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CONDITIONAL STABILIZING ESTIMATION FOR AN INTEGRAL EQUATION OF FIRST KIND WITH ANALYTIC KERNEL

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Dedicated to the memory of
Professor Dr. rer. nat. habil. Siegfried Prößdorf

ABSTRACT. We discuss an integral equation of first kind which arises from an inverse problem for detecting steel reinforcement bars. Since the kernel of this integral equation is analytic, our problem is severely ill-posed. We transform the problem for this integral equation into a Cauchy problem for Laplace's equation. By using the conditional estimation of the Cauchy problem for the elliptic operator, we prove that, under suitable hypotheses, a logarithmic stabilizing estimation holds for the solution of the integral equation. In addition, this method can be used to determine discontinuous points of the solution of the integral equation.

1. Introduction. In [**3**], Engl and Isakov discuss an inverse problem for detecting steel reinforcement bars inside of concrete. Under suitable hypotheses, they transform the problem into a first kind integral equation with analytic kernel and prove the uniqueness of the solution of the integral equation. From the theory of integral operators, we know this integral equation is a severely ill-posed problem, since the singular values of the integral operator decrease rapidly, e.g., [**5, 13**]. This kind of integral equation is also proposed in geophysics, e.g., Ramm [**9, 10**]. The same integral equation is discussed by Lavrent'ev [**6**], Bukhgeim [**1**] and Serikbaev [**11**].

In this paper, we transform the integral equation into a Cauchy problem for Laplace's equation, which has been studied extensively.

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The uniqueness in the integral equation is derived directly from the uniqueness in the solution of a Cauchy problem for Laplace's equation. Next we discuss stability in solving the integral equation. For this, our estimation requires some additional hypotheses such as that the solution of the integral equation should have some extra smoothness. Otherwise, the estimation will fail. In [**11**], under suitable hypotheses, an estimate for a similar integral equation is obtained. By using Payne's results on the conditional stabilizing estimation of a Cauchy problem for an elliptic equation, a logarithmic conditional estimate is obtained. The key of our proof is to extend the conditional estimation in a Cauchy problem for Laplace's equation up to the boundary. Our method can be used to distinguish the discontinuous points of the solution of the integral equation, see Remark 4.2, and also for the original inverse problem. This research is in progress.

This paper is organized as follows. In Section 2 we will introduce some notations and pose the problem in a simplified manner. In Section 3 we transform the integral equation into a Cauchy problem for Laplace's equation by adding one variable. Then our main results are stated as Theorems 3.1–3.3 and proved. In the last section, we prove a conditional stabilizing estimate for the original problem in [**3**] by means of Theorem 3.2 and we also give some remarks on our method.

2. Notations and problems. Let D_1 and D be bounded domains in R^3 and $x = (x_1, x_2, x_3) \in R^3$. Without loss of generality for our problem, we can assume that D and D_1 are simply connected domains in R^3 and D is compactly contained in the ball $B_R = \{x \in R^3 | \quad |x| < R\}$. Throughout this paper, we assume that $D_1 \cap B_R = \varnothing$. $W^{n,p}(\Omega)$ and $W_0^{n,p}(\Omega)$ are the usual Sobolev spaces, $L^p = W^{0,p}$.

In [**3**], the problem of identifying steel reinforcement bars in concrete is transformed into the following first kind integral equation with analytic kernel:

$$\int_D k(x-y)\mu(y)\,dy = f(x), \quad x \in D_1$$

where the kernel is

$$k(x) = \frac{3}{16\pi^2}\left(\frac{x_2}{|x|^8} + 4\frac{x_1^2 x_2}{|x|^{10}}\right), \quad x \in R^3 \setminus \{0\}.$$

By simple calculation, the kernel function k can be written in the form of

$$k(x) = L_x\left(\frac{1}{|x|^2}\right), \quad \text{for} \quad x \in R^3 \setminus \{0\}$$

where $x = (x_1, x_2, x_3)$, $|x|^2 = x_1^2 + x_2^2 + x_3^2$ and L_x is a partial differential operator defined by

$$L_x = -\frac{1}{256\pi^2}\frac{\partial}{\partial x_2}\Delta_x^2 - \frac{1}{512\pi^2}\frac{\partial}{\partial x_2}\frac{\partial^2}{\partial x_1^2}\Delta_x$$

where Δ_x is the Laplace operator with respect to x.

Since $D \cap D_1 = \varnothing$, the above integral equation can be rewritten as

$$L_x \int_D \frac{1}{r_{xy}^2}\mu(y)\,dy = f(x), \quad x \in D_1$$

where $r_{xy} = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$.

The original integral equation is then reduced to

$$\int_D \frac{1}{r_{xy}^2}\mu(y)\,dy = f(x), \quad x \in D_1. \tag{2.1}$$

On the basis of the results for (2.1), in Section 4 we will study the original integral equation.

Our present problem is then: given $f(x)$ defined on D_1, can we get a conditional estimate for the solution of the equation (2.1)?

Since $D \cap D_1 = \varnothing$, the kernel $(1/r_{xy}^2)$ is an analytic function with respect to x and y. From the theory of ill-posed problems, see [**5, 13**], our problem is severely ill-posed and its numerical treatment is difficult.

3. Results and proofs. First we transform our problem into a Cauchy problem for Laplace's equation in a four-dimensional space.

We define a new function in the four-dimensional space $R^3 \times R$ by

$$G(x, \xi) = \int_D \frac{1}{r_{xy}^2 + \xi^2}\mu(y)\,dy, \quad (x, \xi) \in R^3 \times R. \tag{3.1}$$

It is easy to verify that the kernel function $g(x, y, \xi) = 1/(r_{xy}^2 + \xi^2)$ with respect to $(x, \xi) \in R^4$ satisfies the following equation and boundary conditions:

$$\left(\Delta_x + \frac{\partial^2}{\partial \xi^2}\right) g(x, y, \xi) = 0, \quad R^4 \setminus \{x = y, \xi = 0\}$$

$$g(x, y, 0) = \frac{1}{r_{xy}^2}, \quad x \neq y$$

$$\frac{\partial g}{\partial \xi}(x, y, 0) = 0, \quad x \neq y$$

where $\Delta_x = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$.

Henceforth, let $\widehat{\Omega} = R^4 \setminus \{D \times \{\xi = 0\}\}$. Then by (2.1), the above properties of $g(x, y, \xi)$ and $D \cap D_1 = \varnothing$, we find the function $G(x, \xi)$ satisfies the following Laplace's equation and Cauchy boundary conditions

$$\left(\Delta_x + \frac{\partial^2}{\partial \xi^2}\right) G(x, \xi) = 0, \quad (x, \xi) \in \widehat{\Omega} \tag{3.2}$$

$$G(x, 0) = \int_D \frac{1}{r_{xy}^2} \mu(y)\, dy \equiv f(x), \quad x \in D_1 \tag{3.3}$$

$$\frac{\partial G}{\partial \xi}(x, 0) = 0, \quad x \in D_1. \tag{3.4}$$

Notice that the boundary value (3.3) is nothing but the right side of our integral equation (2.1).

Next we will establish the relation between the function $G(x, \xi)$ and the solution $\mu(y)$ of the integral equation.

Lemma 3.1. *For $x \in D$ and $\mu \in L^p(D)$, $p > 1$, we have, for $\xi > 0$,*

$$\frac{\partial G(\cdot, \xi)}{\partial \xi} \longrightarrow -\omega_4 \mu(\cdot) \quad \text{in} \quad L^p(D) \quad \text{as} \quad \xi \to +0 \tag{3.5}$$

where ω_4 is the area of the unit sphere in R^4.

Proof. Since $G(x, \xi)$ is the single layer potential for Laplace's operator on the boundary $D \times \{\xi = 0\}$ in R^4, the conclusion (3.5) follows straightforwardly from the potential theory, e.g., [**2, 5**] for a Hölder continuous function μ. For $\mu \in L^p(D)$, we can derive (3.5) by L^p boundedness of the double layer potential operator, e.g., [**12**] and density of the space of Hölder continuous functions in $L^p(D)$. $\square$

On the basis of the above result, our problem for the integral equation can be treated as a Cauchy problem (3.2)–(3.4) for Laplace's equation. So our problem can be formulated as

Problem. *Given a function $f(x)$ defined on D_1, find a harmonic function $G(x,\xi)$ which satisfies the equations* (3.2)–(3.4). *Then by* (3.5)*, the solution of the original integral equation can be found from* $\lim_{\xi\to+0}(\partial G(x,\xi)/\partial\xi)$*, $x \in D$.*

Therefore we can directly prove the uniqueness for the integral equation (2.1).

Theorem 3.1. *There is at most one solution $\mu(x)$ in $L^p(D)$, $p > 1$, to the equation* (2.1).

Proof. Since the equation is a linear equation, when proving the uniqueness, we can assume $f(x) = 0$. By using the uniqueness in the Cauchy problem (3.2)–(3.4), e.g., [**7**], we know that $G(x,\xi) = 0$, $\xi > 0$. From Lemma 3.1, we obtain $\mu(x) = 0$. $\square$

Next we give some results on a Cauchy problem for Laplace's equation which will be used for our estimates below.

Lemma 3.2. *Let Ω be an n-dimensional domain which is bounded by a closed surface S, and Σ a part of S, $W(z)$ a function defined in Ω which satisfies*

$$\Delta W(z) = 0, \quad z \in \Omega$$

and

$$|W(z)| \le M_1, \quad z \in \Omega$$

with a constant $M_1 > 0$.

Then, for a given point $\hat{z}$ *inside* Ω, *the following inequality holds:*

$$\mathrm{Max}\,\{|W(\hat{z})|, |\nabla W(\hat{z})|\} \le K_0 M_1^{2(1-\delta)} [\varepsilon_1 + \varepsilon_2]^{\delta}$$

where $\delta \in (0,1)$ *and* K_0 *are constants which depend on* Σ *and* $d(\hat{z}, S)$, *the distance between* $\hat{z}$ *and* S, *and*

$$\varepsilon_1 = \int_\Sigma W^2\, d\sigma,$$

$$\varepsilon_2 = \int_\Sigma \left(\frac{\partial W}{\partial z_1}\frac{\partial W}{\partial z_1} + \frac{\partial W}{\partial z_2}\frac{\partial W}{\partial z_2} + \frac{\partial W}{\partial z_3}\frac{\partial W}{\partial z_3} \right) d\sigma = \int_\Sigma |\nabla W|^2\, d\sigma.$$

The proof can be found in Payne [**7**, p. 37].

It should be remarked that, if $d(\hat{z}, S)$ tends to zero, then δ will tend to zero and the constant K_0 may be unbounded. This result is not enough for our use, because the solution $\mu(x)$ is the boundary value of the ξ-derivative of the harmonic function $G(x,\xi)$. So we need to obtain an estimate up to the boundary for the Cauchy problem.

Remark 3.1. The same estimation holds for $\partial^2 W(\hat{z})/\partial z_i \partial z_j$ and higher derivatives of W at $\hat{z}$, [**7**, p. 43].

Henceforth we assume

$$\mu \in L^1(D), \quad \|\mu\|_{L^1(D)} \le M. \tag{3.6}$$

Here $M > 0$ is an a priori given constant.

We apply Lemma 3.2 to get estimates for the functions $(\partial G/\partial \xi)(x,\xi)$, $(\partial^2 G/\partial \xi^2)(x,\xi)$ and $(\partial^3 G/\partial \xi^3)(x,\xi)$.

Lemma 3.3. *Let*

$$g_1(x) = \frac{\partial G}{\partial \xi}(x,1), \quad x \in B_R$$

$$g_2(x) = \frac{\partial^2 G}{\partial \xi^2}(x,1), \quad x \in B_R$$

$$g_3(x,\xi) = \frac{\partial G}{\partial \xi}(x,\xi), \quad (x,\xi) \in \partial B_R \times \{0 \le \xi \le 1\}.$$

Then we have

$$\|g_j\|_{L^\infty(B_R)}, \|g_3\|_{C^3(\partial B_R\times[0,1])} \le M_2\left(\int_{D_1} (|f(x)|^2 + |\nabla f(x)|^2)\,dx\right)^{\delta}.$$

where $j = 1, 2$; $M_2 > 0$ *and* $\delta \in (0, 1)$ *are constants which depend on* D_1 *and* $d(\partial B_R, D)$*, the distance between* ∂B_R *and* D. (*Since* D *is compactly contained in* B_R*,* $d(\partial B_R, D)$ *is a positive constant.*)

Lemma 3.4. *Let*

$$\begin{aligned} h_1(x) &= \frac{\partial^2 G}{\partial \xi^2}(x,1), \quad x \in B_R \\ h_2(x) &= \frac{\partial^3 G}{\partial \xi^3}(x,1), \quad x \in B_R \\ h_3(x,\xi) &= \frac{\partial^2 G}{\partial \xi^2}(x,\xi), \quad (x,\xi) \in \partial B_R \times \{0 \le \xi \le 1\}. \end{aligned}$$

Then we have

$$\|h_j\|_{L^\infty(B_R)}, \|h_3\|_{C^3(\partial B_R\times[0,1])} \le M_3\left(\int_{D_1} (|f(x)|^2 + |\nabla f(x)|^2)\,dx\right)^{\delta_1}$$

where $j = 1, 2$; $M_3 > 0$ *and* $\delta_1 \in (0, 1)$ *are constants which depend on* D_1 *and* $d(\partial B_R, D)$.

Remark 3.2. In Lemmas 3.3 and 3.4, the functions $g_2(x)$ and $h_1(x)$ are the same. For our convenience, we use the different notations.

For proofs of these lemmas, we can first verify that

$$|G(x,\xi)| \le M_4\{d((x,\xi), D\times\{\xi = 0\})\}^{-2}, \quad (x,\xi) \in \widehat{\Omega}$$

with some constant $M_4 > 0$. Here $d((x,\xi), D\times\{\xi = 0\})$ is the distance from (x,ξ) to $D\times\{\xi = 0\}$ and $(x,\xi) \in \widehat{\Omega}$ implies that $d((x,\xi), D\times\{\xi = 0\}) > 0$. In fact, this can be proved directly from the expression of $G(x,\xi)$. Therefore we can apply Lemma 3.2 and obtain Lemmas 3.3 and 3.4.

Next we get an estimate for the Cauchy problem of Laplace's equation. In this estimation, we can explicitly express the constants which depend on the distance between the points and the boundary. We set $\Gamma = \partial B_R \times \{0 \le \xi \le 1\}$.

Lemma 3.5. *Consider the Cauchy problem for Laplace's equation in the domain* $B_R \times \{0 < \xi < 1\}$:

$$\begin{aligned} \Delta w(x,\xi) &= 0 \quad \text{in} \quad B_R \times \{0 < \xi < 1\} \\ w(x,\xi) &= a_3(x,\xi), \quad (x,\xi) \in \Gamma \\ w(x,1) &= a_1(x), \quad x \in B_R \\ \frac{\partial w}{\partial \xi}(x,1) &= a_2(x), \quad x \in B_R. \end{aligned}$$

If $w \in C^3(\overline{B_R} \times (0,1])$, $\|w(\cdot,0)\|_{L^2(B_R)} \le C_1$, $\|a_i\|_{L^2(B_R)} \le C_2$, $i = 1,2$, *and* $\|a_3\|_{C^3(\Gamma)} \le C_3$ *with constants* C_1, C_2 *and* C_3, *then we have*

$$\|w(\cdot,\xi)\|_{L^2(B_R)} \le C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)})^\xi$$

where the constant C *depends on* C_i, $1 \le i \le 3$, *but not on* ξ.

Proof. Let Ξ be a bounded domain in R^4 with C^3-boundary such that $\partial B_R \times \{0 \le \xi \le 1\} \subset \partial \Xi$. Since $a_3 \in C^3(\Gamma)$, by results on unique solvability of the boundary value problems for elliptic equations and the maximum principle, e.g., [**4, 8**], we can construct a harmonic function $v_0(x,\xi)$ in Ξ such that

$$v_0(x,\xi) = a_3(x,\xi), \quad x \in \Gamma \tag{3.7}$$

and

$$\|v_0\|_{C^1(\overline{\Xi})} \le \widetilde{C}\|a_3\|_{C^3(\Gamma)} \tag{3.8}$$

where $\widetilde{C} > 0$ is a constant which depends on Ξ.

Let $w_1(x,\xi) = w(x,\xi) - v_0(x,\xi)$. We see that

$$\begin{aligned} \Delta w_1(x,\xi) &= 0, \quad x \in B_R \times \{0 < \xi < 1\} \\ w_1(x,\xi) &= 0, \quad x \in \partial B_R \times \{0 \le \xi \le 1\} \end{aligned}$$

$$w_1(x,1) = a_1(x) - v_0(x,1),$$
$$\frac{\partial w_1}{\partial \xi}(x,1) = a_2(x) - \frac{\partial v_0}{\partial \xi}(x,1), \quad x \in B_R.$$

Let us set $\hat{a}_1(x) = a_1(x) - v_0(x,1)$, $\hat{a}_2(x) = a_2(x) - (\partial v_0/\partial \xi)(x,1)$, $x \in B_R$. We then have

$$\|\hat{a}_1\|_{L^2(B_R)} \le \|a_1\|_{L^2(B_R)} + C_4\|a_3\|_{C^3(\Gamma)} \tag{3.9}$$
$$\|\hat{a}_2\|_{L^2(B_R)} \le \|a_2\|_{L^2(B_R)} + C_4\|a_3\|_{C^3(\Gamma)}. \tag{3.10}$$

For $\alpha \in H^2(B_R)$, we define a function $\Psi(\xi)$ by

$$\Psi(\xi) = \int_{B_R} \alpha(x) w_1(x,\xi)\, dx, \quad 0 < \xi \le 1. \tag{3.11}$$

Since w_1 is a harmonic function in $B_R \times \{0 < \xi \le 1\}$ and vanishes on $\partial B_R \times \{0 \le \xi \le 1\}$, by Green's formula $\Psi(\xi)$ satisfies

$$\begin{aligned} \frac{d^2\Psi(\xi)}{d\xi^2} = &-\int_{B_R} \Delta_x \alpha(x) w_1(x,\xi)\, dx \\ &- \int_{\partial B_R} \frac{\partial w_1(x,\xi)}{\partial n} \alpha(x)\, dx, \quad 0 < \xi \le 1. \end{aligned} \tag{3.12}$$

Let us choose $\alpha = \alpha(x)$ such that

$$\Delta_x \alpha(x) = -\lambda \alpha(x), \quad x \in B_R \tag{3.13}$$
$$\alpha(x) = 0, \quad x \in \partial B_R \tag{3.14}$$
$$\|\alpha\|_{L^2(B_R)} = 1 \tag{3.15}$$

where $\Delta_x = (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) + (\partial^2/\partial x_3^2)$.

Then the equation (3.12) can be transformed to

$$\frac{d^2\Psi(\xi)}{d\xi^2} = \lambda \Psi(\xi), \quad 0 < \xi \le 1. \tag{3.16}$$

Moreover the initial conditions for $\Psi(\xi)$ are

$$\Psi(1) = \int_{B_R} \alpha(x) \hat{a}_1(x)\, dx, \tag{3.17}$$
$$\frac{d\Psi}{d\xi}(1) = \int_{B_R} \alpha(x) \hat{a}_2(x)\, dx. \tag{3.18}$$

The problem (3.13)–(3.15) is an eigenvalue problem for Laplace's equation on the ball B_R. We enumerate the eigenvalues repeatedly according to their multiplicities:

$$0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$$

and denote the solution of (3.13)–(3.15) with $\lambda = \lambda_n$ by α_n, $n \ge 0$.

The solution of the equation (3.16) with initial conditions (3.17) and (3.18) can be written as

$$\Psi(\xi) = c\exp(\sqrt{\lambda}(\xi - 1)) + d\exp(-\sqrt{\lambda}(\xi - 1)) \tag{3.19}$$

where

$$\begin{aligned} c &= \frac{1}{2}\left(\int_{B_R} \alpha(x)\hat{a}_1(x)\,dx + \frac{\int_{B_R} \alpha(x)\hat{a}_2(x)\,dx}{\sqrt{\lambda}}\right) \\ d &= \frac{1}{2}\left(\int_{B_R} \alpha(x)\hat{a}_1(x)\,dx - \frac{\int_{B_R} \alpha(x)\hat{a}_2(x)\,dx}{\sqrt{\lambda}}\right). \end{aligned}$$

For $\lambda = \lambda_n$, we denote the function $\Psi(\xi)$ by $\Psi_n(\xi)$ and c, d by c_n, d_n.

We notice that $\{\alpha_n(x) | n \ge 0\}$ is complete in $L^2(B_R)$ and $\Psi_n(\xi)$ is the Fourier coefficient by (3.11), and for fixed $\xi \in (0, 1]$, we have

$$w_1(\cdot, \xi) = \sum_{n=0}^{\infty} (w_1(\cdot, \xi), \alpha_n)_{L^2(B_R)} \alpha_n = \sum_{n=0}^{\infty} \Psi_n(\xi)\alpha_n \quad \text{in} \quad L^2(B_R).$$

For $\xi \in (0, 1]$, we write $\Psi_n(\xi)$ as

$$\begin{aligned} \Psi_n(\xi) &= c_n \exp(\sqrt{\lambda_n}(\xi - 1)) + d_n \exp(-\sqrt{\lambda_n}(\xi - 1)) \\ &\equiv \Psi_n^{(1)}(\xi) + \Psi_n^{(2)}(\xi) \end{aligned}$$

and we set

$$\begin{aligned} w_1^{(1)}(x, \xi) &= \sum_{n=0}^{\infty} \Psi_n^{(1)}(\xi)\alpha_n(x) \\ &= \sum_{n=0}^{\infty} c_n e^{\sqrt{\lambda_n}(\xi - 1)} \alpha_n(x) \end{aligned}$$

and

$$\begin{aligned} w_1^{(2)}(x,\xi) &= \sum_{n=0}^{\infty} \Psi_n^{(2)}(\xi)\alpha_n(x) \\ &= \sum_{n=0}^{\infty} d_n e^{\sqrt{\lambda_n}(1-\xi)}\alpha_n(x). \end{aligned}$$

Then w_1 can be written as

$$w_1(x,\xi) = w_1^{(1)}(x,\xi) + w_1^{(2)}(x,\xi).$$

We first estimate $w_1^{(2)}$. By the Parseval equality, we have

$$\begin{aligned} \|w_1^{(2)}(\cdot,\xi)\|_{L^2(B_R)}^2 &= \sum_{n=0}^{\infty} |\Psi_n^{(2)}(\xi)|^2 \\ &= \sum_{n=0}^{\infty} |d_n|^2 \exp(-2\sqrt{\lambda_n}(\xi-1)) \\ &= \sum_{n=0}^{\infty} |d_n|^{2\xi}(d_n \exp(\sqrt{\lambda_n}))^{2(1-\xi)}. \end{aligned}$$

The Hölder inequality yields

$$\|w_1^{(2)}(\cdot,\xi)\|_{L^2(B_R)}^2 \le \left(\sum_{n=0}^{\infty} d_n^2\right)^{\xi}\left(\sum_{n=0}^{\infty}(d_n \exp(\sqrt{\lambda_n}))^2\right)^{1-\xi}, \quad 0 \le \xi \le 1.$$

We rewrite this as

$$\|w_1^{(2)}(\cdot,\xi)\|_{L^2(B_R)} \le \|w_1^{(2)}(\cdot,1)\|_{L^2(B_R)}^{\xi}\|w_1^{(2)}(\cdot,0)\|_{L^2(B_R)}^{1-\xi}, \quad 0 \le \xi \le 1. \tag{3.20}$$

On the other hand, from the expressions for c_n and d_n, noting (3.9) and (3.10), we obtain

$$\begin{aligned} \|w_1^{(2)}(\cdot,1)\|_{L^2(B_R)}^2 &= \sum_{n=0}^{\infty} d_n^2 \le C(\|\hat{a}_1\|_{L^2(B_R)} + \|\hat{a}_2\|_{L^2(B_R)}) \\ &\le C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)})^2 \qquad (3.21) \\ \|w_1^{(1)}(\cdot,1)\|_{L^2(B_R)}^2 &= \sum_{n=0}^{\infty} c_n^2 \le C(\|\hat{a}_1\|_{L^2(B_R)} + \|\hat{a}_2\|_{L^2(B_R)}) \\ &\le C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)})^2. \qquad (3.22) \end{aligned}$$

Here and henceforth $C > 0$ denotes a generic constant which is independent of ξ.

Therefore, since $\xi - 1 \le 0$, for $w_1^{(1)}$, by (3.22) we have

$$\begin{aligned} \|w_1^{(1)}(\cdot,\xi)\|_{L^2(B_R)} &\le \|w_1^{(1)}(\cdot,1)\|_{L^2(B_R)} \\ &\le C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)}). \end{aligned} \tag{3.23}$$

For $\|w_1^{(2)}(\cdot,0)\|_{L^2(B_R)}$, using the assumption $\|w(\cdot,0)\|_{L^2(B_R)} \le C_1$, and by (3.8) and (3.22), we have

$$\begin{aligned} \|w_1^{(2)}(\cdot,0)\|_{L^2(B_R)} &\le \|w_1(\cdot,0)\|_{L^2(B_R)} + \|w_1^{(1)}(\cdot,0)\|_{L^2(B_R)} \\ &\le \|w(\cdot,0)\|_{L^2(B_R)} + \|v_0(\cdot,0)\|_{L^2(B_R)} + \|w_1^{(1)}(\cdot,1)\|_{L^2(B_R)} \\ &\le C_1 + 2\|a_3\|_{C^3(\Gamma)} \\ &\quad + C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)}) \equiv M_5. \end{aligned} \tag{3.24}$$

Therefore, by (3.20), (3.21) and (3.24), we have

$$\begin{aligned} \|w_1^{(2)}(\cdot,\xi)\|_{L^2(B_R)} &\le \|w_1^{(2)}(\cdot,1)\|^{\xi}_{L^2(B_R)} \|w_1^{(2)}(\cdot,0)\|^{1-\xi}_{L^2(B_R)} \\ &\le M_5(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^3(\Gamma)})^{\xi}. \end{aligned} \tag{3.25}$$

By (3.23) and (3.25), we have the conclusion

$$\begin{aligned} \|w_1(\cdot,\xi)\|_{L^2(B_R)} &\le \|w_1^{(1)}(\cdot,\xi)\|_{L^2(B_R)} + \|w_1^{(2)}(\cdot,\xi)\|_{L^2(B_R)} \\ &\le C(\|a_1\|_{L^2(B_R)} + \|a_2\|_{L^2(B_R)} + \|a_3\|_{C^2(\Gamma)})^{\xi}. \end{aligned}$$

By recalling $w = w_1 + v_0$ and (3.8), the proof is complete. $\square$

Applying Lemma 3.5, we can get estimates for $\partial G(x,\xi)/\partial\xi$ and $\partial^2 G(x,\xi)/\partial\xi^2$.

Lemma 3.6. *Let $q > 3$. If $\mu(x)$ is in $W_0^{2,q}(D)$, then*

$$\sup_{0<\xi\le 1} \left\| \frac{\partial^2 G(\cdot,\xi)}{\partial \xi^2} \right\|_{L^\infty(B_R)} \le M_6 \|\Delta\mu\|_{L^q(B_R)}. \tag{3.26}$$

Here the constant M_6 depends on q, but does not depend on μ. Here and henceforth for $\mu \in W_0^{2,q}(D)$, by taking the 0-extension of μ to B_R, we assume $\mu \in W_0^{2,q}(B_R)$.

Proof. Let p satisfy $(1/p) + (1/q) = 1$. Then $1 \le p < 3/2$. We recall that

$$g(x, y, \xi) = (|x - y|^2 + \xi^2)^{-1}.$$

We will show that there exists $M_7 > 0$ such that

$$\|g(x, \cdot, \xi)\|_{L^p(B_R)} \le M_7, \quad x \in D_1, \xi \in (0, 1).$$

In fact, since $1 \le p < 3/2$, we obtain

$$\begin{aligned}
\int_{|y|\le R} \left(\frac{1}{|x-y|^2+\xi^2}\right)^p dy &= \int_{|x+\xi z|\le R} \left(\frac{\xi^3}{\xi^{2p}(|z|^2+1)^p}\right) dz \\
&\le \xi^{3-2p} \int_{|z|\le (R+|x|)/\xi} \frac{1}{(|z|^2+1)^p}\, dz \\
&= C_5 \xi^{3-2p} \int_0^{(R+|x|)/\xi} \frac{r^2 dr}{(r^2+1)^p} \\
&\le C_5 \xi^{3-2p} \int_0^1 \frac{r^2 dr}{(r^2+1)^p} \\
&\quad + C_5 \xi^{3-2p} \int_1^{(R+|x|)/\xi} \frac{r^2 dr}{(r^2+1)^p}.
\end{aligned}$$

The first term is bounded, since $3 - 2p \ge 0$ and ξ is bounded. By $3 - 2p > 0$, the second term can be estimated as

$$\begin{aligned}
\xi^{3-2p} \int_1^{(R+|x|)/\xi} \frac{r^2 dr}{(r^2+1)^p} &\le \xi^{3-2p} \int_1^{(R+|x|)/\xi} r^{2-2p}\, dr \\
&= \xi^{3-2p} \frac{1}{3-2p}((R+|x|)^{3-2p}\xi^{2p-3} - 1) \\
&= \frac{1}{3-2p}((R+|x|)^{3-2p} - \xi^{3-2p}) < \infty.
\end{aligned}$$

In addition, for $\xi \geq 0$, since the kernel function $g(x, y, \xi)$ satisfies $\Delta_x g(x, y, \xi) + (\partial^2 g(x, y, \xi)/\partial \xi^2) = 0$, we have

$$\begin{aligned}\frac{\partial^2 G}{\partial \xi^2}(x, \xi) &= \int_{B_R} \mu(y) \frac{\partial^2 g(x, y, \xi)}{\partial \xi^2}\, dy \\ &= -\int_{B_R} \mu(y) \Delta_x g(x, y, \xi)\, dy.\end{aligned}$$

Since $\mu \in W_0^{2,q}(B_R)$, by using the integration by parts, we have

$$\begin{aligned}\frac{\partial^2 G}{\partial \xi^2}(x, \xi) &= -\int_{B_R} \mu(y) \Delta_x g(x, y, \xi)\, dy \\ &= -\int_{B_R} \mu(y) \Delta_y g(x, y, \xi)\, dy \\ &= -\int_{B_R} \Delta_y \mu(y) g(x, y, \xi)\, dy.\end{aligned}$$

Therefore, we get

$$\begin{aligned}|\frac{\partial^2 G}{\partial \xi^2}(x, \xi)| &\leq \int_{B_R} |g(x, y, \xi) \Delta_y \mu(y)|\, dy \\ &\leq \left(\int_{B_R} |g(x, y, \xi)|^p\, dy \right)^{1/p} \left(\int_{B_R} |\Delta_y \mu(y)|^q\, dy \right)^{1/q} \\ &= \|g(x, \cdot, \xi)\|_{L^p(B_R)} \|\Delta \mu\|_{L^q(B_R)}.\end{aligned}$$

and the lemma is proved. □

Now we state our main theorem.

Theorem 3.2. *Suppose $\mu(y)$ is the solution of the equation (2.1), and let $q > 3$. If $\mu \in W_0^{2,q}(D)$ and $\|\Delta \mu\|_{L^q(D)} \leq M_0$, where M_0 is a given constant, then we have the following conditional estimate for μ*

$$\|\mu\|_{L^2(B_R)} \leq C \frac{1}{|\ln(1/\varepsilon)|}$$

where $\varepsilon = \int_{D_1} |f(x)|^2\, dx + \int_{D_1} |\nabla f(x)|^2\, dx$. The constant C depends on M_0.

Proof. Without loss of generality, we may assume that $\varepsilon < 1$. By Lemma 3.1, for $x \in D$, we have

$$\frac{\partial G(x,\xi)}{\partial \xi} \longrightarrow -\omega_4 \mu(x), \quad \xi \to +0. \tag{3.27}$$

It follows that

$$-\omega_4 \mu(x) = \frac{\partial G(x,\xi)}{\partial \xi} - \int_0^\xi \frac{\partial^2 G(x,\rho)}{\partial \rho^2}\, d\rho.$$

So we have

$$\begin{aligned}\omega_4 \|\mu\|_{L^2(B_R)} &\le \left\| \frac{\partial G(\cdot,\xi)}{\partial \xi} \right\|_{L^2(B_R)} + \left\| \int_0^\xi \frac{\partial^2 G(\cdot,\rho)}{\partial \rho^2}\, d\rho \right\|_{L^2(B_R)} \\ &\le \left\| \frac{\partial G(\cdot,\xi)}{\partial \xi} \right\|_{L^2(B_R)} + \int_0^\xi \left\| \frac{\partial^2 G(\cdot,\rho)}{\partial \rho^2} \right\|_{L^2(B_R)} d\rho.\end{aligned}$$

We use Minkowski's inequality, [**12**, p. 271], in the last inequality.

Since $\|\partial G(\cdot,0)/\partial \xi\|_{L^2(B_R)}$ is bounded, by Lemmas 3.3–3.5, we have

$$\left\| \frac{\partial G(\cdot,\xi)}{\partial \xi} \right\|_{L^2(B_R)} \le C_6 \varepsilon^{\delta \xi}$$

where $\delta \in (0,1)$ is chosen in Lemma 3.3. In view of Lemma 3.6, we can obtain $\sup_{0<\xi\le 1} \|\partial^2 G(\cdot,\xi)/\partial \xi^2\|_{L^\infty(B_R)} \le M_0 M_6$ and applying Lemmas 3.3–3.5, we have

$$\left\| \frac{\partial^2 G(\cdot,\rho)}{\partial \rho^2} \right\|_{L^2(B_R)} \le C_7 \varepsilon^{\delta_1 \rho}$$

where $\delta_1 \in (0,1)$ is chosen in Lemma 3.4. Therefore,

$$\begin{aligned}\|\mu\|_{L^2(B_R)} &\le C_6 \varepsilon^{\delta \xi} + C_7 \int_0^\xi \varepsilon^{\delta_1 \rho}\, d\rho \\ &\le C_6 \varepsilon^{\delta} + \frac{C_7}{\delta_1}\left(-\frac{1}{\ln \varepsilon} + \frac{\varepsilon^{\delta_1 \xi}}{\ln \varepsilon} \right) \\ &\le C\left(-\frac{1}{\ln \varepsilon} \right)\end{aligned}$$

with a constant C independent of μ. Thus the theorem is proved. □

Next we will show that under certain conditions the estimate in our theorem may be the best possible estimate.

Theorem 3.3. *Consider the following Cauchy problem for Laplace's equation on the domain* $B_R \times \{0 < \xi < 1\}$

$$\Delta u = 0, \quad B_R \times \{0 < \xi < 1\} \tag{3.28}$$

$$u = 0, \quad \partial B_R \times \{0 \le \xi \le 1\} \tag{3.29}$$

$$u = g_1, \quad B_R \times \{\xi = 1\} \tag{3.30}$$

$$\frac{\partial u}{\partial \xi} = g_2 \qquad B_R \times \{\xi = 1\}. \tag{3.31}$$

If $\|u(\cdot,0)\|_{L^2(B_R)} \le M_8$ *and* $\|\partial u(\cdot,0)/\partial\xi\|_{L^2(B_R)} \le M_8$ *with some constant* $M_8 > 0$*, then we have*

$$\|u(\cdot,0)\|_{L^2(B_R)} \le C\frac{1}{|\ln(1/\varepsilon)|}$$

where C *is a constant which depends on* M_8*, and we set* $\varepsilon = \|g_1\|^2_{L^2(B_R)} + \|g_2\|^2_{L^2(B_R)}$*. Moreover, this is the best possible estimate.*

Proof. The first part can be proved by the same way as the proof of Theorem 3.2. We omit it here.

We will prove the second part. If this is not the best possible estimate, then we can assume that the following estimate holds for $u(x,0)$:

$$\|u(\cdot,0)\|_{L^2(B_R)} = o\left(\frac{1}{\ln(1/\varepsilon)}\right) \tag{3.32}$$

for all $u \ne 0$ satisfying (3.28)–(3.31) and $\|u(\cdot,0)\|_{L^2(B_R)} \le M_8$, $\|\partial u(\cdot,0)/\partial\xi\|_{L^2(B_R)} \le M_8$ with some constant $M_8 > 0$.

Let

$$u_n(x,\xi) = \frac{1}{\sqrt{\lambda_n}} e^{-\sqrt{\lambda_n}\xi}\alpha_n(x)$$

where $\alpha_n(x)$ and λ_n are defined in the proof of Lemma 3.5, see also (3.13)–(3.15).

Then we have

$$\|u_n(\cdot,\xi)\|_{L^2(B_R)} = \frac{1}{\sqrt{\lambda_n}} e^{-\sqrt{\lambda_n}\xi}, \qquad \left\| \frac{\partial u_n}{\partial \xi}(\cdot,\xi) \right\|_{L^2(B_R)} = e^{-\sqrt{\lambda_n}\xi}$$

$$\|u_n(\cdot,0)\|_{L^2(B_R)} = \frac{1}{\sqrt{\lambda_n}}.$$

Therefore we have

$$\varepsilon = \left(\frac{1}{\sqrt{\lambda_n}} + 1 \right) e^{-\sqrt{\lambda_n}}.$$

So we have

$$\|u_n(\cdot,0)\|_{L^2(B_R)} \ln \frac{1}{\varepsilon} = \frac{\sqrt{\lambda_n} - \ln((1/\sqrt{\lambda_n}) + 1)}{\sqrt{\lambda_n}} \longrightarrow 1,$$
$$n \to \infty.$$

This contradicts (3.32), completing the proof. □

Remark 3.3. It should be noticed that Theorem 3.3 indicates the best possible estimate only for the Cauchy problem (3.28)–(3.31), not for μ. However this theorem suggests that the estimate in Theorem 3.2 is the best possible.

4. Discussion of the inverse problem of detecting reinforcement bars. In this section we will show the uniqueness and a conditional stabilizing estimate for the problem in [**3**] by means of Theorem 3.2. Furthermore, we give some remarks on our problems and method.

4.1. Uniqueness. By our method, we can give an alternative proof of uniqueness in $L^p(D)$ ($p > 1$) for the linearized inverse problem in [**3**] as follows.

We can define $H(x,\xi)$ as

$$H(x,\xi) = -L_x \int_D \frac{1}{r_{xy}^2 + \xi^2} \mu(y)\, dy$$

and the relation between $H(x,\xi)$ and $\mu(y)$ is

$$\int_{R^3} \psi(x)\frac{\partial H(x,\xi)}{\partial \xi}\,dx \longrightarrow -\omega_4 \int_{R^3} L_x\psi(x)\mu(x)\,dx, \quad \xi \to +0$$

where $\psi(x)$ is an arbitrarily rapidly decreasing function, e.g., [**14**].

Similarly to (3.2)–(3.4), we can verify that $H(x,\xi)$ satisfies the following equations.

$$\begin{aligned} \Delta H(x,\xi) &= 0, \quad (x,\xi) \in R^4 \setminus \{D \times \{\xi = 0\}\} \\ H(x,0) &= f(x), \quad x \in D_1 \\ \frac{\partial H}{\partial \xi}(x,0) &= 0, \quad x \in D_1. \end{aligned}$$

If $f(x) = 0$, from the uniqueness of the Cauchy problem for Laplace's equation, we have

$$H(x,\xi) = 0, \quad (x,\xi) \in R^4 \setminus \{D \times \{\xi = 0\}\}.$$

So we have

$$\int_{R^3} L_x\psi(x)\mu(x)\,dx = 0.$$

Since the support set of $\mu(x)$ is contained in B_R, we have that, for any rapidly decreasing function $\psi(x)$ in R^3,

$$\int_{R^3} L_x\psi(x)\mu(x)\,dx = \int_{B_R} L_x\psi(x)\mu(x)\,dx = 0. \tag{4.1}$$

Finally we will show that $\mu(x) = 0$, $x \in B_R$.

From the expression of L_x, we have

$$L_x = L_1 L_2 L_2$$

where $L_1 = -1/256\pi(\partial/\partial x_2)$, $L_2 = \Delta_x + 1/2(\partial^2/\partial x_1^2)$ and $L_3 = \Delta_x$.

Let the fundamental solutions for L_2 and L_3 be E_2 and E_3, respectively. For any given function $\widehat{\psi}(x) \in C^\infty(B_{R_1})$, where $R_1 > R$, we construct a rapidly decreasing function $\psi(x)$ such that

$$L_x\psi(x) = \widehat{\psi}(x), \quad x \in B_R.$$

Suppose that $\rho(x) \in C_0^\infty(R^3)$, $0 \le \rho(x) \le 1$ satisfies

$$\begin{aligned} \rho(x) &= 1, \quad |x| \le R \\ \rho(x) &= 0, \quad |x| \ge R_1. \end{aligned}$$

Let

$$\psi(x) = \rho(x) \int_{R^3} \rho(z) E_3(x,z) \int_{R^3} E_2(z,y)\psi_1(y)\,dy\,dz$$

where

$$\psi_1(y) = -256\pi\rho(y) \int_0^{y_2} \widehat{\psi}(y_1,\sigma,y_3)\,d\sigma.$$

It is easy to verify $\psi(x) \in C_0^\infty(R^3)$ and

$$L_x\psi(x) = \widehat{\psi}(x), \quad x \in B_R.$$

Therefore, from (4.1), we can find that $\mu(x) = 0$, $x \in B_R$.

4.2. Stabilizing estimate. We turn to the conditional stability for the original integral equation

$$L_x \int_D \frac{1}{r_{xy}^2}\mu(y)\,dy = f(x), \quad x \in D_1. \tag{4.2}$$

In this case we have to assume more regularity on μ:

$$\mu \in W_0^{7,q}(B_R) \tag{4.3}$$

$$\|\Delta L_x \mu\|_{L^q(B_R)} \le M_9 \tag{4.4}$$

where $q > 3$ and $M_9 > 0$ are constants.

Then we have

Theorem 4.1. *Under* (4.3) *and* (4.4), *if* μ *is the solution of the integral equation* (4.2), *then*

$$\|\mu\|_{W^{4,2}(B_R)} \le C\frac{1}{|\ln(1/\varepsilon)|} \tag{4.5}$$

where $\varepsilon = \int_{D_1} |f(x)|^2\,dx + \int_{D_1} |\nabla f(x)|^2\,dx$ *and* C *is a constant which depends on* M_9.

Proof. Similarly to the proof of Theorem 3.2, we may assume that $\varepsilon < 1$. Since

$$-L_x \frac{1}{r_{xy}^2} = L_y \frac{1}{r_{xy}^2}, \quad x \in D_1, y \in D.$$

By (4.2), we have

$$\int_D -L_y \left(\frac{1}{r_{xy}^2}\right) \mu(y)\,dy = f(x), \quad x \in D_1.$$

Recalling (4.3) and the definition of L_y, and applying integration by parts, we obtain

$$\int_D \frac{1}{r_{xy}^2} L_y \mu(y)\,dy = f(x), \quad x \in D_1.$$

Therefore, by (4.2) and (4.3), we can apply Theorem 3.2 for $(L_y\mu)(y)$, not $\mu(y)$, so that

$$\left\| \left(\Delta^2 + \frac{1}{2}\frac{\partial^2}{\partial x_1^2}\Delta\right)\phi \right\|_{L^2(B_R)} \le C\frac{1}{\ln(1/\varepsilon)} \tag{4.6}$$

where $\phi(x) = (\partial\mu(x)/\partial x_2)$, $x \in B_R$.

We set

$$\tilde{\phi} = \left(\Delta + \frac{1}{2}\frac{\partial^2}{\partial x_1^2}\right)\phi.$$

By (4.3) and (4.6), we see that

$$\|\Delta\tilde{\phi}\|_{L^2(B_R)} \le C\frac{1}{\ln(1/\varepsilon)}$$

$$\tilde{\phi}|_{\partial B_R} = 0.$$

Therefore, by the regularity properties of elliptic equations, e.g., Theorems 8.12 and 8.13 in [**4**] , we obtain

$$\|\tilde{\phi}\|_{W^{2,2}(B_R)} \le C\frac{1}{\ln(1/\varepsilon)},$$

namely,

$$\left\| \left(\Delta + \frac{1}{2} \frac{\partial^2}{\partial x_1^2} \right) \phi \right\|_{W^{2,2}(B_R)} \leq C \frac{1}{\ln(1/\varepsilon)}.$$

By (4.3), we have $\phi|_{\partial B_R} = 0$. Applying the regularity of elliptic equations, we get

$$\|\phi\|_{W^{4,2}(B_R)} \leq C \frac{1}{\ln(1/\varepsilon)}.$$

Next we get the estimate for μ. For any point $x = (x_1, x_2, x_3) \in B_R$, we have a point $(x_1, \tilde{x_2}, x_3) \in \partial B_R$. Then we have

$$\mu(x) = \int_{\tilde{x_2}}^{x_2} \phi(x_1, z, x_3)\, dz. \tag{4.7}$$

Applying the estimate for ϕ, we have

$$\|\mu\|_{W^{4,2}(B_R)} \leq C \frac{1}{\ln(1/\varepsilon)}$$

and the proof is complete. □

4.3. Remarks.

Remark 4.1. From our treatment, we know that the local values of function $\mu(x)$ can be obtained pointwise from $G(x, \xi)$. It is possible to get some local conditional estimate for $\mu(x)$, in the case where the global estimation will fail. Here "global estimate" means a uniform one for all $x \in D$.

Remark 4.2. Our treatment also can distinguish discontinuous points of μ.

Let $D = B_r$, $r < R$, and $\mu \in C^\infty(\overline{D})$, $\mu(x_0) \neq 0$ and $\mu(x) = 0$, $|x| > r$, where $x_0 = (r, 0, 0)$. By Lemma 3.1, we have

$$\frac{\partial G((x_1, 0, 0), \xi)}{\partial \xi} \longrightarrow 0, \quad x_1 > r$$

$$\frac{\partial G((x_1, 0, 0), \xi)}{\partial \xi} \longrightarrow -\omega_4 \mu(x_1, 0, 0), \quad x_1 < r,$$

as $\xi \longrightarrow +0$. (Since $\mu \in C^\infty(D)$, the convergence is pointwise, e.g., [**5**]).

If we define $g_\xi(x_1) = \partial G((x_1,0,0),\xi)/\partial\xi$, then we have

$$g_\xi(x_1) \longrightarrow \hat{g}(x_1), \quad \xi \longrightarrow +0$$

where $\hat{g}(x_1) = -\omega_4\mu(x_1,0,0)$, $x_1 < r$ and $\hat{g}(x_1) = 0$, $x_1 > r$. Since $\mu(x_1,0,0) \neq 0$, $\hat{g}(x_1)$ has a jump at $x_1 = r$. It can be seen that, for $x_1 = r$, $\lim_{\xi\to+0}(\partial g_\xi(x_1)/\partial x_1)$ is unbounded, i.e., $\lim_{\xi\to+0}(\partial^2 G(x,\xi)/(\partial x_1 \partial\xi))|_{(r,0,0)}$ is unbounded. Moreover for other points $x_1 \neq r$ in the neighborhood of $x_1 = r$, it can be proved that $\lim_{\xi\to+0}(\partial g_\xi(x_1)/\partial x_1) = -\omega_4(\partial\mu(x_1,0,0)/\partial x_1)$ which is bounded.

This means that, at a discontinuous point x_0, if ξ is small enough, the value $\nabla(\partial G(x_0,\xi)/\partial\xi)$ is greater than the value $\nabla(\partial G(x,\xi)/\partial\xi)$ for the continuous points $x \neq x_0$. So we can find the locations of discontinuous points from values at the points on the plane $\xi = \zeta$, where ζ is small enough. Since these points are interior points for the Cauchy problem we consider, we can have the Hölder estimate for these points. So it is possible to develop a numerical technique for locating discontinuities in this way.

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REFERENCES

1. A.L. Bukhgeim, *Volterra equations and inverse problems*, Nauka, Novosibirsk, 1983 (in Russian).

2. R. Courant and D. Hilbert, *Methods of mathematical physics*, Vol. 2, Inter-Science Publishers, New York, 1962.

3. H.W. Engl and V. Isakov, *On the identifiability of steel reinforcement bars in concrete from magnetostatic measurements*, European J. Appl. Math. **3** (1992), 255–262.

4. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Springer-Verlag, New York, 1983.

5. R. Kress, *Linear integral equations*, Springer-Verlag, New York, 1989.

6. M.M. Lavrent'ev, *On an inverse problem for the wave equation*, Dokl. Akad. Nauk SSSR **157** (1964), 520–521.

7. L.E. Payne, *Bounds in the Cauchy problem for the Laplace equation*, Arch. Rational Mech. Anal. **5** (1960), 35–45.

8. M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1967.

9. A.G. Ramm, *Scattering by obstacles*, D. Reidel Publishing Company, Dordrecht, 1986.

10. ———, *Some inverse scattering problem of geophysics*, Inverse Problems, **1** (1985), 133–172.

11. A.U. Serikbaev, *On a multidimensional inverse problem for the Weber potential*, J. Inverse Ill-posed Problems **2** (1994), 323–326.

12. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, New Jersey, 1970.

13. A.N. Tikhonov and V.Y. Arsenin, *Solutions of ill-posed problems*, Winston and Sons, Washington, 1977.

14. K. Yosida, *Functional analysis*, Springer-Verlag, Berlin, 1971.

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