(in which case L_0 is an odd operator of order β plus a constant operator). In either case L_0 is a pseudodifferential operator of order β and symbol $[n]_\beta$, and is an isometry from H^s to $H^{s-\beta}$ for all $s \in \mathbf{R}$.

The operator K is required to be a continuous mapping

$$K : H^s \to H^{s-\beta+\eta} \tag{2.3}$$

for all $s \in \mathbf{R}$ and for some $\eta > 0$. We then have $L_0^{-1}K$ bounded from H^s to $H^{s+\eta}$ and compact on H^s for all real values of s. We also assume that L is one-to-one and thus, by the Fredholm alternative,

$$(I + L_0^{-1}K)^{-1} : H^s \longrightarrow H^s$$

is bounded for all $s \in \mathbf{R}$.

We seek an approximate solution to the problem: Given $f \in H^{t-\beta}$, find the unique $u \in H^t$ such that

$$Lu = f. \tag{2.4}$$

Appropriate restrictions on t will be imposed later.

To define the qualocation approximations to (2.4), we first define a uniform mesh,

$$x_i = ih, \quad i \in \mathbf{Z}, \qquad h = 1/N.$$

For any integer $m \geq 1$ we then denote by S^m the space of 1-periodic, complex-valued, smoothest splines of order m, with breakpoints x_i. (By a smoothest spline of order m we mean a piecewise polynomial of degree $\leq m-1$ belonging, if $m \geq 2$, to the class C^{m-2}.) The qualocation method to solve (2.4) may be described as a modified Petrov-Galerkin method with trial space S^r and test space $S^{r'}$, $r, r' \geq 1$, in which the outer integral on both sides of the Galerkin equation is performed by a special quadrature rule of composite type

$$Q_h g := h \sum_{i=0}^{N-1} \sum_{j=1}^{J} w_j g((i+\xi_j)h), \tag{2.5}$$

obtained by repeating a specially constructed J-point 'elementary' rule

$$Qg := \sum_j w_j g(\xi_j) \approx \int_0^1 g(x)\,dx, \tag{2.6}$$

where

$$0 \le \xi_1 < \xi_2 < \cdots < 1 \quad \text{and} \quad \sum_j w_j = 1, \qquad w_j > 0. \tag{2.7}$$

(Here and in the sequel j runs from 1 to J.) For example, the 3/7, 4/7 rule referred to in the introduction is defined by $J = 2$, $w_1 = 3/7$, $w_2 = 4/7$, $\xi_1 = 0$ and $\xi_2 = 1/2$.

A discretized form of the inner product

$$\langle v, w \rangle := \int_0^1 v(x)\overline{w(x)}\,dx$$

may now be defined by

$$\langle v, w \rangle_h := Q_h(v\bar{w}).$$

The original qualocation approximation to (2.4) is then given by

$$u_h \in S^r \quad \text{and} \quad \langle Lu_h, \psi' \rangle_h = \langle f, \psi' \rangle_h \quad \forall\, \psi' \in S^{r'}, \tag{2.8}$$

whereas the tolerant qualocation method to be considered here is given by

$$u_h \in S^r \quad \text{and} \quad \langle Lu_h, \psi' \rangle_h = \langle f, \psi' \rangle \quad \forall\, \psi' \in S^{r'}, \tag{2.9}$$

with an exact inner product on the righthand side.

The qualocation method (2.8) requires f to be continuous, at least at the quadrature points. There is no such restriction for the tolerant qualocation method (2.9). For both methods we also require that they be *well-defined*, cf. [**3**], meaning that either

$$r > \beta + 1 \tag{2.10}$$

or

$$r > \beta + 1/2 \quad \text{and} \quad \xi_1 > 0. \tag{2.11}$$

These conditions ensure that Lu_h is well-defined at the quadrature points.

We recall here some results from [**3**] which will be useful in our analysis. Let

$$\Lambda := \left\{ \mu \in \mathbf{Z} : -\frac{N}{2} < \mu \leq \frac{N}{2} \right\} \quad \text{and} \quad \Lambda^* := \Lambda \backslash \{0\}.$$

By $\{\psi_\mu : \mu \in \Lambda\}$ we denote a basis for S^r which is defined as in [**3**] by

$$\hat{\psi}_\mu(m) = \begin{cases} 0 & \text{if } m \not\equiv \mu, \\ 1 & \text{if } m = \mu = 0, \\ (\mu/m)^r & \text{if } m \equiv \mu,\ m \neq 0, \end{cases} \tag{2.12}$$

where by $m \equiv \mu$ we mean $m \equiv \mu \pmod N$, or

$$m = \mu + lN \quad \text{for some } l \in \mathbf{Z}.$$

In the same way, $\{\psi'_\mu : \mu \in \Lambda\}$ denotes a basis for $S^{r'}$. The following functions occur frequently in the analysis of the qualocation and related methods:

$$\begin{gathered} F^{\pm}_\alpha(x,y) := \sum_{l \neq 0} \left\{ \begin{matrix} 1 \\ \operatorname{sign} l \end{matrix} \right\} \frac{1}{|l+y|^\alpha} \phi_l(x), \quad x \in \mathbf{R}, \\ y \in [-1/2, 1/2], \quad \alpha > 1/2 \end{gathered} \tag{2.13}$$

(where the upper and lower values correspond to the $+$ and $-$ cases, respectively),

$$\Delta'(\xi, y) := y^{r'} F^{\tau'}_{r'}(\xi, y), \qquad \tau' = \begin{cases} + & \text{if } r' \text{ even}, \\ - & \text{if } r' \text{ odd}, \end{cases} \tag{2.14}$$

and

$$\Omega(\xi, y) := \left\{ \begin{matrix} 1 \\ \operatorname{sign} y \end{matrix} \right\} |y|^{r-\beta} F^{\tau}_{r-\beta}(\xi, y), \tag{2.15}$$

where

$$\tau = \begin{cases} + & \text{if } r,\ L_0 \text{ are both even or both odd}, \\ - & \text{if } r,\ L_0 \text{ are of opposite parity}. \end{cases}$$

It was proved in [**3**, Lemma 1] that for $\mu, \nu \in \Lambda$ there holds

$$\langle L_0\psi_\nu, \psi'_\mu\rangle_h = \begin{cases} 0 & \text{if } \nu \neq \mu, \\ 1 & \text{if } \nu = \mu = 0, \\ [\mu]_\beta D(\mu/N) & \text{if } \nu = \mu \neq 0, \end{cases} \tag{2.16}$$

where

$$D(y) := \sum_j w_j(1 + \Omega(\xi_j, y))(1 + \overline{\Delta'(\xi_j, y)}), \qquad y \in [-1/2, 1/2]. \tag{2.17}$$

In the same manner one can prove easily from (2.2) that

$$\langle L_0 u, \psi'_\mu\rangle = \begin{cases} \hat{u}(0) & \text{if } \mu = 0, \\ [\mu]_\beta(\hat{u}(\mu) + R(\mu)) & \text{if } \mu \in \Lambda^*, \end{cases} \tag{2.18}$$

where

$$R(\mu) = \sum_{l \neq 0} \left[\frac{\mu + lN}{\mu}\right]_\beta \left(\frac{\mu}{\mu + lN}\right)^{r'} \hat{u}(\mu + lN). \tag{2.19}$$

Hence for the case $L = L_0$ we find, on writing

$$u_h = \sum_{\nu \in \Lambda} \hat{u}_h(\nu)\psi_\nu \tag{2.20}$$

and using (2.9) and (2.16), that

$$\langle L_0\psi_\mu, \psi'_\mu\rangle_h \hat{u}_h(\mu) = \langle L_0 u, \psi'_\mu\rangle \quad \forall\, \mu \in \Lambda. \tag{2.21}$$

This together with (2.16) and (2.18) implies

$$\hat{u}_h(\mu) - \hat{u}(\mu) = \begin{cases} 0 & \text{if } \mu = 0, \\ -\dfrac{E(\mu h)}{D(\mu h)}\hat{u}(\mu) + \dfrac{R(\mu)}{D(\mu h)} & \text{if } \mu \in \Lambda^*, \end{cases} \tag{2.22}$$

where

$$E(y) := E_1(y) + E_2(y), \tag{2.23}$$

with

$$E_1(y) := \sum_j w_j \Omega(\xi_j, y)[1 + \overline{\Delta'(\xi_j, y)}]$$

and

$$E_2(y) := \sum_j w_j \overline{\Delta'(\xi_j, y)}. \tag{2.24}$$

The new ingredient in the error expression (2.22), compared with the qualocation analysis of [**3**], is the term E_2 in (2.23). On the other hand, the expression for $R(\mu)$ is now simpler than in the ordinary qualocation analysis.

We recall the following definitions of stability and (qualocation) order from [**3**]:

Definition 2.1. (i) The method is *stable* if

$$\inf\{|D(y)| : y \in [-1/2, 1/2]\} > 0. \tag{2.25}$$

(ii) The method is of *qualocation order* $r - \beta + b$ (and the *additional order* is $b \geq 0$) if

$$|E_1(y)| \leq c|y|^{r-\beta+b} \quad \text{for } |y| \leq 1/2. \tag{2.26}$$

We shall continue to use these definitions in this paper, but now the function $E_1(y)$ does not tell us the whole story about order of convergence, because there is a second function controlling the order of convergence, namely $E_2(y)$.

Definition 2.2. The method is of *polynomial order* $r' + b'$ (and the *additional polynomial order* is $b' \geq 0$) if

$$|E_2(y)| \leq c|y|^{r'+b'} \quad \text{for } |y| \leq 1/2. \tag{2.27}$$

Remark 2.1. The 'polynomial order' is at least r', as follows from the definitions of $E_2(y)$ and $\Delta'(\xi, y)$.

The reason for the 'polynomial' tag lies in the following proposition.

Lemma 2.1. *Let the rule Q, see* (2.6)*, be exact for all polynomials of degree $\leq r' + b' - 1$, with $b' \geq 0$. Then the method is of polynomial order $r' + b'$.*

Proof. See Appendix. □

Definition 2.3. The *tolerant order* σ of the method is the minimum of the qualocation order and the polynomial order, i.e., σ is the maximum number satisfying

$$|E_1(y)| \leq c|y|^\sigma \quad \text{and} \quad |E_2(y)| \leq c|y|^\sigma \quad \text{for } |y| \leq 1/2. \tag{2.28}$$

We note that, from the definitions of $E_1(y)$ and $E_2(y)$, there follows $\sigma \geq \min(r - \beta, r')$.

3. Analysis for the $K = 0$ case. We consider in this section the special case where $K = 0$, i.e., $L = L_0$. The following result will be extended to more general L in Theorem 4.1.

Theorem 3.1. *Let* (2.9) *with $L = L_0$, i.e., $K = 0$, be a well-defined method which is stable and of tolerant order $\sigma \geq \min(r - \beta, r')$. Then $u_h \in S^r$ is uniquely defined and satisfies the error estimate*

$$\|u_h - u\|_s \leq ch^{t-s}\|u\|_t, \tag{3.1}$$

if $\beta - r' \leq s < r - 1/2$, $\beta - r' + 1/2 < t \leq r$ and $0 \leq t - s \leq \sigma$.

Proof. The existence and uniqueness of u_h, under the assumption in the theorem that the method is stable, is given by (2.20), (2.21), (2.16) and (2.25). We only have to prove (3.1). It follows from the definition of the norm $\|\cdot\|_s$ and from $\hat{u}_h(0) = \hat{u}(0)$, see (2.22), that

$$\begin{aligned}\|u_h - u\|_s^2 &= \sum_{\mu\in\Lambda^*} |\mu|^{2s} |\hat{u}_h(\mu) - \hat{u}(\mu)|^2 \\ &\quad + \sum_{\mu\in\Lambda} {\sum_{n\equiv\mu}}' |n|^{2s} |\hat{u}_h(n) - \hat{u}(n)|^2 \\ &= T_1 + T_2,\end{aligned} \tag{3.2}$$

where

$$ {\sum_{n\equiv\mu}}' = \sum_{\substack{n\equiv\mu \\ n\neq\mu}} .$$

By using again (2.22), together with (2.25), we obtain

$$\begin{aligned} T_1 &= \sum_{\mu\in\Lambda^*} |\mu|^{2s} \left(-\frac{E(\mu h)}{D(\mu h)} \hat{u}(\mu) + \frac{R(\mu)}{D(\mu h)} \right)^2 \\ &\leq c \sum_{\mu\in\Lambda^*} |\mu|^{2s} |E(\mu h)|^2 |\hat{u}(\mu)|^2 + c \sum_{\mu\in\Lambda^*} |\mu|^{2s} |R(\mu)|^2 \\ &= T_{11} + T_{12}. \end{aligned} \tag{3.3}$$

From (2.28) we infer, using $t - s \leq \sigma$ and $|\mu h| \leq 1/2$,

$$\begin{aligned} T_{11} &\leq c \sum_{\mu\in\Lambda^*} |\mu|^{2s} |\mu h|^{2\sigma} |\hat{u}(\mu)|^2 \\ &= ch^{2(t-s)} \sum_{\mu\in\Lambda^*} |\mu h|^{2(\sigma - t + s)} |\mu|^{2t} |\hat{u}(\mu)|^2 \\ &\leq ch^{2(t-s)} \sum_{\mu\in\Lambda^*} |\mu|^{2t} |\hat{u}(\mu)|^2 \\ &\leq ch^{2(t-s)} \|u\|_t^2. \end{aligned} \tag{3.4}$$

In the following we use, as usual, the fact that for $\alpha > 1$,

$$\sum_{l\neq 0} |l + y|^{-\alpha} \leq \sum_{l\neq 0} |l + 1/2|^{-\alpha} < \infty \quad \forall\, y \in [-1/2, 1/2]. \tag{3.5}$$

Because $t > \beta - r' + 1/2$ and $s \geq \beta - r'$ (implying $|\mu/N|^{s+r'-\beta} \leq 1$), the definition (2.19) of $R(\mu)$ and the Cauchy-Schwarz inequality give

$$
\begin{aligned}
T_{12} &\leq c \sum_{\mu\in\Lambda^*} |\mu|^{2(s+r'-\beta)} \\
&\quad \cdot \left(\sum_{l\neq 0} |\mu + lN|^{-r'+\beta-t} |\mu + lN|^t |\hat{u}(\mu + lN)| \right)^2 \\
&\leq cN^{2(s-t)} \sum_{\mu\in\Lambda^*} \left|\frac{\mu}{N}\right|^{2(s+r'-\beta)} \left(\sum_{l\neq 0} |l + \mu/N|^{-2(r'-\beta+t)} \right) \\
&\qquad \cdot \left(\sum_{l\neq 0} |\mu + lN|^{2t} |\hat{u}(\mu + lN)|^2 \right) \\
&\leq ch^{2(t-s)} \|u\|_t^2.
\end{aligned}
\tag{3.6}
$$

Thus $T_1 = T_{11} + T_{12}$ satisfies the required bound.

For the term T_2, it follows from (2.20) and (2.12) that $m^r \hat{u}_h(m) = \mu^r \hat{u}_h(\mu)$ if $m \equiv \mu$, thus from the Cauchy-Schwarz inequality there follows

$$
\begin{aligned}
T_2 &= \sum_{\mu\in\Lambda} \sum_{n\equiv\mu}{}' |n|^{2s} \left| \left(\frac{\mu}{n}\right)^r \hat{u}_h(\mu) - \hat{u}(n) \right|^2 \\
&\leq 3 \sum_{\mu\in\Lambda} \sum_{n\equiv\mu}{}' |n|^{2s} \left(\left|\frac{\mu}{n}\right|^{2r} |\hat{u}_h(\mu) - \hat{u}(\mu)|^2 + \left|\frac{\mu}{n}\right|^{2r} |\hat{u}(\mu)|^2 + |\hat{u}(n)|^2 \right) \\
&= T_{21} + T_{22} + T_{23}.
\end{aligned}
\tag{3.7}
$$

By using again (3.5) and noting that $r - s > 1/2$, $|\mu/N|^{r-s} \leq 1$ and $|\mu/N|^{r-t} \leq 1$, we obtain

$$
\begin{aligned}
T_{21} &= 3 \sum_{\mu\in\Lambda^*} \left(|\mu|^{2s} |\hat{u}_h(\mu) - \hat{u}(\mu)|^2 \sum_{n\equiv\mu}{}' \left|\frac{\mu}{n}\right|^{2(r-s)} \right) \\
&\leq c \sum_{\mu\in\Lambda^*} |\mu|^{2s} |\hat{u}_h(\mu) - \hat{u}(\mu)|^2 \\
&= cT_1 \leq ch^{2(t-s)} \|u\|_t^2,
\end{aligned}
\tag{3.8}
$$

and

$$
(3.9)\qquad \begin{aligned}
T_{22} &= 3\sum_{\mu\in\Lambda^*}\sum_{n\equiv\mu}{}' \left|\frac{\mu}{n}\right|^{2(r-t)} |n|^{2(s-t)}|\mu|^{2t}|\hat{u}(\mu)|^2 \\
&= 3N^{2(s-t)}\sum_{\mu\in\Lambda^*}|\mu|^{2t}|\hat{u}(\mu)|^2\left|\frac{\mu}{N}\right|^{2(r-t)}\sum_{l\neq 0}\left|l+\frac{\mu}{N}\right|^{-2(r-s)} \\
&\le ch^{2(t-s)}\|u\|_t^2.
\end{aligned}
$$

Finally, on noting that if $n \equiv \mu$ and $n \neq \mu$ then $|n| \ge N/2$, we obtain

$$
(3.10)\qquad T_{23} = 3\sum_{\mu\in\Lambda}\sum_{n\equiv\mu}{}' |n|^{2(s-t)}|n|^{2t}|\hat{u}(n)|^2 \le ch^{2(t-s)}\|u\|_t^2.
$$

Thus T_2 and T_1 both satisfy the required bound, and the theorem is proved. □

Remark 3.1. The increased regularity requirement in the standard qualocation method (see Remark 4.2) can be traced to an additional term, which may be identified as a quadrature error in the quantity in [**3**] that corresponds to $R(\mu)$.

4. Perturbation argument. We consider now the full operator $L = L_0 + K$, with K satisfying (2.3). A perturbation argument is used to generalize the result of Theorem 3.1 to this case. While the perturbation argument follows standard lines (see, for example, [**6**]), extra care is needed here because we are seeking to preserve the full order of convergence of the Petrov-Galerkin method.

Following [**9**], we define an operator $P : H^t \to \mathcal{T}_h$ for $t > 1/2$ by

$$
(4.1)\qquad \langle Pf, v\rangle = \langle f, v\rangle_h \quad \forall\, f \in H^t, \quad v \in S^{r'},
$$

where

$$
\mathcal{T}_h = \operatorname{span}\{\phi_\mu : \mu \in \Lambda\},
$$

so that $\mathcal{T}_h$ is a space of trigonometric polynomials of degree at most $N/2$. Lemmas 1 and 2 in the Appendix of [**9**] show that P is well defined, and a modification of the proofs of these lemmas yields:

Lemma 4.1. *If the method is of polynomial order $r' + b'$ for some $b' \geq 0$, then for $0 \leq s \leq t$, $t - s \leq r' + b'$ and $t > 1/2$, there holds*

$$\|Pf - f\|_s \leq ch^{t-s}\|f\|_t.$$

Proof. See Appendix. □

Remark 4.1. In [**9**] the elementary rule Q, see (2.6), was just a 1-point rule. The novel aspect of the present result resides in its exploitation of the polynomial order of the elementary rule to lift the maximum order of convergence above r'.

Using this lemma we shall prove the following.

Theorem 4.1. *Let* (2.9) *with $L = L_0 + K$ be a well-defined tolerant qualocation method which is stable and of tolerant order $\sigma \geq \min(r', r - \beta)$. If K satisfies* (2.3) *with some $\eta > r' + 1/2$, then for h sufficiently small u_h is uniquely defined and satisfies*

$$\|u_h - u\|_s \leq ch^{t-s}\|u\|_t \tag{4.2}$$

if $\beta - r' \leq s < r - 1/2$, $\beta - r' + 1/2 < t \leq r$ and $0 \leq t - s \leq \sigma$.

Remark 4.2. (i) If $\sigma \geq r + r' - \beta$, then the condition $t - s \leq \sigma$ is redundant, and the convergence result is exactly as for the Petrov-Galerkin method, see [**8**, Theorem 3.1]. (Most treatments of Galerkin methods require $t \geq \beta/2$, but in the present setting of smoothest splines and uniform meshes it is easy to see that t can be allowed the same lower bound as above; for example, one can follow the arguments of the present paper with the quadrature rule Q replaced by the exact integral.)

(ii) In the standard qualocation case the method is assumed to be of qualocation order $r - \beta + b$ for some $b > 0$, and (3.1) is replaced, see [**3**, Theorem 2], by

$$\|u_h - u\|_s \leq ch^{t-s}\|u\|_{t+\max(\beta-s,0)}$$

if $\beta - b \le s \le t \le r$, $s < r - 1/2$ and $t > \beta + 1/2$. Note the strengthened regularity requirement on the exact solution u when compared to (4.2).

Proof. Let t, s and η satisfy the constraints in the theorem, and assume $u \in H^t$ and for the moment also that $u_h \in S^r$ satisfying (2.9) exists. Since $(I + L_0^{-1}K)^{-1}$ is bounded on H^s, we have

$$\begin{aligned} \|u_h - u\|_s &\le c\|(I + L_0^{-1}K)(u_h - u)\|_s \\ &\le c(\|u_h - u - L_0^{-1}Ku + L_0^{-1}PKu_h\|_s \\ &\qquad + \|L_0^{-1}Ku_h - L_0^{-1}PKu_h\|_s) \\ &= T_5 + T_6. \end{aligned} \tag{4.3}$$

(It can be seen that PKu_h is well defined. Indeed, given that $S^r \subset H^{r-1/2-\varepsilon}$ for arbitrary $\varepsilon > 0$, there follows $Ku_h \in H^{r-1/2-\varepsilon+\eta-\beta}$ where $r - 1/2 + \eta - \beta > r + r' - \beta > 1/2$. With $\varepsilon > 0$ chosen sufficient small, we therefore have $r - 1/2 - \varepsilon + \eta - \beta > 1/2$.) With $\mu \in \Lambda$, from the equation

$$\langle (L_0 + K)u_h, \psi'_\mu \rangle_h = \langle (L_0 + K)u, \psi'_\mu \rangle$$

and the definition (4.1) of P we can write

$$\langle L_0 u_h, \psi'_\mu \rangle_h = \langle L_0[u + L_0^{-1}(Ku - PKu_h)], \psi'_\mu \rangle.$$

Thus u_h is the solution of the approximate problem studied in Section 3 if the exact solution is $u + L_0^{-1}(Ku - PKu_h)$. Hence Theorem 3.1 and the boundedness of L_0^{-1} from $H^{t-\beta}$ to H^t imply

$$\begin{aligned} T_5 &\le ch^{t-s}\|u + L_0^{-1}(Ku - PKu_h)\|_t \\ &\le ch^{t-s}(\|u\|_t + \|Ku - PKu_h\|_{t-\beta}) \\ &\le ch^{t-s}(\|u\|_t + \|(PK - K)(u_h - u)\|_{t-\beta} \\ &\qquad + \|PKu - Ku\|_{t-\beta} + \|K(u_h - u)\|_{t-\beta}) \\ &= ch^{t-s}(\|u\|_t + T_{51} + T_{52} + T_{53}). \end{aligned} \tag{4.4}$$

(Note that PKu is well defined since $Ku \in H^{t+\eta-\beta}$ and $t + \eta - \beta > -r' + 1/2 + r' + 1/2 = 1$.) It also follows from the isometric property of L_0 that

$$\begin{aligned} T_6 &= c\|PKu_h - Ku_h\|_{s-\beta} \\ &\le c(\|(PK - K)(u_h - u)\|_{s-\beta} + \|PKu - Ku\|_{s-\beta}) \\ &= T_{61} + T_{62}. \end{aligned} \tag{4.5}$$

The assumption $t+r'-\beta > 1/2$ assures us that we can use Lemma 4.1 with the norm $\|\cdot\|_{t+r'-\beta}$ on both left and right. Since $t-\beta < t-\beta+r'$, this together with (2.3) gives

$$
\begin{aligned}
T_{51} &\le \|(PK-K)(u_h-u)\|_{t+r'-\beta} \\
&\le c\|K(u_h-u)\|_{t+r'-\beta} \\
&\le \|u_h-u\|_{t+r'-\eta},
\end{aligned} \tag{4.6}
$$

$$
\begin{aligned}
T_{52} &\le \|PKu-Ku\|_{t+r'-\beta} \\
&\le c\|Ku\|_{t+r'-\beta} \\
&\le c\|u\|_{t+r'-\eta} \\
&\le c\|u\|_{t},
\end{aligned} \tag{4.7}
$$

and

$$
T_{53} \le c\|u_h-u\|_{t-\eta} \le c\|u_h-u\|_{t+r'-\eta}. \tag{4.8}
$$

Similarly, because $s \ge \beta - r'$, $t+r'-\beta > 1/2$ and $t-s \le \sigma$ we obtain from Lemma 4.1 and (2.3)

$$
\begin{aligned}
T_{61} &\le c\|(PK-K)(u_h-u)\|_{s+r'-\beta} \\
&\le ch^{t-s}\|K(u_h-u)\|_{t+r'-\beta} \\
&\le ch^{t-s}\|u_h-u\|_{t+r'-\eta},
\end{aligned} \tag{4.9}
$$

and

$$
\begin{aligned}
T_{62} &\le \|PKu-Ku\|_{s+r'-\beta} \\
&\le ch^{t-s}\|Ku\|_{t+r'-\beta} \\
&\le ch^{t-s}\|u\|_{t+r'-\eta} \\
&\le ch^{t-s}\|u\|_{t}.
\end{aligned} \tag{4.10}
$$

Inequalities (4.3)–(4.10) imply

$$
\begin{aligned}
\|u_h-u\|_s &\le ch^{t-s}(\|u\|_t + \|u_h-u\|_{t+r'-\eta}) \\
&\le ch^{t-s}(\|u\|_t + \|u_h-u\|_{s'}),
\end{aligned} \tag{4.11}
$$

where s' is any number satisfying

$$
\beta - r' \le s' < r - 1/2, \qquad 0 < t - s' \le \min(\sigma, \eta - r'). \tag{4.12}
$$

(It is easily seen that a suitable number s' exists: if t satisfies $\beta - r' + 1/2 < t \leq r - 1/2$, then (4.12) is satisfied whenever s' satisfies $0 < t - s' \leq 1/2$; and if $r - 1/2 < t \leq r$, then it is satisfied by $s' = r - 1/2 - \varepsilon$ for ε a sufficiently small positive number, because $\sigma \geq \min(r', r - \beta) > 1/2$ and $\eta - r' > 1/2$.) Because s' satisfies the conditions (4.12), it therefore satisfies the conditions on s in the theorem, allowing us to apply (4.11) with $s = s'$, giving

$$\|u_h - u\|_{s'} \leq ch^{t-s'}(\|u\|_t + \|u_h - u\|_{s'})$$

and hence, because $s' < t$,

$$\|u_h - u\|_{s'} \leq ch^{t-s'}\|u\|_t \leq c\|u\|_t,$$

if h is sufficiently small. Thus, finally, (4.11) gives

$$\|u_h - u\|_s \leq ch^{t-s}\|u\|_t,$$

so that (4.2) holds.

The above a priori estimate implies that the homogeneous equation (2.9) has only the trivial solution. This in turn yields (since the linear system is square) existence and uniqueness of the nonhomogeneous equation, completing the proof. □

Remark 4.3. For the integral equation of the first kind for the Helmholtz equation, L_0 is an even operator of order -1, see Definitions 2.1 and 2.2, and for a smooth curve K satisfies, see [**6**],

$$K : H^s \to H^{s+3} \quad \text{for any } s \in \mathbf{R}.$$

Hence the condition on η in the theorem indicates that we can solve this equation with the same order as the Petrov-Galerkin method if we use piecewise-constant test functions. Such rules can be designed by using the table in Section 5.

5. Stability and order of the method. It was proved in [**3**, Theorem 3] that a well-defined method with $\tau = \tau'$ is stable unless either

(i) $J = 1$, $\xi_1 = 1/2$ and $\tau = \tau' = +$, or

(ii) $J = 1$, $\xi_1 = 0$ and $\tau = \tau' = -$.

In these two cases the method is unstable. In line with the Petrov-Galerkin method, no result was proved for the case $\tau \neq \tau'$. These results carry on immediately to the tolerant qualocation method, because the lefthand sides of (2.8) and (2.9) are the same.

The convergence behavior of the method, we recall from Definition 2.1, is determined by the behavior for small values of y of

$$E_1(y) = \sum_j w_j \Omega(\xi_j, y)[1 + \overline{\Delta'(\xi_j, y)}]$$

and

$$E_2(y) = \sum_j w_j \overline{\Delta'(\xi_j, y)}.$$

From now on, we assume that the quadrature rule Q in (2.6) is symmetric, i.e., if $\xi \in (0,1)$ is a quadrature point then so is $1-\xi$, and the weights associated with the two points are equal. In this case the functions E_1 and E_2 take real values and are even, see [**3**]. Let $G^{\pm}_{\alpha}(x,y)$ and $H^{\pm}_{\alpha}(x,y)$ be the real and imaginary parts of $F^{\pm}_{\alpha}(x,y)$. We then have

$$E_1(y) = \sum_j w_j y^{r-\beta} G^{\tau}_{r-\beta}(\xi_j, y) + O(y^{r-\beta+r'}), \tag{5.1}$$

and

$$E_2(y) = \sum_j w_j y^{r'} G^{\tau'}_{r'}(\xi_j, y). \tag{5.2}$$

It was proved in [**3**, Lemma A.2] that, for any fixed $\xi \in (0,1)$ and $\alpha > 0$, $G^{\pm}_{\alpha}(x,y)$ and $H^{\pm}_{\alpha}(x,y)$ have the power-series expansions

$$G^{+}_{\alpha}(\xi, y) = \sum_{k=0}^{\infty} \binom{-\alpha}{2k} G_{\alpha+2k}(\xi) y^{2k},$$
$$G^{-}_{\alpha}(\xi, y) = \sum_{k=1}^{\infty} \binom{-\alpha}{2k-1} G_{\alpha+2k-1}(\xi) y^{2k-1},$$

$$H_\alpha^+(\xi, y) = \sum_{k=1}^{\infty} \binom{-\alpha}{2k-1} H_{\alpha+2k-1}(\xi) y^{2k-1},$$
$$H_\alpha^-(\xi, y) = \sum_{k=0}^{\infty} \binom{-\alpha}{2k} H_{\alpha+2k}(\xi) y^{2k},$$

where
$$\binom{-\alpha}{j} = \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-j+1)}{j!},$$

$$G_\alpha(\xi) = 2\sum_{l=1}^{\infty} \frac{1}{l^\alpha} \cos 2\pi l\xi, \tag{5.3}$$

and

$$H_\alpha(\xi) = 2\sum_{l=1}^{\infty} \frac{1}{l^\alpha} \sin 2\pi l\xi. \tag{5.4}$$

There are two cases for us to consider.

1. *The case* $\tau = \tau' = +$. Equations (5.1) and (5.2) can be rewritten as

$$\begin{aligned} E_1(y) = \sum_{l=0}^{\infty} \binom{-r+\beta}{2l} \Big(\sum_j w_j G_{r-\beta+2l}(\xi_j)\Big) \\ \cdot y^{r-\beta+2l} + O(y^{r-\beta+r'}), \end{aligned} \tag{5.5}$$

and

$$E_2(y) = \sum_{l=0}^{\infty} \binom{-r'}{2l} \Big(\sum_j w_j G_{r'+2l}(\xi_j)\Big) y^{r'+2l}. \tag{5.6}$$

It is obvious from (5.5) and (5.6) that the tolerant order is at least $\min(r-\beta, r')$. On the other hand, since the last term in (5.5) is of order $O(y^{r-\beta+r'})$ the method is of tolerant order $\sigma \in [\min(r-\beta, r'), r-\beta+r']$ if and only if all the coefficients of y up to the order $\sigma - 1$ in the series on the righthand sides of (5.5) and (5.6) vanish. Therefore we have

Theorem 5.1. *Let $\tau = \tau' = +$. The qualocation method with a symmetric rule* (2.5) *is of tolerant order $\sigma \in [\min(r-\beta, r'), r-\beta+r']$ if the rule satisfies*

$$\sum_j w_j G_{r-\beta+2l}(\xi_j) = 0 \quad \text{for } l = 0, 1, \dots, \left\lfloor \frac{\sigma - 1 - r + \beta}{2} \right\rfloor, \tag{5.7}$$

and

$$\sum_j w_j G_{r'+2l}(\xi_j) = 0 \quad \text{for } l = 0, 1, \dots, \left\lfloor \frac{\sigma - 1 - r'}{2} \right\rfloor. \tag{5.8}$$

Remark 5.1. Note that the left side of (5.7) or (5.8) may be written as QG_α for appropriate values of α. Note also that the exact integral of G_α is zero, thus the conditions state that the elementary rule Q is exact for certain functions G_α.

Remark 5.2. We note that if $r - \beta + r'$ is even there may be overlap between the two sets of conditions (5.7) and (5.8).

2. *The case $\tau = \tau' = -$.* Equations (5.1) and (5.2) can be rewritten as

$$\begin{aligned} E_1(y) = \sum_{l=1}^{\infty} \binom{-r+\beta}{2l-1} \Big(\sum_j w_j G_{r-\beta+2l-1}(\xi_j) \Big) \\ \cdot y^{r-\beta+2l-1} + O(y^{r-\beta+r'}), \end{aligned} \tag{5.9}$$

and

$$E_2(y) = \sum_{l=1}^{\infty} \binom{-r'}{2l-1} \Big(\sum_j w_j G_{r'+2l-1}(\xi_j) \Big) y^{r'+2l-1}. \tag{5.10}$$

Similarly to the previous case, we have the following theorem:

Theorem 5.2. *Let $\tau = \tau' = -$. The qualocation method with a symmetric rule* (2.5) *is of tolerant order $\sigma \in [\min(r-\beta, r'), r-\beta+r']$ if the rule satisfies*

$$\sum_j w_j G_{r-\beta+2l-1}(\xi_j) = 0 \quad \text{for } l = 1, 2, \dots, \left\lfloor \frac{\sigma - r + \beta}{2} \right\rfloor, \tag{5.11}$$

and

$$\sum_j w_j G_{r'+2l-1}(\xi_j) = 0 \quad \textit{for } l = 1, 2, \ldots, \left\lfloor \frac{\sigma - r'}{2} \right\rfloor . \tag{5.12}$$

In Table 1 we list sets of equations we should solve to achieve a highest possible tolerant order σ for a given number of quadrature points J. We note that, for each J, since the rule is symmetric and $\sum_j w_j = 1$, there are $J-1$ degrees of freedom. While there is as yet no theory which assures us that these (nonlinear) equations have solution points and weights satisfying (2.7), numerical experiments, see [**16**] where complete tables will appear, have found that in every case yet examined they do. In Table 2 we show two specific examples of such rules for the case $J=4$, $\beta=-1$, $\tau=+$, $r=r'=2$ and $\sigma=5$.

We may see implications of Theorems 5.1 and 5.2 in a more explicit way by considering some examples. In these examples we assume for simplicity that the tolerant order has its maximum value, i.e., $\sigma = r - \beta + r'$. Thus, in these examples, the convergence conditions are exactly as for the Petrov-Galerkin method.

Example 1. Let L_0 be the identity operator, thus L_0 is an even operator with $\beta = 0$. If we choose $r = r'$ to be even, i.e., $\tau = \tau' = +$, then Theorem 5.1 implies that for the tolerant order to be $\sigma = 2r$, the rule should satisfy

$$\sum_j w_j G_k(\xi_j) = 0 \quad \text{for } k = r, r+2, \ldots, 2r-4, 2r-2.$$

In particular, if $r = r' = 2$, i.e., the test and trial functions are piecewise linear, then the tolerant order of the resulting method is 4, because the rule so designed has degree of precision 3, i.e., it integrates exactly polynomials of degree up to 3. Similarly, if we choose $r = r'$ to be odd, then Theorem 5.2 implies that the rule should satisfy

$$\sum_j w_j G_k(\xi_j) = 0 \quad \text{for } k = r+1, r+3, \ldots, 2r-4, 2r-2,$$

to obtain a method of tolerant order $2r$. It may be helpful to remember that for a symmetric rule Q the degree of precision is automatically an odd number. If α is even, then G_α is to within a constant multiple the even Bernoulli polynomial, cf. [**2**].

Example 2. Let L_0 be the logarithmic-kernel integral operator, i.e., L_0 is an even operator with $\beta = -1$. If we choose $r = r'$ to be even, then Theorem 5.1 implies that, for a tolerant order of $2r+1$, the rule should satisfy

$$\sum_j w_j G_k(\xi_j) = 0 \quad \text{for } k = r, r+1, \dots, 2r-1, 2r.$$

In particular, if $r = r' = 2$, then the tolerant order of the resulting method is 5. If we choose $r = r'$ to be odd, then Theorem 5.2 implies that the rule should satisfy

$$\sum_j w_j G_k(\xi_j) = 0 \quad \text{for } k = r+1, r+2, \dots, 2r-1, 2r,$$

to obtain a method of tolerant order $2r+1$.

6. Numerical experiments. In this section we test the tolerant qualocation methods when the operator L is the integral operator with logarithmic kernel, for which the principal part L_0 is an even operator of order $\beta = -1$.

This operator arises in the boundary integral reformulation of the Dirichlet problem for Laplace's equation, using the single-layer potential representation. Let $\Omega \subset \mathbf{R}^2$ be a bounded open region whose boundary Γ is a simple smooth closed curve. To avoid the problem of 'Γ-contour,' see [**5**], we assume that the transfinite diameter of Γ is different from 1. We want to find a continuous function $U : \overline{\Omega} \to \mathbf{R}$ such that

$$\Delta U = 0 \quad \text{in } \Omega, \quad \text{and} \quad U|_\Gamma = g, \tag{6.1}$$

where g is given. By expressing U as a single-layer potential of an unknown density $w : \Gamma \to \mathbf{R}$,

$$U(X) = -\frac{1}{\pi} \int_\Gamma \log |X - Y| w(Y)\, ds_Y, \quad X \in \Omega, \tag{6.2}$$

TABLE 1. Equations to be solved to achieve tolerant order σ for various values of $r-\beta$ and r'.

J	$r-\beta$	r'	$\tau=\tau'$	σ	$\sum_j w_j G_\alpha(\xi_j)=0$ with $\alpha=$
2	1	2	+	2	1
		3	−	4	2
	2	1	−	3	2
		2	+	4	2
		3	−	4	3
		4	+	4	2
	3	1	−	4	2
		2	+	3	2
		3	−	6	4
		4	+	4	3
	4	1	−	4	2
		2	+	4	2
		3	−	5	4
		4	+	6	4
3	1	2	+	3	1,2
		3	−	4	2
		4	+	4	1,3
	2	1	−	3	2
		2	+	4	2
		3	−	5	3,4
		4	+	6	2,4
	3	1	−	4	2
		2	+	4	2,3
		3	−	6	4
		4	+	5	3,4

TABLE 1. Continued.

J	$r-\beta$	r'	$\tau=\tau'$	σ	$\sum_j w_j G_\alpha(\xi_j) = 0$ with $\alpha =$
	4	2	$+$	6	2,4
		3	$-$	6	4,5
		4	$+$	8	4,6
	5	1	$-$	6	2,4
		2	$+$	5	2,4
		3	$-$	8	4,6
		4	$+$	6	4,5
4	1	2	$+$	3	1,2
		3	$-$	4	2
		4	$+$	5	1,3,4
	2	1	$-$	3	2
		2	$+$	4	2
		3	$-$	5	3,4
		4	$+$	6	2,4
	3	1	$-$	4	2
		2	$+$	5	2,3,4
		3	$-$	6	4
		4	$+$	6	3,4,5
	4	1	$-$	5	2,4
		2	$+$	6	2,4
		3	$-$	7	4,5,6
		4	$+$	8	4,6
	5	1	$-$	6	2,4
		2	$+$	6	2,4,5
		3	$-$	8	4,6
		4	$+$	7	4,5,6

TABLE 2. Quadrature points and weights used in experiments.

Rule 1		Rule 2	
ξ_j	w_j	ξ_j	w_j
0.0596687364534889	0.1562419769232472	0.0	0.0798080681520659
0.3181117948932994	0.3437580230767528	0.1557316965548238	0.2673850860744639
0.6818882051067006	0.3437580230767528	0.5	0.3854217596990064
0.9403312635465111	0.1562419769232472	0.8442683034451762	0.2673850860744639

where ds_Y is the element of arc length, and taking the limit as X approaches Γ, we reformulate the problem (6.1) into a first kind integral equation with logarithmic kernel in the unknown w,

$$Vw(X) = g(X), \quad X \in \Gamma,$$

where

$$Vw(X) := -\frac{1}{\pi}\int_\Gamma \log|X-Y|w(Y)\,ds_Y, \quad X \in \Gamma.$$

Introducing a parametrization $\gamma : [0,1] \to \Gamma$ we obtain from the above equation

$$-2\int_0^1 \log|\gamma(x)-\gamma(y)|u(y)\,dy = f(x), \quad x \in [0,1], \tag{6.3}$$

where $u(x) = (2\pi)^{-1}w(\gamma(x))|\gamma'(x)|$ and $f(x) = g(\gamma(x))$. It is well known that, see, for example, [**13**], (6.3) is of the form (2.4) where L is given by (2.1)–(2.3) with L_0 being an even operator of order -1.

In the following experiments, the curve Γ is taken to be an ellipse centered at the origin with major axis of length 4 along the x-axis and minor axis of length 2. We solve (6.3) using piecewise-linear splines as trial and test functions in the tolerant qualocation method specified in Section 5, Example 2, and also in the original qualocation method with the 3/7, 4/7 rule. For the tolerant quadrature method we use the two four-point rules defined in Table 2 (which are taken from [**16**]). The difference between the rules is that rule 2 includes 0 and 1/2 among its points, rule 1 does not. Both rules integrate exactly G_2, G_3 and G_4, see Table 1.

TABLE 3. Errors in the potential and experimental order of convergence at $X = (1.0, 0.3)$ for Experiment 1.

	Tol. qual., rule 1		Tol. qual., rule 2		Qual., 3/7–4/7 rule	
N	$\lvert U(X)-U_N(X)\rvert$	Eoc	$\lvert U(X)-U_N(X)\rvert$	Eoc	$\lvert U(X)-U_N(X)\rvert$	Eoc
8	3.76e-02		3.75e-02		3.53e-02	
16	2.22e-04	7.40	2.29e-04	7.35	6.32e-04	5.81
32	1.39e-05	4.00	1.42e-05	4.02	1.70e-05	5.22
64	3.87e-07	5.17	3.94e-07	5.17	5.17e-07	5.04
128	1.20e-08	5.02	1.22e-08	5.01	1.60e-08	5.01
256	3.73e-10	5.00	3.80e-10	5.00	5.00e-10	5.00

TABLE 4. Errors in the potential and experimental order of convergence at $X = (1.0, 0.3)$ for Experiment 2.

	Tol. qual., rule 1		Tol. qual., rule 2		Qual., 3/7–4/7 rule	
N	$\lvert U_{1024}(X)-U_N(X)\rvert$	Eoc	$\lvert U_{1024}(X)-U_N(X)\rvert$	Eoc	$\lvert U_{1024}(X)-U_N(X)\rvert$	Eoc
8	1.91e-03		1.91e-03		1.84e-03	
16	4.44e-04	2.10	4.44e-04	2.10	2.69e-04	2.78
32	6.61e-06	6.07	6.59e-06	6.07	2.78e-05	3.27
64	3.73e-08	7.47	3.71e-08	7.47	1.15e-05	1.28
128	3.08e-09	3.60	3.07e-09	3.59	3.98e-06	1.53
256	2.89e-10	3.41	2.89e-10	3.41	1.29e-06	1.62

In the first experiment we check that when g is smooth all methods give the $O(h^5)$ order of convergence for the approximate potential at the point $X = (1.0, 0.3) \in \Omega$. In the second experiment we take g not smooth, which results in a nonsmooth solution u of (2.4). Here the advantage of the new method over the original qualocation method is shown.

Experiment 1. We take $g(X) = X_1^2 - X_2^2$, $X \in \Gamma$, where $X = (X_1, X_2)$, so that the exact solution U of (6.1) is given by $U(X) = X_1^2 - X_2^2$, $X \in \Omega$. The numbers in Table 3 show that, as predicted by Theorem 4.1 and [**11**], an order of convergence of $O(h^5)$ is achieved for all rules. (The column Eoc shows the estimated orders of convergence computed from the errors themselves.)

Experiment 2. In this experiment we choose g so that $f(x) = \sqrt{x(1-x)}$. Since $f \in H^{1-\varepsilon}$ for any $\varepsilon > 0$, we have $u \in H^{-\varepsilon}$. This lack of smoothness of u results in a decrease in the order of convergence, in particular for the original qualocation method. We compare the four-point rules in Table 2 with the 3/7, 4/7 rule. It is predicted by Theorem 4.1 that the four-point rules yield convergence order of $O(h^{3-\varepsilon})$, whereas the 3/7, 4/7 rule for the original method is predicted, see [**11**], to yield only order $O(h^{1-\varepsilon})$. The numbers in Table 4 suggest better than predicted orders in both cases, perhaps $O(h^{7/2})$ for the tolerant method, and $O(h^{3/2})$ for the original method. (In this experiment, since the exact solution U is not known we computed the errors in the potential as $|U_{1024}(X) - U_N(X)|$ for $N = 8, 16, 32, 64, 128, 256$).

In summary the two qualocation methods of Table 2 perform as well as the original qualocation method if u is smooth. They perform far better than the original qualocation method when u is not smooth, at the expense of a doubling of the computational work in setting up the matrix.

Appendix

In the following we prove Lemmas 2.1 and 4.1.

Proof of Lemma 2.1. We recall that, see (2.24) and (2.14),

$$E_2(y) = \sum_j w_j \overline{\Delta'(\xi_j, y)},$$

where

$$\Delta'(\xi, y) = y^{r'} F_{r'}^{\tau'}(\xi, y) \quad \text{with} \quad \tau' = \begin{cases} + & \text{if } r' \text{ even}, \\ - & \text{if } r' \text{ odd}. \end{cases}$$

It was proved in [**3**, Lemma A2] that, for r' even,

$$y^{r'} F_{r'}^{+}(\xi, y) = y^{r'} G_{r'}(\xi) + \sum_{k=1}^{\infty} \binom{-r'}{2k} G_{r'+2k}(\xi) y^{r'+2k} + \iota \sum_{k=1}^{\infty} \binom{-r'}{2k-1} H_{r'+2k-1}(\xi) y^{r'+2k-1},$$

and for r' odd,

$$y^{r'}F^-_{r'}(\xi,y) = \iota y^{r'}H_{r'}(\xi) + \sum_{k=1}^{\infty}\binom{-r'}{2k-1}G_{r'+2k-1}(\xi)y^{r'+2k-1} + \iota\sum_{k=1}^{\infty}\binom{-r'}{2k}H_{r'+2k}(\xi)y^{r'+2k},$$

with G_α and H_α defined by (5.3) and (5.4). Since G_{2m} and H_{2n-1} are, to within constant multiples, the even and odd Bernoulli polynomials, see [**3**], all the coefficients in the above formulas are polynomials of degree indicated by the subscript. If a rule Q is exact for polynomials up to degree $r'+b'-1$, then we have

$$QG_{2m} = 0 \quad \text{and} \quad QH_{2n-1} = 0,$$

for all m and n such that $0 \le 2m \le r'+b'-1$ and $0 \le 2n-1 \le r'+b'-1$, which immediately implies $E_2(y) = O(y^{r'+b'})$. □

Proof of Lemma 4.1. We recall from [**3**, Lemma 1 (ii)] that

$$\langle \phi_n, \psi'_\mu\rangle_h = \begin{cases} 0 & \text{if } n \not\equiv \mu, \\ \sum_j w_j\phi_l(\xi_j) & \text{if } n = lN,\ \mu = 0, \\ \sum_j w_j\phi_l(\xi_j)(1+\overline{\Delta'(\xi_j,\mu/N)}) & \text{if } n = \mu + lN,\ \mu \ne 0, \end{cases} \tag{7.1}$$

and from (2.12), with ψ_μ replaced by ψ'_μ, that

$$\langle \phi_\nu, \psi'_\mu\rangle = \begin{cases} 1 & \text{if } \nu = \mu, \\ 0 & \text{if } \nu \ne \mu. \end{cases} \tag{7.2}$$

On writing f and Pf as $f = \sum_{n\in\mathbf{Z}}\hat f(n)\phi_n$ and $Pf = \sum_{\mu\in\Lambda}\widehat{Pf}(\mu)\phi_\mu$ and using the defining equation (4.1) with (7.1) and (7.2), we obtain

$$\widehat{Pf}(\mu) = \begin{cases} \hat f(0) + \sum_{l\ne 0}(\sum_j w_j\phi_l(\xi_j))\hat f(lN) & \text{if } \mu = 0, \\ \hat f(\mu) + (\sum_j w_j\overline{\Delta'(\xi_j,\mu h)})\hat f(\mu) & \\ \quad + \sum_{l\ne 0}(\sum_j w_j\phi_l(\xi_j)[1+\overline{\Delta'(\xi_j,\mu h)}]) & \\ \qquad \cdot\hat f(\mu+lN) & \text{if } \mu \ne 0. \end{cases} \tag{7.3}$$

Hence, since $\sum_{l\neq 0} |l|^{-2t} < \infty$ when $t > 1/2$,

$$|\widehat{Pf}(0) - \hat{f}(0)|^2 \leq \left(\sum_{l\neq 0} |lN|^{-t}|lN|^t|\hat{f}(lN)| \right)^2 \leq ch^2\|f\|_t^2. \tag{7.4}$$

On the other hand, for $\mu \neq 0$ we have

$$\begin{aligned} |\widehat{Pf}(\mu) - \hat{f}(\mu)|^2 &\leq 2|E_2(\mu h)|^2|\hat{f}(\mu)|^2 \\ &\quad + 2\left| \sum_{l\neq 0} \left(\sum_j w_j\phi_l(\xi_j)[1+\overline{\Delta'(\xi_j,\mu h)}] \right) \hat{f}(\mu + lN) \right|^2 \\ &= T_3(\mu) + T_4(\mu), \end{aligned} \tag{7.5}$$

where E_2 is defined in (2.24). It follows from (2.27) and the assumptions in the lemma that

$$T_3(\mu) \leq c|\mu h|^{2(r'+b')}|\hat{f}(\mu)|^2. \tag{7.6}$$

Noting that $|1+\overline{\Delta'(\xi_j,\mu h)}| \leq c$, $t > 1/2$, and using (3.5) together with the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} T_4(\mu) &\leq c\left(\sum_{l\neq 0} |\mu + lN|^{-t}|\mu + lN|^t|\hat{f}(\mu + lN)| \right)^2 \\ &\leq ch^{2t} \sum_{l\neq 0} |\mu + lN|^{2t}|\hat{f}(\mu + lN)|^2. \end{aligned} \tag{7.7}$$

Inequalities (7.4)–(7.7), $s \geq 0$, $r' + b' + s - t \geq 0$, $|\mu| \leq N/2$ and $|l + \mu/N|^{2(s-t)} \leq c$ imply

$$\begin{aligned} \|Pf - f\|_s^2 &= |\widehat{Pf}(0) - \hat{f}(0)|^2 + \sum_{\mu\in\Lambda^*} |\mu|^{2s}|\widehat{Pf}(\mu) - \hat{f}(\mu)|^2 \\ &\quad + \sum_{\mu\in\Lambda} {\sum_{n\equiv\mu}}' |n|^{2s}|\hat{f}(n)|^2 \\ &\leq ch^{2t}\|f\|_t^2 + ch^{2(t-s)} \sum_{\mu\in\Lambda^*} |\mu h|^{2(r'+b'+s-t)}|\mu|^{2t}|\hat{f}(\mu)|^2 \\ &\quad + ch^{2t} \sum_{\mu\in\Lambda^*} |\mu|^{2s} \sum_{l\neq 0} |\mu + lN|^{2t}|\hat{f}(\mu + lN)|^2 \\ &\quad + \sum_{\mu\in\Lambda} \sum_{l\neq 0} |\mu + lN|^{2(s-t)}|\mu + lN|^{2t}|\hat{f}(\mu + lN)|^2 \\ &\leq ch^{2(t-s)}\|f\|_t^2. \end{aligned}$$

Thus the lemma is proved. □

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