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RIGOROUS RESULTS ON THE ASYMPTOTIC SOLUTIONS OF SINGULARLY PERTURBED NONLINEAR VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. This paper studies singularly perturbed Volterra integral equations of the form

$$\varepsilon u(t) = f(t;\varepsilon) + \int_0^t g(t,s,u(s))\,ds, \quad 0 \le t \le T,$$

where ε is a small parameter. The function $f(t;\varepsilon)$ is defined for $0 \le t \le T$ and $g(t,s,u)$ for $0 \le s \le t \le T$. There are many existence and uniqueness results known that ensure that a unique continuous solution $u(t;\varepsilon)$ exists for all small $\varepsilon > 0$. The aim is to find asymptotic approximations to these solutions and rigorously prove the asymptotic correctness. This work is restricted to problems where there is an *initial layer*; various hypotheses are placed on g to exclude other behaviors.

1. Introduction. A singular perturbation problem is a problem which depends on a parameter (or parameters) in such a way that solutions of the problem behave nonuniformly as the parameter tends toward some limiting value of interest. The nature of the nonuniformity of the solutions can vary. This article concerns solutions of nonlinear Volterra integral equations in which such nonuniformity occurs in an isolated narrow region called the initial (or boundary) layer. The thickness of the layer vanishes as the parameter tends to zero.

In particular, we consider the nonlinear singularly perturbed Volterra integral equation

$$\varepsilon u(t) = f(t;\varepsilon) + \int_0^t g(t,s,u(s))\,ds, \quad 0 \le t \le T, \tag{1.1}$$

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where $0 < \varepsilon \ll 1$. The function $f(t;\varepsilon)$ is C^∞ and defined for $0 \le t \le T$ and $0 \le \varepsilon \le 1$; $g(t,s,u)$ is also C^∞ and defined for $0 \le s \le t \le T$ and $-\infty < u < \infty$. Also we require that $\lim_{\varepsilon\to 0} f(0;\varepsilon) = 0$. f has an asymptotic power series expansion,

$$f(t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j f_j(t) \quad \text{as } \varepsilon \to 0,$$

where each $f_j(t)$ is C^∞. Furthermore, we require that $f_0(0) = 0$ and $f_1(0)$ is nontrivial. The function g can also depend regularly on ε but we assume here that it is independent of ε.

Problem (1.1) depends on the parameter ε in such a way that the reduced equation

$$0 = f_0(t) + \int_0^t g(t,s,v(s))\,ds, \quad 0 \le t \le T,$$

is a Volterra equation of the first kind. For this to have a continuous solution, $f_0(t)$ cannot be merely continuous. Assuming that a stability condition for the boundary layer holds, we show that $u(t;\varepsilon)$ converges uniformly to $v(t)$ as $\varepsilon \to 0$. The fact that we need more smoothness on g and f implies there is a loss of regularity as ε changes from positive values to zero. The loss of regularity here is due to the dependence of $u(t;\varepsilon)$ on ε. Our task is to discover the nature of this dependence by working with suitable approximate integral equations and then express $u(t;\varepsilon)$ in terms of solutions of these equations for small ε.

Singularly perturbed Volterra integral equations occur in various areas of interest ranging from engineering, physics, chemistry and ecology to epidemiology. A comprehensive survey of literature to singularly perturbed Volterra models is found in [**17**].

Angell and Omstead [**2**] used the additive decomposition method to obtain the first few terms in a formal solution of (1.1). However, their approach has some short shortcomings including the fact that the general equations for the coefficients in the formal solution cannot be determined and their results are not rigorously presented. This paper aims at improving their results. Lange and Smith [**19**], [**20**] used the additive decomposition method in their study of singularly perturbed linear Fredholm and Volterra integral equations. They deduced the general expansion for the formal solution and rigorous estimates to show its closeness to the exact solution. Skinner [**32**] developed a method of generating all the terms of the formal solution and showed that the formal solution is an asymptotic solution. The approach in [**32**] is somehow more complicated than the presentation given here. However, an adaptation of Skinner's method of deriving the equations for the formal solution is included here. This work builds on that of Smith [**33**, Chapter 6], O'Malley [**28**, Chapter 4] and O'Malley [**29**, Chapter 2] on singularly perturbed initial value problems for nonlinear ordinary differential equations. Kauthen [**15**] provided a survey of analytical and numerical solutions to singularly perturbed Volterra and Fredholm integral and integro-differential equations. The analysis in [**17**] for solutions of Volterra equations is similar to the one presented here. Kauthen constructed asymptotic solutions to linear problems with convolution kernels and hence one will find that the present work is general and is more detailed.

In recent years there has been an increasing interest to solutions of singularly perturbed integral equations, both analytically and numerically. The reader is advised to consult Kauthen [**17**] for references on singularly perturbed integral equations, some of which are included in this paper. Many researchers seem to have focused their interest on singularly perturbed Volterra integro-differential equations and Fredholm integral equations. These include the work by Angell and Olmstead [**1**]–[**3**], Lange and Smith [**18**]–[**19**], Lomov [**24**], Liu [**22**]–[**23**], Kauthen [**15**]–[**16**] and Horvat and Rogina [**14**] on integro-differential equations. Fredholm equations have been studied by Lange and Smith [**18**]–[**19**], Angell and Olmstead [**4**], Gautesen [**9**]–[**10**], Olmstead and Gautesen [**27**], Wills and Nemat-Nasser [**35**], Georgiou [**9**], Ramm and Shifrin [**30**]–[**31**], Smith [**33**] and Liqun and Nasser [**25**] which generalizes the results in [**35**].

In the use of additive decomposition method, one imposes the boundary layer stability condition. This forces the inner layer solution to decay (as the parameter tends to zero) exponentially and thus simplifies the analysis. When the boundary layer stability condition fails, care has to be taken especially when using the additive decomposition method. In this case there is no exponential decay for the inner layer solution. Lange [**18**] demonstrated this phenomenon using an example. It is typical for the inner layer solution corresponding to a weakly singular kernel to decay algebraically. Indeed, singularly perturbed Volterra integral equations with weakly singular kernels are not well studied. The problem on the heat conduction however has received some attention, (see [**17**] for the details). Bijura [**5**] investigated solutions to a problem of the form

$$\varepsilon u(t) = f(t) + \frac{1}{\Gamma(\beta)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\beta}} u(s)\, ds, \quad 0 \le t \le T, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ and $0 < \beta < 1$. It is assumed in [**5**] that the data is continuous and $k(t,t) = -1$. By pointing out possible technical difficulties, Bijura constructed the leading order asymptotic solution and the corresponding proof of asymptotic correctness. The main interest in [**5**] is on the decay of the inner layer solution. The nonlinear version of (1.2) with a convolution kernel is discussed by Kauthen [**17**] using numerical methods. Chen [**7**] derived the asymptotic expansion of the solution to a certain nonlinear singularly perturbed Fredholm integral equations with weakly singular kernels. The existence of its solution is also proved.

The paper is organised as follows: In Section 2, we construct a formal solution for (1.1) using the additive decomposition method. We start by introducing the inner layer variable using the dominant balance argument. The solution is thought in the form

$$U_N(t;\varepsilon) = \sum_{j=0}^{N} \varepsilon^j \left[y_j(t) + z_j\left(\frac{t}{\varepsilon}\right) \right], \quad \frac{t}{\varepsilon} = \tau, \tag{1.3}$$

where $y_j(t)$ represents the outer solution and $z_j\left(\frac{t}{\varepsilon}\right)$ represents the inner layer solution. In Section 3 we prove that $y_j(t)$ and $z_j(\tau)$ have the properties assumed in their derivation. Then in Section 4 we prove, using the Banach fixed point theorem, that

$$|u(t;\varepsilon) - U_N(t;\varepsilon)| = O(\varepsilon^{N+1}) \quad \text{as } \varepsilon \to 0,$$

uniformly for $0 \le t \le T$. To demonstrate the methodology developed in previous sections, in Section 5 we illustrate it by solving two examples, one of which has been considered in Angell and Olmstead [**2**] and another one from population growth modeling.

2. Derivation of the formal solution. We derive in this section a formal solution for the integral equation (1.1) using the additive decomposition method. We suppose that the solution of (1.1) can be represented in the form

$$u(t;\varepsilon) = y(t;\varepsilon) + \phi(\varepsilon)z(t/\mu(\varepsilon);\varepsilon), \tag{2.1}$$

where

$$y(t;\varepsilon) = y_0(t) + o(1), \quad z(\tau;\varepsilon) = z_0(\tau) + o(1) \quad \text{as } \varepsilon \to 0.$$

Firstly we determine formally the width $\mu(\varepsilon)$ and the magnitude $\phi(\varepsilon)$ of the initial boundary layer, supposing that $\mu(\varepsilon) \to 0$. For this argument we assume that $g(0,0,u)$ is nontrivial. Substituting (2.1) into (1.1) gives

$$\begin{aligned} \varepsilon y(t;\varepsilon) + \varepsilon\phi(\varepsilon)z(t/\mu(\varepsilon);\varepsilon) &= f(t;\varepsilon) \\ &\quad + \int_0^t g(t,s,y(s;\varepsilon) + \phi(\varepsilon)z(s/\mu(\varepsilon);\varepsilon))\,ds, \end{aligned} \tag{2.2}$$

which, letting $\tau = t/\mu(\varepsilon)$, is equivalent to

$$\begin{aligned} &\varepsilon y(\mu(\varepsilon)\tau;\varepsilon) + \varepsilon\phi(\varepsilon)z(\tau;\varepsilon) \\ &\qquad = f(\mu(\varepsilon)\tau;\varepsilon) + \mu(\varepsilon)\int_0^\tau g(\mu(\varepsilon)\tau, \mu(\varepsilon)\sigma, y(\mu(\varepsilon)\sigma;\varepsilon) \\ &\qquad\quad + \phi(\varepsilon)z(\sigma;\varepsilon))\,d\sigma. \end{aligned}$$

Hence, fixing $\tau > 0$ and letting $\varepsilon \to 0$,

$$\begin{aligned} &\varepsilon y_0(0) + \varepsilon\phi(\varepsilon)z_0(\tau) \\ &\quad = \varepsilon f_1(0) + \mu(\varepsilon)\int_0^\tau g(0,0,y_0(0) + \phi(\varepsilon)z_0(\sigma))\,d\sigma + o(\varepsilon) + o(\mu(\varepsilon)). \end{aligned}$$

Dominant terms can be balanced if we take

$$\mu(\varepsilon) = \varepsilon, \quad \phi(\varepsilon) = 1.$$

To obtain a formal solution we now suppose that $y(t;\varepsilon)$ and $z(\tau;\varepsilon)$ have the asymptotic expansions

$$y(t;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j y_j(t), \quad z(\tau;\varepsilon) \sim \sum_{j=0}^{\infty} \varepsilon^j z_j(\tau)$$

as $\varepsilon \to 0$. $y(t;\varepsilon)$ represents the outer solution, which approximates the solution outside the initial layer, while $z\left(\frac{t}{\varepsilon};\varepsilon\right)$ represents the inner correction term which is required for uniform approximation of the solution of (1.1) inside the initial layer but is negligible outside the initial layer. We required for each $j \geq 0$ that

$$z_j(\tau) = o(\tau^{-r}) \quad \text{as } \tau \to \infty \tag{2.3}$$

for all $r \geq 0$. The rapid decay in the initial layer is crucial for the application of the method of additive decomposition because then transcendentally small terms can be omitted from the asymptotic expansions.

Since Theorem 2.1 from Skinner [**32**] is used later in this section, it is stated here.

Lemma 2.1. *Suppose that $\eta(t,\tau;\varepsilon)$ is a C^∞ function on $[0,T] \times [0,\infty) \times [0,1]$ and $\eta(t,\tau;\varepsilon) = o(\tau^{-r})$ as $\tau \to \infty$ for all $r \geq 0$. Then*

$$\eta\left(t, \frac{t}{\varepsilon};\varepsilon\right) = \sum_{j=0}^{N} \varepsilon^j \eta_j\left(\frac{t}{\varepsilon}\right) + O(\varepsilon^{N+1}),$$

where $\eta_j(\tau)$ is a C^∞ function on $[0,\infty)$ and is the coefficient of ε^j in the Taylor expansion of $e \mapsto \eta(\varepsilon\tau,\tau,\varepsilon)$. Also $\eta_j(\tau) = o(\tau^{-r})$ as $\tau \to \infty$ for all $r \geq 0$.

Another result which will be used later in this section and whose proof is omitted is:

Lemma 2.2. *For each integer $j \geq 0$, let $p_j(t)$ be a continuous function on $[0,T]$ and $q_j(\tau)$ a continuous function on $[0,\infty)$ such that $q_j(\tau) \to 0$ as $\tau \to \infty$. Suppose that, for every integer $N \geq 1$,*

$$\sum_{j=0}^{N-1} \left\{p_j(t) + q_j\left(\frac{t}{\varepsilon}\right)\right\}\varepsilon^j = O(\varepsilon^N), \tag{2.4}$$

uniformly as $\varepsilon \to 0$. Then $p_j = 0$ and $q_j = 0$ for every $j \geq 0$.

We shall substitute (1.3) into (1.1). Therefore, for a fixed integer $N \geq 0$, we first consider the term

$$\int_0^t g(t, s, U_N(s; \varepsilon))\, ds.$$

We introduce

$$H(t, s; \varepsilon) = g\left(t, s, \sum_{j=0}^{N} \varepsilon^j y_j(s)\right),$$

$$K(t, s, \sigma; \varepsilon) = g\left(t, s, \sum_{j=0}^{N} \varepsilon^j (y_j(s) + z_j(\sigma))\right) - g\left(t, s, \sum_{j=0}^{N} \varepsilon^j y_j(s)\right),$$

By (2.3) and the mean value theorem, $K(t, s, \sigma; \varepsilon) = o(\sigma^{-r})$ as $\sigma \to \infty$ for all $r \geq 0$. By applying Lemma 2.1 to $(s, \sigma; \varepsilon) \mapsto K(t, s, \sigma; \varepsilon)$, we deduce that

$$K(t, \varepsilon\sigma, \sigma; \varepsilon) = \sum_{j=0}^{N} \varepsilon^j k_j(t, \sigma) + O(\varepsilon^{N+1}), \tag{2.6}$$

with $k_j(t, \sigma) = o(\sigma^{-r})$ for all $r \geq 0$. Also straightforward Taylor expansions yields

$$H(t, s; \varepsilon) = \sum_{j=0}^{N} \varepsilon^j h_j(t, s) + O(\varepsilon^{N+1}), \tag{2.7}$$

$$K(\varepsilon\tau, \varepsilon\sigma, \sigma; \varepsilon) = \sum_{j=0}^{N} \varepsilon^j l_j(\tau, \sigma) + O(\varepsilon^{N+1}). \tag{2.8}$$

The coefficients $h_j(t, s)$ in (2.7) are given by

$$h_0(t, s) = g(t, s, y_0(s)), \quad h_1(t, s) = \partial_3 g(t, s, y_0(s)) y_1(s),$$

and in general for $j \geq 1$,

$$h_j(t, s) = \partial_3 g(t, s, y_0(s)) y_j(s) + \Phi_j(t, s),$$

where $\Phi_j(t,s)$ is determined by $y_i(s)$ for $0 \le i \le j-1$. The first two terms of Φ_j are given by

$$\Phi_1(t,s) = 0, \quad \Phi_2(t,s) = \frac{1}{2}\partial_3^2 g(t,s,y_0(s))y_1^2(s).$$

The coefficients $k_j(t,\sigma)$ in (2.6) are given by

$$\begin{aligned} k_0(t,\sigma) &= g(t,0,y_0(0)+z_0(\sigma)) - g(t,0,y_0(0)),\\ k_1(t,\sigma) &= \partial_3 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma) + \Psi_1(t,\sigma), \end{aligned}$$

and in general, for $j \ge 1$,

$$k_j(t,\sigma) = \partial_3 g(t,0,y_0(0)+z_0(\sigma))z_j(\sigma) + \Psi_j(t,\sigma).$$

Here the function $\Psi_j(t,\sigma)$ is determined by $y_i(s)$ for $0 \le i \le j$ and $z_i(\sigma)$ for $0 \le i \le j-1$. The first two Ψ_j are given by

$$\begin{aligned} \Psi_1(t,\sigma) &= \{\partial_2 g(t,0,y_0(0)+z_0(\sigma) - \partial_2 g(t,0,y_0(0))\}\sigma\\ &\quad + \{\partial_3 g(t,0,y_0(0)+z_0(\sigma)) - \partial_3 g(t,0,y_0(0))\}(y_0'(0)\sigma + y_1(0)),\\ \Psi_2(t,\sigma) &= \bigg\{\partial_3 g(t,0,y_0(0)+z_0(\sigma))\\ &\qquad - \partial_3 g(t,0,y_0(0))\bigg\}\bigg(y_2(0) + y_1(0)\sigma + \frac{1}{2}y_0''(0)\sigma^2\bigg)\\ &\quad + \{\partial_2\partial_3 g(t,0,y_0(0)+z_0(\sigma))\\ &\qquad - \partial_2\partial_3 g(t,0,y_0(0))\}(y_0'(0)\sigma^2 + y_1(0)\sigma)\\ &\quad + \partial_2\partial_3 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma)\sigma\\ &\quad + \partial_3^2 g(t,0,y_0(0)+z_0(\sigma))z_1(\sigma)y_1(0)\\ &\quad + \frac{1}{2}\{\partial_3^2 g(t,0,y_0(0)+z_0(\sigma))\\ &\qquad - \partial_3^2 g(t,0,y_0(0))\}(y_0'(0)^2\sigma^2 + y_1^2(0)\\ &\quad + 2y_0'(0)y_1(0)\sigma)\\ &\quad + \frac{1}{2}\partial_3^2 g(t,0,y_0(0)+z_0(\sigma))\{z_1^2(\sigma) + z_1(\sigma)y_0'(0)\sigma\}\\ &\quad + \frac{1}{2}\{\partial_2^2 g(t,0,y_0(0)+z_0(\sigma)) - \partial_2^2 g(t,0,y_0(0))\}\sigma^2. \end{aligned}$$

The coefficients $l_j(\tau,\sigma)$ in (2.8) are given by

$$l_0(\tau,\sigma) = g(0,0,y_0(0)+z_0(\sigma)) - g(0,0,y_0(0)),$$
$$l_1(\tau,\sigma) = \partial_3 g(0,0,y_0(0)+z_0(\sigma))z_1(\sigma) + \Xi_1(\tau,\sigma),$$

and in general, for $j \geq 1$,

$$l_j(\tau,\sigma) = \partial_3 g(0,0,y_0(0)+z_0(\sigma))z_j(\sigma) + \Xi_j(\tau,\sigma),$$

where $\Xi_j(\tau,\sigma)$ is determined by y_i for $i \leq j$ and z_i for $i \leq j-1$. In particular,

$$\begin{aligned}\Xi_1(\tau,\sigma) = {} & \{\partial_1 g(0,0,y_0(0)+z_0(\sigma)) - \partial_1 g(0,0,y_0(0))\}\tau \\ & + \{\partial_2 g(0,0,y_0(0)+z_0(\sigma)) - \partial_2 g(0,0,y_0(0))\}\sigma \\ & + \{\partial_3 g(0,0,y_0(0)+z_0(\sigma)) - \partial_3 g(0,0,y_0(0))\} \\ & \times (y_0'(0)\sigma + y_1(0)).\end{aligned}$$

It follows from (2.5) that

(2.9)
$$\begin{aligned}\int_0^t g(t,s,U_N(s;\varepsilon))\,ds = {} & \sum_{j=0}^{N} \varepsilon^j \left(\int_0^t h_j(t,s)\,ds + \varepsilon \int_0^\infty k_j(t,\sigma)\,d\sigma \right) \\ & - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{\frac{t}{\varepsilon}}^\infty k_j(t,\sigma)\,d\sigma + O(\varepsilon^{N+1}).\end{aligned}$$

Since $k_j(t,\sigma) = o(\sigma^{-r})$ for all $r \geq 0$,

$$\int_\tau^\infty k_j(t,\sigma)\,d\sigma = o(\tau^{-r}),$$

for all $r \geq 0$, and Lemma 2.1 implies that

$$\int_{\frac{t}{\varepsilon}}^\infty k_j(t,\sigma)\,d\sigma = \int_{\frac{t}{\varepsilon}}^\infty \sum_{i=0}^{j} \varepsilon^i \tilde{k}_{j,i}\left(\frac{t}{\varepsilon},\sigma\right) d\sigma + O(\varepsilon^{N+1}),$$

where $\tilde{k}_{j,i}(\tau,\sigma)$ is the coefficient of ε^i in the Taylor expansion of $\varepsilon \mapsto k_j(\varepsilon\tau,\sigma)$. Of course, Lemma 2.1 also assures us that

$$\int_\tau^\infty \tilde{k}_{j,i}(\tau,\sigma)\,d\sigma = o(\tau^{-r}) \quad \text{as } \tau \to \infty$$

for all $r \geq 0$. Note also that if

$$K(\varepsilon\tau, \varepsilon\sigma, \sigma; \varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j l_j(\tau, \sigma),$$

then

$$\sum_{i=0}^{j} \tilde{k}_{j-i,i}(\tau, \sigma) = l_j(\tau, \sigma). \tag{2.10}$$

It follows that (2.9) becomes

$$\begin{aligned}\int_0^t g(t, s, U_N(s; \varepsilon))\, ds = &\sum_{j=0}^{N} \varepsilon^j \left(\int_0^t h_j(t, s)\, ds + \varepsilon \int_0^\infty k_j(t, \sigma)\, d\sigma \right) \\ &- \sum_{j=0}^{N-1} \varepsilon^{j+1} \int_\tau^\infty l_j\left(\frac{t}{\varepsilon}, \sigma\right) d\sigma + O(\varepsilon^{N+1}).\end{aligned}$$

Next we define the *residual* $\rho_N(t; \varepsilon)$ by

$$\varepsilon U_N(t; \varepsilon) = f(t; \varepsilon) + \int_0^t g(t, s, U_N(s; \varepsilon))\, ds - \rho_N(t; \varepsilon). \tag{2.11}$$

Then, putting $y_{-1}(t) = 0$ and $k_{-1}(t, \sigma) = 0$, we see that

(2.12)

$$\begin{aligned}\rho_N(t; \varepsilon) = &\sum_{j=0}^{N} \varepsilon^j \left(\int_0^t h_j(t, s)\, ds + \int_0^\infty k_{j-1}(t, \sigma)\, d\sigma + f_j(t) - y_{j-1}(t) \right) \\ &- \sum_{j=0}^{N-1} \varepsilon^{j+1} \left(z_j\left(\frac{t}{\varepsilon}\right) + \int_{\frac{t}{\varepsilon}}^\infty l_j\left(\frac{t}{\varepsilon}, \sigma\right) \right) + O(\varepsilon^{N+1}).\end{aligned}$$

If $U_N(t; \varepsilon)$ is a formal solution for all $N \geq 0$, then $\rho_N(t; \varepsilon) = O(\varepsilon^{N+1})$ as $\varepsilon \to 0$ for all $N \geq 0$, in which case the argument of Lemma 2.2 shows that, for every $j \geq 0$, $y_j(t)$ and $z_j(\tau)$ satisfy

(2.13)

$$y_{j-1}(t) = f_j(t) + \int_0^t h_j(t, s)\, ds + \int_0^\infty k_{j-1}(t, \sigma)\, d\sigma,$$

(2.14)

$$z_j(\tau) = -\int_\tau^\infty l_j(\tau, \sigma)\, d\sigma$$

There is also an initial condition for solutions of (2.14), obtained from $\varepsilon u(0;\varepsilon) = f(0;\varepsilon)$; namely, that for all $j \geq 0$,

$$z_j(0) = f_{j+1}(0) - y_j(0). \tag{2.15}$$

Remark 2.3. There is considerable simplification in the case $g(t,s,u) = a(t,s)u$ for which (1.1) is a linear equation. It is found that

$$h_j(t,s) = a(t,s)y_j(s), \quad k_j(t,\sigma) = \sum_{i=0}^{j} e_i(t,\sigma)z_{j-i}(\sigma),$$

where

$$e_i(t,\sigma) = \frac{1}{i!}\partial_2^i a(t,0)\sigma^i.$$

Remark 2.4. Equation (2.12) for the residual has been derived only assuming that (2.3) is true. It follows that if (2.3) holds and (2.13) and (2.14) hold for $0 \leq j \leq N$, then $|\rho_N(t;\varepsilon)| = O(\varepsilon^{N+1})$ as $\varepsilon \to 0$.

3. Properties of the formal solution. In this section it is shown that there are unique solutions $y_j(t)$ and $z_j(\tau)$ of (2.13) and (2.14), and that they have the important properties assumed in their derivation. It is convenient to rewrite these equations as

$$0 = f_0(t) + \int_0^t g(t,s,y_0(s))\,ds, \tag{3.1}$$

$$z_0(\tau) = -\int_\tau^\infty (g(0,0,y_0(0)+z_0(\sigma)) - g(0,0,y_0(0)))\,d\sigma, \tag{3.2}$$

and $j \geq 1$,

$$0 = \phi_j(t) + \int_0^t \partial_3 g(t,s,y_0(s))y_j(s)\,ds, \tag{3.3}$$

$$z_j(\tau) = -\int_\tau^\infty \partial_3 g(0,0,y_0(0)+z_0(\sigma))z_j(\sigma)\,d\sigma + \psi_j(\tau). \tag{3.4}$$

Here we used the definitions

(3.5)
$$\phi_j(t) = f_j(t) + \int_0^t \Phi_j(t,s)\,ds + \int_0^\infty k_{j-1}(t,\sigma)\,d\sigma - y_{j-1}(t),$$

(3.6)
$$\psi_j(\tau) = -\int_\tau^\infty \Xi_j(\tau,\sigma)\,d\sigma.$$

We see that the leading order solutions (outer and inner correction) are given by nonlinear equations while the higher order terms are given by linear equations.

We use the following hypotheses on the functions $f(t;\varepsilon)$ and the kernel $g(t,s,u)$. They are based on the assumptions used in O'Malley [**29**, Chapter 4].

($\mathbf{H}_f$) The function $f:[0,T]\times[0,1]\to\mathbf{R}$ is C^∞ and $f(0;0)=0$. Also $g:\Delta_T\times\mathbf{R}\to\mathbf{R}$ is a C^∞ function where

$$\Delta_T = \{(t,s); 0\le s\le t\le T\}.$$

($\mathbf{H}_{y_0}$) A C^∞ solutions $y_0:[0,T]\to\mathbf{R}$ exists on (3.1) which is unique in the class of continuous functions on $[0,T]$.

($\mathbf{H}_g$) There is a positive constant α such that

$$\begin{aligned}\partial_3 g(t,t,y_0(t)) &\le -\alpha < 0 \quad \text{for all } 0\le t\le T,\\ \partial_3 g(0,0,v) &\le -\alpha < 0,\end{aligned}$$

for all v between $y_0(0)$ and $y_0(0)+f_1(0)$.

Remark 3.1. If ($\mathbf{H}_f$) holds, $f(t;\varepsilon)$ has the asymptotic expansion

$$f(t;\varepsilon) \sim \sum_{j=0}^\infty \varepsilon^j f_j(t), \quad \text{as } \varepsilon\to 0,$$

where each $f_j(t)$ is C^∞ on $[0,T]$.

Remark 3.2. Equation (3.1) is a Volterra integral equation of the first kind for $y_0(t)$. An existence and uniqueness theorem for this equation is given in Linz [**21**, Ch. 5, Theorem 5.2]. It is obtained by applying the method of successive approximations to the differentiated version of (3.1).

Proposition 3.3. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold. Then* (3.2) *and* (2.15) *have a* C^∞ *solution* z_0 *satisfying*

$$|z_0(\tau)| \le c_0 e^{-\alpha\tau}, \quad \tau \ge 0, \tag{3.7}$$

for some positive constant c_0.

Proof. The problem of solving (3.2) subject to (2.15) is equivalent to the initial-value problem

$$\begin{aligned} z_0'(\tau) &= g(0,0,y_0(0)+z_0(\tau)) - g(0,0,y_0(0)), \\ z(0) &= f_1(0) - y_0(0). \end{aligned} \tag{3.8}$$

By standard theory of ordinary differential equations, (see for example, Hirsch and Smale [**13**], Ch. 8), (3.8) has a unique continuous solution defined on a maximal interval $[0,S)$ such that $\lim_{\tau\uparrow S}|z_0(\tau)| = \infty$ if $S<\infty$. By the mean value theorem there is a function $\omega(\tau)$ such that

$$z_0'(\tau) = \partial_3 g(0,0,(1-\omega(\tau))y_0(0) + \omega(\tau)z_0(\tau))z_0(\tau).$$

Assumption $(\mathbf{H}_g)$ implies that $z_0(\tau)$ decreases if $z_0(0)>0$ and increases if $z_0(0)<0$ and that $z_0(\tau)+y_0(0)$ lies between $y_0(0)$ and $y_0(0)+f_1(0)$. Therefore,

$$z_0'(\tau)z_0(\tau) \le -\alpha z_0(\tau)^2,$$

and hence $|z_0(\tau)| \le |z_0(0)|e^{-\alpha\tau}$ for all $0\le\tau<S$. Hence $S=\infty$ and (3.7) holds. $\square$

Proposition 3.4. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold. Then for every integer* $j\ge 1$, (3.3) *has a* C^∞ *solution* $y_j(t)$ *on* $[0,T]$, *and equations* (3.4) *and* (2.15) *have a* C^∞ *solution* z_j *on* $[0,\infty)$ *satisfying*

$$|z_j(\tau)| \le c_j e^{-\beta\tau}, \quad \tau \ge 0, \tag{3.9}$$

for some positive constants c_j and $\beta < \alpha$.

Proof. Consider the hypothesis that there is an integer $N \geq 0$ such that there are C^∞ solutions $y_j(t)$ of (3.3) for $0 \leq j \leq N$ and C^∞ solutions $z_j(\tau)$ for $0 \leq j \leq N$ of (3.4) and (2.15) such that

$$|z_j(\tau)| \leq c_j e^{-\beta\tau}, \quad \tau \geq 0. \tag{3.10}$$

Due to Proposition 3.3 and $(\mathbf{H}_{y_0})$, this hypothesis is true for $N = 0$.

Suppose now this hypothesis is true for $M > 0$. Then $\Phi_{M+1}(t,s)$ and $k_M(t,\sigma)$ are determined and, by (3.5), $\phi_{M+1}(t)$ is a well-defined C^∞ function on $[0,T]$. Assumption $(\mathbf{H}_{y_0})$ implies that $\partial_3 g(t,t,y_0(t)) \neq 0$ for all $0 \leq t \leq T$. Then it makes sense to consider the differentiated version of (3.3), namely,

$$\begin{aligned} y_{M+1}(t) = &-\frac{\phi'_{M+1}(t)}{\partial_3 g(t,t,y_0(t))} \\ &- \frac{1}{\partial_3 g(t,t,y_0(t))}\int_0^t \partial_3\partial_1 g(t,s,y_0(s))y_{M+1}(s)\,ds. \end{aligned} \tag{3.11}$$

This is a linear Volterra integral equation of the second kind in y_{M+1} and has a C^∞ solution on $[0,T]$ which can be written in terms of the resolvent kernel. The theory can be found for example in Gripenberg, London and Staffan [**12**, Chapter 2] or Miller [**26**, Chapter 4]. It follows from (3.11) that

$$\text{constant} = \phi_{M+1}(t) + \int_0^t \partial_3 g(t,s,y_0(s))y_{M+1}(s)\,ds. \tag{3.12}$$

But since $z_M(0) = f_{M+1}(0) - y_M(0)$ and $l_M(0,\sigma) = k_M(0,\sigma)$, (3.4) implies that

$$\begin{aligned} \phi_{M+1}(0) &= f_{M+1}(0) - y_M(0) + \int_0^\infty k_M(0,\sigma)\,d\sigma \\ &= z_M(0) + \int_0^\infty k_M(0,\sigma)\,d\sigma = 0. \end{aligned}$$

Thus the constant in (3.12) vanishes and (3.3) holds in the case $j = M+1$.

Now that $y_{M+1}(t)$ has been found, it follows from (3.6) that $\psi_{M+1}(\tau)$ is a well-defined C^∞ function. An argument like that of O'Malley [**28**, pp. 84–85] shows that

$$|\psi_j(\tau)| \le \gamma_j e^{-\beta\tau}, \quad \tau \ge 0, \tag{3.13}$$

can be deduced from (3.10) for $0 \le j \le M$. The details are omitted. Equation (3.4) is equivalent to the linear scalar equation

$$\begin{aligned} z'_{M+1}(\tau) &= \partial_3 g(0,0,y_0(0)+z_0(\tau))z_{M+1}(\tau) + \psi'_{M+1}(\tau), \\ z_{M+1}(0) &= f_{M+1}(0) - y_M(0). \end{aligned}$$

It easily follows from the exact solution, $(\mathbf{H}_g)$ and (3.13) that (3.10) holds for $J = M+1$. This completes our proof that the induction hypothesis holds for $M+1$. The proposition then follows. $\square$

Lemma 3.5. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold. Then the residual* ρ_N *given by* (2.11) *satisfies*

$$|\rho_N(t;\varepsilon)| = O(\varepsilon^{N+1}) \quad \textit{as } \varepsilon \to 0, \tag{3.14}$$

uniformly for all $0 \le t \le T$. *Moreover,*

$$|\rho'_N(t;\varepsilon)| = O(\varepsilon^{N+1}) \quad \textit{as } \varepsilon \to 0, \tag{3.15}$$

uniformly for all $0 \le t \le T$, *and*

$$|\rho_N(0;\varepsilon)| = O(\varepsilon^{N+2}). \tag{3.16}$$

Proof. Since Propositions 3.3 and 3.4 have established (2.3), the proof of (3.14) follows from Remark 2.4. To prove (3.16)

$$\rho_N(0;\varepsilon) = f(0;\varepsilon) - \varepsilon U_N(0;\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j f_j(0) - \sum_{j=0}^{N} \varepsilon^{j+1}(y_j(0) + z_j(0)).$$

Using the initial conditions in (2.15) and the fact that $f_0(0) = 0$, we have

$$\rho_N(0;\varepsilon) = \sum_{j=N+1}^{\infty} f_{j+1}(0)\varepsilon^{j+1} = O(\varepsilon^{N+2}).$$

Differentiation of (2.11) gives

$$\begin{aligned}\rho_N'(t;\varepsilon) = f'(t;\varepsilon) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^{j} z_j'\left(\frac{t}{\varepsilon}\right) \\ + g\left(t, t, \sum_{j=0}^{N} \varepsilon^{j} \left(y_j(t) + z_j\left(\frac{t}{\varepsilon}\right)\right)\right) \\ + \int_0^t \partial_1 g\left(t, s, \sum_{j=0}^{N} \varepsilon^{j} \left(y_j(s) + z_j\left(\frac{s}{\varepsilon}\right)\right)\right) ds.\end{aligned}$$

Introducing the new notations

$$\begin{aligned}H^*(t,s;\varepsilon) &= \partial_1 g\left(t, s, \sum_{j=0}^{N} \varepsilon^{j} y_j(s)\right),\\ K^*(t,s,\sigma;\varepsilon) &= \partial_1 g\left(t, s, \sum_{j=0}^{N} \varepsilon^{j} (y_j(s) + z_j(\sigma))\right) - \partial_1 g\left(t, s, \sum_{j=0}^{N} \varepsilon^{j} y_j(s)\right),\end{aligned}$$

we have

$$\begin{aligned}\rho_N'(t;\varepsilon) = \sum_{j=0}^{N} \varepsilon^{j} f_j'(t) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^{j} z_j'\left(\frac{t}{\varepsilon}\right) \\ + H(t,t;\varepsilon) + K(t,t,\frac{t}{\varepsilon};\varepsilon) + \int_0^t H^*(t,s;\varepsilon)\, ds \\ + \varepsilon \int_0^{\frac{t}{\varepsilon}} K^*(t, \varepsilon\sigma, \sigma; \varepsilon)\, d\sigma + O(\varepsilon^{N+1}).\end{aligned} \tag{3.17}$$

Two useful Taylor expansions are

$$\begin{aligned}H^*(t,s;\varepsilon) &= \sum_{j=0}^{N} \varepsilon^{j} h_j^*(t,s) + O(\varepsilon^{N+1})\\ K^*(t,\varepsilon\sigma,\sigma;\varepsilon) &= \sum_{j=0}^{N} \varepsilon^{j} k_j^*(t,\sigma) + O(\varepsilon^{N+1}),\end{aligned}$$

where the coefficients satisfy

$$k_j^*(t,\sigma) = \partial_1 k_j(t,\sigma), \qquad h_j^*(t,s) = \partial_1 h_j(t,s).$$

Therefore (3.17) is equivalent to

$$\begin{aligned}
\rho_N'(t;\varepsilon) = {} & \sum_{j=0}^{N} \varepsilon^j f_j'(t) - \sum_{j=0}^{N} \varepsilon^{j+1} y_j'(t) - \sum_{j=0}^{N} \varepsilon^j z_j'\left(\frac{t}{\varepsilon}\right) \\
& + \sum_{j=0}^{N} \varepsilon^j h_j(t,t) + \sum_{j=0}^{N} \varepsilon^j k_j\left(t, \frac{t}{\varepsilon}\right) + \sum_{j=0}^{N} \varepsilon^j \int_0^t h_j^*(t,s)\, ds \\
& + \sum_{j=0}^{N} \varepsilon^{j+1} \int_0^\infty k_j^*(t,\sigma)\, d\sigma - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{\frac{t}{\varepsilon}}^\infty k_j^*(t,\sigma)\, d\sigma \\
& + O(\varepsilon^{N+1}).
\end{aligned}$$

Then, substituting the differentiated version of (2.13), we get

$$\begin{aligned}
\rho_N'(t;\varepsilon) = {} & \varepsilon^{N+1}\left(\int_0^\infty k_N^*(t,\sigma)\, d\sigma - y_N'(t) \right) \\
& - \sum_{j=0}^{N} \varepsilon^j z_j'\left(\frac{t}{\varepsilon}\right) + \sum_{j=0}^{N} \varepsilon^j k_j\left(t, \frac{t}{\varepsilon}\right) \\
& - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{\frac{t}{\varepsilon}}^\infty k_j^*(t,\sigma)\, d\sigma + O(\varepsilon^{N+1}).
\end{aligned} \tag{3.18}$$

By substituting the differentiated version of (2.14), one gets

$$\begin{aligned}
\rho_N'(t;\varepsilon) = {} & \varepsilon^{N+1}\left(\int_0^\infty k_N^*(t,\sigma)\, d\sigma - y_N'(t) \right) + \sum_{j=0}^{N} \varepsilon^j k_j\left(t, \frac{t}{\varepsilon}\right) \\
& - \sum_{j=0}^{N} \varepsilon^j l_j\left(\frac{t}{\varepsilon}, \frac{t}{\varepsilon}\right) + \sum_{j=0}^{N} \varepsilon^j \int_{\frac{t}{\varepsilon}}^\infty \partial_1 l_j\left(\frac{t}{\varepsilon}, \sigma\right) d\sigma \\
& - \sum_{j=0}^{N} \varepsilon^{j+1} \int_{\frac{t}{\varepsilon}}^\infty k_j^*(t,\sigma)\, d\sigma + O(\varepsilon^{N+1}).
\end{aligned}$$

Using Lemma 2.1,

$$\begin{aligned}\rho_N'(t;\varepsilon) &= \varepsilon^{N+1}\left(\int_0^\infty k_N^*(t,\sigma)\,d\sigma - y_N'(t)\right) + \sum_{j=0}^N \varepsilon^j \sum_{i=0}^j \varepsilon^i \tilde{k}_{j,i}\left(\frac{t}{\varepsilon},\frac{t}{\varepsilon}\right)\\ &\quad - \sum_{j=0}^N \varepsilon^j l_j\left(\frac{t}{\varepsilon},\frac{t}{\varepsilon}\right) + \sum_{j=0}^N \varepsilon^j \int_{\frac{t}{\varepsilon}}^\infty \partial_1 l_j\left(\frac{t}{\varepsilon},\sigma\right)d\sigma\\ &\quad - \sum_{j=0}^N \varepsilon^{j+1}\int_{\frac{t}{\varepsilon}}^\infty \sum_{i=0}^j \varepsilon^i \tilde{k}_{j,i}^*\left(\frac{t}{\varepsilon},\sigma\right)d\sigma + O(\varepsilon^{N+1}).\end{aligned}$$

Collecting terms together using (2.10) gives

$$\begin{aligned}\rho_N'(t;\varepsilon) = \varepsilon^{N+1}&\left(\int_0^\infty k_N^*(t,\sigma)\,d\sigma - y_N'(t)\right)\\ &+ \sum_{j=1}^N \varepsilon^j \int_{\frac{t}{\varepsilon}}^\infty \partial_1 l_j\left(\frac{t}{\varepsilon},\sigma\right)d\sigma\\ &- \sum_{j=1}^{N+1} \varepsilon^j \int_{\frac{t}{\varepsilon}}^\infty \sum_{i=0}^{j-1} \tilde{k}_{j-i-1,i}^*\left(\frac{t}{\varepsilon},\sigma\right)d\sigma + O(\varepsilon^{N+1}).\end{aligned}$$

We also see that if

$$K^*(\varepsilon\tau,\varepsilon\sigma,\sigma;\varepsilon) = \sum_{j=0}^\infty \varepsilon^j l_j^*(\tau,\sigma),$$

then

$$\sum_{i=0}^j \tilde{k}_{j-i,i}^*(\tau,\sigma) = l_j^*(\tau,\sigma),$$

where the coefficients obey

$$l_{j-1}^*(\tau,\sigma) = \partial_1 l_j(\tau,\sigma), \quad j \geq 1.$$

Therefore,

$$|\rho_N'(t;\varepsilon)| = O(\varepsilon^{N+1}), \tag{3.19}$$

uniformly for all $0 \le t \le T$. □

4. Existence of asymptotic solution. In this section we establish that $U_N(t;\varepsilon)$ defined in (1.3) is an asymptotic solution. Our method is to adapt the theory in Section 6.3 of Smith [**33**] for systems of singularly perturbed ordinary differential equations. Skinner [**32**] employed a similar method. The analysis here also has benefited from the general discussion in Eckhaus [**8**, Section 6.1] on developing a rigorous theory of singular perturbation. The main result in this paper is the following.

Theorem 4.1. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold. Then* (1.1) *has a continuous solution* $u(t;\varepsilon)$ *with the property that there are constants* C_N *and* ε_N^* *such that*

$$|u(t;\varepsilon) - U_N(t;\varepsilon)| \le C_N \varepsilon^{N+1}$$

for all $0 \le t \le T$ *and* $0 < \varepsilon \le \varepsilon_N^*$.

It is natural to introduce $r_N(t;\varepsilon) = u(t;\varepsilon) - U_N(t;\varepsilon)$ which satisfies the equation

(4.1)

$$\varepsilon r_N(t;\varepsilon) = \rho_N(t;\varepsilon) + \int_0^t [g(t,s,U_N(s;\varepsilon)+r_N(s;\varepsilon)) - g(t,s,U_N(s;\varepsilon))]\, ds.$$

However, if the functions r_N and ρ_N are scaled, a mapping considered later becomes a *uniform* contraction rather than just a contraction. For this reason let

$$\theta(t;\varepsilon) = \varepsilon^{-(N+1)} \rho_N(t;\varepsilon), \quad x(t;\varepsilon) = \varepsilon^{-(N+1)} r_N(t;\varepsilon),$$

where, for simplicity, the dependence on the fixed integer N is omitted from the notation. Then, for $\varepsilon > 0$, (4.1) is equivalent to

(4.2)

$$\varepsilon x(t;\varepsilon) = \theta(t;\varepsilon) + \int_0^t \partial_3 g(t,s,U_N(s,\varepsilon)) x(s;\varepsilon)\, ds + \int_0^t h(t,s,x(s;\varepsilon);\varepsilon)\, ds,$$

where

$$\begin{aligned} h(t,s,x;\varepsilon) := {} & \varepsilon^{-(N+1)} g(t,s,U_N(s;\varepsilon)+x) \\ & - \varepsilon^{-(N+1)} g(t,s,U_N(s;\varepsilon)) \\ & - \partial_3 g(t,s,U_N(s;\varepsilon)) x. \end{aligned}$$

By Taylor's theorem $h(t,s,x;\varepsilon) = \varepsilon^{(N+1)} h_1(t,s,x;\varepsilon)$, where

$$h_1(t,s,x;\varepsilon) = x^2 \int_0^1 (1-v)\partial_3^2 g(t,s,U_N(s;\varepsilon) + v\varepsilon^{(N+1)}x)\, dv.$$

Hence, because $|\theta(t;\varepsilon)| = O(1)$ as $\varepsilon \to 0$ uniformly by Lemma 3.5, we expect the nonlinear term

$$\int_0^t h(t,s,x(s;\varepsilon);\varepsilon)\, ds$$

to be of higher order than other terms in (4.2). Therefore we first consider the approximate equation

$$\varepsilon w(t;\varepsilon) = \xi(t;\varepsilon) + \int_0^t \partial_3 g(t,s,U_N(s;\varepsilon)) w(s;\varepsilon)\, ds, \tag{4.3}$$

where $\xi(t;\varepsilon) = O(1)$ uniformly as $\varepsilon \to 0$ and $\xi(0;\varepsilon) = O(\varepsilon)$.

Lemma 4.2. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold for each* $0 < \varepsilon \le \varepsilon_0$. *Also suppose that* $\xi(\cdot;\varepsilon) : [0,T] \to \mathbf{R}$ *is a continuously differentiable function with* $\|\xi'(\cdot,;\varepsilon)\| = O(1)$ *and* $|\xi(0;\varepsilon)| = O(\varepsilon)$. *Then* (4.3) *has a unique continuous solution* $w(\cdot;\varepsilon)$ *satisfying* $\|w(\cdot;\varepsilon)\| = O(1)$ *for all* ε *in some interval* $(0,\varepsilon_1] \subset (0,\varepsilon_0]$.

Proof. The standard theory of linear Volterra equations of the second kind ensures that for each $0 < \varepsilon < \varepsilon_0$ (4.3) has a continuous solution $t \mapsto w(t;\varepsilon)$ on $[0,T]$ and that $w(\cdot;\varepsilon)$ is continuously differentiable because $\xi(\cdot;\varepsilon)$ is. Let $0 < \beta < \alpha$. It follows from $(\mathbf{H}_g)$that there is a number $0 < \varepsilon_1 \le \varepsilon_0$ such that

$$p(t;\varepsilon) = \partial_3 g(t,t,U_N(t;\varepsilon)) \le -\beta$$

for all $0 \le t \le T$ and $0 \le \varepsilon \le \varepsilon_1$. Equation (4.3) can be differentiated to get an equation of the form

$$\varepsilon w'(t;\varepsilon) - p(t;\varepsilon) w(t;\varepsilon) = \xi_1(t;\varepsilon), \tag{4.4}$$

where $w(0;\varepsilon) = \xi(0;\varepsilon)/\varepsilon$ and

$$\xi_1(t;\varepsilon) = \xi'(t;\varepsilon) + \int_0^t \partial_1\partial_3 g(t,s,U_N(s;\varepsilon)) w(s;\varepsilon)\, ds.$$

Since the solution of (4.4) satisfies

$$w(t;\varepsilon) = w(0;\varepsilon)e^{(1/\varepsilon)\int_0^t p(v;\varepsilon)\,dv} + \frac{1}{\varepsilon}\int_0^t e^{(1/\varepsilon)\int_s^t p(v;\varepsilon)\,dv}\xi_1(s;\varepsilon)\,ds$$

and

$$e^{(1/\varepsilon)\int_0^t p(v;\varepsilon)\,dv} \le e^{-\beta\frac{t}{\varepsilon}}, \quad e^{(1/\varepsilon)\int_s^t p(v;\varepsilon)\,dv} \le e^{-\beta(t-s)/\varepsilon},$$

we see that

$$|w(t;\varepsilon)| \le C_1 + \frac{C_2}{\beta} + \frac{M}{\beta}\int_0^t |w(s;\varepsilon)|\,ds,$$

where

$$\begin{aligned} C_1 &= \sup_{0<\varepsilon\le\varepsilon_0} |\xi(0;\varepsilon)|/\varepsilon,\\ C_2 &= \sup_{0<\varepsilon\le\varepsilon_0} \|\xi'(\cdot;\varepsilon)\|,\\ M &= \sup_{\substack{(t,s)\in\Delta_T\\ 0\le\varepsilon\le\varepsilon_0}} |\partial_1\partial_3 g(t,s,U_N(s;\varepsilon))|. \end{aligned}$$

By Gronwall's inequality

$$|w(t;\varepsilon)| \le \left(C_1 + \frac{C_2}{\beta}e^{\frac{Mt}{\beta}}\right),$$

and the lemma is proved. □

Equation (4.2) can be written as

$$\mathcal{L}(x,\varepsilon) = \theta(\cdot;\varepsilon) + \mathcal{N}(x,\varepsilon), \tag{4.5}$$

where $\mathcal{L},\mathcal{N} : C[0,T]\times[0,\varepsilon_1] \to C[0,T]$ are defined by

$$\begin{aligned} \mathcal{L}(x,\varepsilon)(t) &= \varepsilon x(t) - \int_0^t \partial_3 g(t,s,U_N(s;\varepsilon))x(s)\,ds,\\ \mathcal{N}(x,\varepsilon)(t) &= \varepsilon^{(N+1)}\int_0^t h_1(t,s,x(s);\varepsilon)\,ds. \end{aligned}$$

It is convenient to introduce the space $\mathcal{X}$ of functions $(t,\varepsilon) \mapsto \xi(t;\varepsilon)$ on $[0,T]\times[0,\varepsilon]$ with $t \mapsto \xi(t;\varepsilon)$ continuously differentiable and $\|\xi'(\cdot;\varepsilon)\|$ and $\xi(0;\varepsilon)/\varepsilon$ are uniformly bounded on $[0,\varepsilon_0]$ and $(0,\varepsilon_0]$, respectively. $\mathcal{X}'$ is given the norm

$$\|\xi\|_{\mathcal{X}} = \sup_{0<\varepsilon\leq\varepsilon_1} |\xi(0;\varepsilon)/\varepsilon| + \sup_{0<\varepsilon\leq\varepsilon_1} \|\xi'(\cdot;\varepsilon)\|.$$

Then $(t,\varepsilon) \mapsto \mathcal{L}(x,\varepsilon)(t)$, $(t,\varepsilon) \mapsto \mathcal{N}(x,\varepsilon)(t)$ and $(t,\varepsilon) \mapsto \theta(t;\varepsilon)$ are in $\mathcal{X}$.

Lemma 4.2 can be reinterpreted as asserting that for $\xi \in \mathcal{X}$ the equation $\mathcal{L}(w,\varepsilon) = \xi(\cdot;\varepsilon)$ is equivalent to $w(\cdot;\varepsilon) = \mathcal{M}(\cdot,\varepsilon)\xi(\cdot;\varepsilon)$ for some linear operator $\mathcal{M}(\cdot,\varepsilon) : \mathcal{X} \to C[0,T]$ and there is a constant μ such that $\|\mathcal{M}(\cdot,\varepsilon)\xi(\cdot;\varepsilon)\| \leq \mu\|\xi\|_{\mathcal{X}}$ uniformly for $0 < \varepsilon \leq \varepsilon_1$. Hence there is a number $\delta > 0$ such that

$$\|\mathcal{M}(\cdot,\varepsilon)\theta(\cdot,\varepsilon)\| \leq \delta.$$

Also (4.5) is equivalent to

$$x = \mathcal{M}(\cdot,\varepsilon)[\theta(\cdot;\varepsilon) + \mathcal{N}(x,\varepsilon)].$$

Thus the problem of finding solutions of (4.5) is equivalent to finding fixed points of a mapping. Let

$$\mathcal{B} = \{x \in C[0,T] : \|x\| \leq 2\delta\}.$$

A simple calculation shows that if x is in $\mathcal{B}$, then

$$\|\mathcal{N}(x,\cdot)\|_{\mathcal{X}} \leq \varepsilon^{N+1} T M_1,$$

where

$$M_1 = \max_{\substack{(t,s)\in\Delta_T \\ |x|\leq 2\delta \\ 0\leq\varepsilon\leq\varepsilon_1}} |h_1(t,s,x;\varepsilon)|$$

Therefore, for each x in $\mathcal{B}$,

$$\|\mathcal{M}(\cdot,\varepsilon)[\theta(\cdot;\varepsilon) + \mathcal{N}(x,\varepsilon)]\| \leq \delta + \mu T M_1 \varepsilon^{N+1} \leq 2\delta,$$

if ε is in some interval $(0, \varepsilon_2]$. It has been shown that the mapping $T_\varepsilon : \mathcal{B} \to \mathcal{B}$ given by

$$\mathcal{T}_\varepsilon(x) = \mathcal{M}(\cdot, \varepsilon)[\theta(\cdot; \varepsilon) + \mathcal{N}(x, \varepsilon)]$$

is well defined.

Next it is shown that $\mathcal{T}_\varepsilon$ is a contraction on $\mathcal{B}$. Note that $\mathcal{N}(x, \varepsilon)(0) = 0$. Let x_1, x_2 be in B. Then

$$\begin{aligned}(\mathcal{N}(x_1, \varepsilon)'(t) - \mathcal{N}(x_2; \varepsilon)'(t)) = \varepsilon^{N+1}\bigg[& h_1(t, t, x_1(t); \varepsilon) \\ & - h_1(t, t, x_2(t); \varepsilon) \\ & + \int_0^t \{\partial_1 h_1(t, s, x_1(s); \varepsilon) \\ & - \partial_1 h_1(t, s, x_2(s); \varepsilon)\}\, ds\bigg],\end{aligned}$$

and, using the mean value theorem,

$$\begin{aligned}&|\mathcal{N}(x_1, \varepsilon)'(t) - \mathcal{N}(x_2, \varepsilon)'(t)| \\ &\qquad \le \varepsilon^{N+1}\left\{M_2|x_1(t) - x_2(t)| + M_3 \int_0^t |x_1(s) - x_2(s)|\, ds\right\}\end{aligned}$$

where

$$\begin{aligned}M_2 &= \max_{\substack{0 \le t \le T \\ |x| \le 2\delta \\ 0 \le \varepsilon \le \varepsilon_0}} |\partial_3 h_1(t, t, x; \varepsilon)|, \\ M_3 &= \max_{\substack{(t,s) \in \Delta_T \\ |x| \le 2\delta \\ 0 \le \varepsilon \le \varepsilon_0}} |\partial_3 \partial_1 h_1(t, s, x; \varepsilon)|.\end{aligned}$$

It follows that

$$\|\mathcal{N}(x_1, \varepsilon) - \mathcal{N}(x_2, \varepsilon)\|_{\mathcal{X}} \le \varepsilon^{N+1}(M_2 + M_3 T)\|x_1 - x_2\|,$$

and hence that $\mathcal{T}_\varepsilon : \mathcal{B} \to \mathcal{B}$ is a uniform contraction for ε in some interval $(0, \varepsilon_3]$ with $0 \le \varepsilon_3 \le \varepsilon_2$. The Banach fixed point theorem implies the following result.

Lemma 4.3. *Suppose that* $(\mathbf{H}_f)$, $(\mathbf{H}_{y_0})$ *and* $(\mathbf{H}_g)$ *hold. Then there is a number* $\varepsilon_3 > 0$ *such that* (4.2) *has a unique solution* $x(\varepsilon)$ *in* $\mathcal{B}$ *for all* $0 < \varepsilon \le \varepsilon_3$.

It is easy to show that since $x(\varepsilon)(t) = x(t;\varepsilon)$ satisfies (4.2)

$$u(t;\varepsilon) := U_N(t;\varepsilon) + \varepsilon^{N+1}x(t;\varepsilon)$$

is a solution of (1.1). Moreover,

$$|u(t;\varepsilon) - U_N(t;\varepsilon)| = \varepsilon^{N+1}|x(t;\varepsilon)| \le 2\delta\varepsilon^{N+1}$$

for all $0 \le t \le T$. This completes the proof of Theorem 4.1.

5. Example.

5.1 Example one. Let us consider the following example from Angell and Olmstead [**2**]

$$\varepsilon u(t) = \int_0^t e^{(t-s)}(u^2(s) - 1)\, ds. \tag{5.1}$$

The exact solution of this is determined by converting the integral equation to a nonlinear first order differential equation subject to the initial condition $u(0) = 0$ is

$$u(t;\varepsilon) = \frac{2}{\varepsilon}\frac{1 - e^{\gamma t}}{(\gamma - 1)e^{\gamma t} + \gamma + 1} \tag{5.2}$$

where

$$\gamma = \frac{1}{\varepsilon}\sqrt{4 + \varepsilon^2}. \tag{5.3}$$

Example (5.1) corresponds to

$$f(t;\varepsilon) = 1 - e^t, \quad g(t,s,u) = e^{(t-s)}u^2,$$

which implies $\partial_3 g(t,t,u) = 2u$. It follows from (3.1) that the leading order outer solution satisfies

$$0 = \int_0^t e^{(t-s)}(y_0^2(s) - 1)\, ds$$

which has solutions $y_0(t) = \pm 1$. But only one of these can be appropriate since (5.1) has a unique solution. $(\mathbf{H}_g)$ cannot be satisfied with $y_0(t) = 1$, but with $y_0(t) = -1$ it holds with $\alpha = 2$, since $\partial_3 g(t, t, y_0) = -2$. Therefore

$$y_0(t) = -1, \quad t \geq 0.$$

The leading order inner correction solution is given by the nonlinear ordinary differential equation

$$z_0'(\tau) = z_0^2(\tau) - 2z_0(\tau), \quad z_0(0) = 1,$$

which has a solution

$$z_0(\tau) = 1 - \tanh \tau, \quad \tau \geq 0.$$

We see from this solution that $z_0(\tau)$ satisfies the requirement that

$$\lim_{\tau \to \infty} z_0(\tau) = 0.$$

To the leading order, the asymptotic solution $U_0(t; \varepsilon)$ of (5.1) is given by

$$U_0(t; \varepsilon) = -\tanh \frac{t}{\varepsilon}.$$

In general, for $j \geq 1$, the outer solution satisfies

$$y_{j-1}(t) = -2 \int_0^t e^{t-s} y_j(s)\, ds + \int_0^t \Phi_j(t, s)\, ds + \int_0^\infty k_{j-1}(t, \sigma)\, d\sigma,$$

where $k_{j-1}(t, \sigma)$ and $\Phi_j(t, s)$ are determined by $y_i(t)$ and $z_i(\tau)$ for $i \leq j - 1$. Since

$$\Phi_1(t, s) = 0, \quad k_0(t, \sigma) = -e^t \operatorname{sech}^2 \sigma,$$

it follows that the first order outer solution satisfies the equation

$$2 \int_0^t e^{t-s} y_1(s)\, ds = 1 - e^t.$$

Solving this by differentiating once gives

$$y_1(t) = -\frac{1}{2}, \quad t \geq 0.$$

From (3.4), the inner correction solution in general satisfies

$$z_j'(\tau) = -2\tanh\tau z_j(\tau) + \psi_j'(\tau),$$

where

$$\psi_j(\tau) = -\int_\tau^\infty \Xi_j(\tau,\sigma)$$

is determined by $y_i(t)$ and $z_i(\tau)$ for $i \leq j$, respectively $i \leq j-1$. Then since

$$\Xi_1(\tau,\sigma) = (\sigma-\tau)\operatorname{sech}^2\sigma + \tanh\sigma - 1,$$

the first order inner correction solution $z_1(\tau)$ satisfies

$$z_1'(\tau) = -2\tanh\tau z_1(\tau), \quad z_1(0) = \frac{1}{2}.$$

Solving this gives

$$z_1(\tau) = \frac{1}{2}\operatorname{sech}^2\tau, \quad \tau \geq 0.$$

Then to the first order, the asymptotic solution $U_1(t;\varepsilon)$ is given by

$$U_1(t;\varepsilon) = -\tanh\frac{t}{\varepsilon} - \frac{\varepsilon}{2}\tanh^2\frac{t}{\varepsilon}.$$

To verify that $U_0(t;\varepsilon)$ is a uniformly valid asymptotic solution, we consider the difference

$$\begin{aligned} u(t;\varepsilon) - U_0(t;\varepsilon) &= \frac{2/\varepsilon(1-e^{\gamma^t})}{(\gamma-1)e^{\gamma t}+\gamma+1} + \frac{e^{2\frac{t}{\varepsilon}}-1}{e^{2\frac{t}{\varepsilon}}+1} \\ &= \frac{2/\varepsilon(1-e^{\gamma t})(e^{2\frac{t}{\varepsilon}}+1)}{(\gamma-1)e^{\gamma t}+\gamma+1(e^{2\frac{t}{\varepsilon}}+1)} \\ &\quad + \frac{(e^{2\frac{t}{\varepsilon}}-1)\{(\gamma-1)e^{\gamma t}+\gamma+1\}}{(\gamma-1)e^{\gamma t}+\gamma+1(e^{2\frac{t}{\varepsilon}}+1)}. \end{aligned} \tag{5.4}$$

Simplifying (5.4) gives

$$u(t;\varepsilon)-U_0(t;\varepsilon)=\frac{e^{\gamma t}+e^{2\frac{t}{\varepsilon}}-e^{\gamma t}e^{2\frac{t}{\varepsilon}}-1}{\gamma e^{\gamma t}+\gamma e^{2\frac{t}{\varepsilon}}+(\gamma-1)e^{\gamma t}e^{2\frac{t}{\varepsilon}}+\gamma+1}.$$

We have from (5.3) that

$$\gamma \sim -\frac{2}{3}+O(\varepsilon), \quad \varepsilon\to 0.$$

Therefore

$$u(t;\varepsilon)-U_0(t;\varepsilon)=\frac{2\varepsilon e^{2\frac{t}{\varepsilon}}-\varepsilon e^{4\frac{t}{\varepsilon}}-\varepsilon}{2\varepsilon e^{2\frac{t}{\varepsilon}}+(2-\varepsilon)e^{4\frac{t}{\varepsilon}}+2+\varepsilon}$$

and

$$|u(t;\varepsilon)-U_0(t;\varepsilon)|\le\left|\frac{2\varepsilon e^{-2\frac{t}{\varepsilon}}-\varepsilon e^{-4\frac{t}{\varepsilon}}-\varepsilon}{2\varepsilon e^{-2\frac{t}{\varepsilon}}+2-\varepsilon+(2+\varepsilon)e^{-4\frac{t}{\varepsilon}}}\right|.$$

It therefore follows that, for $0<\varepsilon\le\varepsilon_0$, we have

$$|u(t;\varepsilon)-U_0(t;\varepsilon)|\le\frac{\varepsilon}{2},$$

for all $0\le t\le T$. Similar calculations show that there exists a positive constant $c_1>0$ such that

$$|u(t;\varepsilon)-U_1(t;\varepsilon)|\le c_1e^2,$$

uniformly for all $0\le t\le T$.

5.2 Example two. Consider the following example which follows from the population growth model. The unperturbed model is discussed by Brauner [**6**] and related models are found in Gripenberg et al. [**12**]

$$\varepsilon u(t)=\varepsilon S(t)+\int_0^t S(t-s)u(s)(1-u(s)/c)\,ds, \tag{5.5}$$

where $c>0$ is a constant. Problem (5.5) is a model for the population growth. The function $u(t)$ is the population size at time t. The survival function $S(t)$ is the fraction of the initial population which is still alive at time t, so $S(0) = 1$. $u(1 - u/c)$ is the rate of reproduction. Since ε is small, (5.5) describes a rapidly growing population. A typical survival function considered here will be nonnegative, nonincreasing and differentiable such as e^{-t}.

Problem (5.5) corresponds to

$$f(t;\varepsilon) = \varepsilon S(t), \quad g(t,s,u) = S(t-s)u(1-u/c).$$

The leading order outer solution, $y_0(t)$, is given by

$$0 = \int_0^t S(t-s)y_0(s)(1 - y_0(s)/c)\,ds \tag{5.6}$$

which implies

$$y_0(t) = 0 \quad \text{or} \quad y_0(t) = c. \tag{5.7}$$

To satisfy $(\mathbf{H}_g)$, the correct leading order outer solution is

$$y_0(t) = c,$$

since then $\partial_3 g(t,t,y_0(t)) = -1$. By (3.1) the leading order inner correction solution $z_0(\tau)$ is given by

$$z_0'(\tau) = -z_0(\tau)\left(1 - \frac{1}{c}z_0(\tau)\right), \quad z_0(0) = 1 - c, \tag{5.8}$$

which has solution

$$z_0(\tau) = \frac{c(1-c)e^{-\tau}}{1 + (c-1)e^{-\tau}}. \tag{5.9}$$

This implies that $\lim_{\tau\to\infty} z_0(\tau) = 0$ and thus to the leading order, the asymptotic solution, $U_0(t;\varepsilon)$ of (5.5) is given by

$$U_0(t;\varepsilon) = \frac{c}{1 + (c-1)e^{-\frac{t}{\varepsilon}}}. \tag{5.10}$$

Thus on a time scale of order ε, the population increases rapidly. Since (5.5) and $y_0(t)$ satisfy the hypotheses imposed on this presentation. The unknown exact solution satisfies

$$\left| u(t;\varepsilon) - \frac{c}{1+(c-1)e^{-\frac{t}{\varepsilon}}} \right| = O(\varepsilon) \tag{5.11}$$

uniformly for $0 \le t \le T$.

The higher order outer solutions are given in general for $j \ge 1$ as (5.12)

$$y_{j-1}(t) = f_j(t) - \int_0^t S(t-s) y_j(s)\, ds + \int_0^t \Phi_j(t,s)\, ds + \int_0^\infty k_{j-1}(t,\sigma)\, d\sigma,$$

where $f_j(t) = S(t)$, $j = 1$ and $f_j(t) = 0$, $j \ge 2$. The functions Φ_j and k_{j-1} are determined by $y_i(t)$ and $z_i(\tau)$ for $i \le j-1$. Since $\Phi_1(t,s) = 0$ and

$$k_0(t,\sigma) = S(t)\left\{ \frac{c(c-1)e^{-\sigma}}{(1+(c-1)e^{-\sigma})^2} \right\},$$

it follows that the first order outer solution, $y_1(t)$ is given by

$$c(1 - S(t)) = -\int_0^t S(t-s) y_1(s)\, ds. \tag{5.13}$$

Equation (5.13) has the unique solution which depends on $s(t)$.

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REFERENCES

1. J.S. Angell and W.E. Olmstead, *Singular perturbation analysis of an integrodifferential equation modelling filament stretching*, Z. Angew. Math. Phys. **36** (1985), 487–490.

2. ———, *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47** (1987), 1–14.

3. ———, *Singularly perturbed Volterra integral equations*, SIAM J. Appl. Math. **47** (1987), 1150–1162.

4. ———, *Singularly perturbed integral equations with end point boundary layers*, SIAM J. Appl. Math. **49** (1989), 1567–1584.

5. A.M. Bijura, *Singularly perturbed Volterra integral equations with weakly singular kernels*, IJMMS, IJMMS, **3** (2002), 129–143.

6. F. Brauner, *On a nonlinear integral equation for population growth problems*, SIAM J. Math. Anal., **6** (1975), 312–317.

7. M. Chen, *Singular perturbation analysis of nonlinear Fredholm integral equation with weak singularity*, ZAMP **23** (1972), 266–271.

8. W. Eckhaus, *Asymptotic analysis of singular perturbations*, Studies Math. Appl., New York, 1979.

9. A.K. Gautesen, *On the asymptotic solution to a class of linear integral equations*, SIAM J. Appl. Math. **48** (1988), 294–306.

10. ———, *Asymptotic solutions to some two-dimensional singularly perturbed Fredholm integral equations*, SIAM J. Appl. Math. **50** (1990), 1507–1515.

11. M. Georgiou and W.E. Olmstead, *Some Fredholm equations with boundary layer resonance*, Z. Angew. Math. Phys. **42** (1991), 460–467.

12. S.-O. Gripenberg, G. Londen and O. Staffans, *Volterra integral and functional equations*, in *Encyclopedia of mathematics and its application*, Cambridge University Press, Cambridge, 1990.

13. W.M. Hirsch and S. Smale, *Differential equations, dynamical systems, and linear algebra*, Academic Press, New York, 1974.

14. V. Horvat and M. Rogina, *Tension spline collocation methods for singularly perturbed Volterra integro-differential and Volterra integral equations*, J. Comp. Appl. Math., to appear.

15. J.P. Kauthen, *Implicit Runge-Kutta methods for some integrodifferential-algebraic equations*, Appl. Numer. Math. **13** (1993), 125–134.

16. ———, *Implicit Runge-Kutta methods for singularly perturbed integro-differential systems*, Appl. Numer. Math. **18** (1995), 201–210.

17. ———, *A survey on singularly perturbed Volterra equations*, Appl. Numer. Math. **24** (1997), 95–114.

18. C.G. Lange, *On spurious solutions of singular perturbation problems*, Stud. Appl. Math. **68** (1983), 227–257.

19. C.G. Lange and D.R. Smith, *Singular perturbation analysis of integral equations*, Stud. Appl. Math. **79** (1988), 1–63.

20. ———, *Singular perturbation analysis of integral equations*, Stud. Appl. Math. **90** (1993), 1–74.

21. P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM, Philadelphia, 1985.

22. J.H. Liu, *A singular perturbation problem in integrodifferential equations*, Electron. J. Differential Equations **2** (1993), 1–10.

23. ———, *Singular perturbations of integrodifferential equations in Banach space*, Proc. Amer. Math. Soc. **122** (1994), 791–799.

24. S.A. Lomov, *Introduction to the general theory of singular perturbations*, Amer. Math. Soc., Providence, RI, 1992.

25. N. Luqun and S. Nemat-Nasser, *Singular perturbation solutions of a class of systems of singular integral equations*, SIAM J. Appl. Math. **61** (2001), 1219–1236.

26. R.K. Miller, *Nonlinear Volterra integral equations*, W.A. Benjamin, California, 1971.

27. W.E. Olmstead and A.K. Gautesen, *Asymptotic solution of some singularly perturbed Fredholm integral equations*, Z. Angew. Math. Phys. **40** (1989), 230–244.

28. R.E. O'Malley, *Introduction to singular perturbation*, Academic Press, New York, 1974.

29. ———, *Singular perturbation methods for ordinary differential equations*, Springer-Verlag, New York, 1991.

30. A.G. Ramm and E.I. Shifrin, *Asymptotics of the solution to a singularly perturbed integral equation*, Appl. Math. Lett. **4** (1991), 67–70.

31. ———, *Asymptotics of the solutions to singularly perturbed multidimensional integral equations*, J. Math. Anal. Appl. **190** (1995), 667–677.

32. L.A. Skinner, *Asymptotic solution to a class of singularly perturbed Volterra integral equations*, Methods Appl. Anal. **2** (1995), 212–221.

33. D.R. Smith, *Singular perturbation theory: An introduction with applications*, Cambridge University Press, Cambridge, New York, 1985.

34. ———, *Linear state regulator and integral equations not reducible to differential equations*, Meth. Appl. Anal. **1** (1994), 415–420.

35. J.R. Willis and S. Nemat-Nasser, *Singular perturbation solution of a class of singular integral equations*, Quart. Appl. Math. **48** (1990), 741–753.

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