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THE ASCENDING CHAIN CONDITION ON PRINCIPAL IDEALS IN COMPOSITE GENERALIZED POWER SERIES RINGS

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ABSTRACT. Let $D \subseteq E$ be an extension of commutative rings with identity, I a nonzero proper ideal of D, $(\Gamma, \leq)$ a strictly totally ordered monoid such that $0 \leq \alpha$ for all $\alpha \in \Gamma$, and $\Gamma^* = \Gamma \setminus \{0\}$. Let $D + [\![E^{\Gamma^*, \leq}]\!] = \{f \in [\![E^{\Gamma, \leq}]\!] \mid f(0) \in D\}$ and $D + [\![I^{\Gamma^*, \leq}]\!] = \{f \in [\![D^{\Gamma, \leq}]\!] \mid f(\alpha) \in I \text{ for all } \alpha \in \Gamma^*\}$. In this paper, we give some conditions for the rings $D + [\![E^{\Gamma^*, \leq}]\!]$ and $D + [\![I^{\Gamma^*, \leq}]\!]$ to satisfy the ascending chain condition on principal ideals.

0. Introduction.

0.1. Generalized power series rings. Let $(\Gamma, \leq)$ be an *ordered monoid*, i.e., Γ is a monoid and $\leq$ is a compatible order relation with the monoid operation: if $\alpha_1, \alpha_2, \beta \in \Gamma$, then $\alpha_1 \leq \alpha_2$ implies $\alpha_1 + \beta \leq \alpha_2 + \beta$. We say that an ordered monoid $(\Gamma, \leq)$ is *artinian* if every decreasing sequence of elements of Γ is finite, and $(\Gamma, \leq)$ is *narrow* if every subset of pairwise order-incomparable elements of Γ is finite. An *ideal* of Γ is a nonempty subset I of Γ such that $I \supseteq \alpha + I := \{\alpha + \gamma | \gamma \in I\}$ for each $\alpha \in \Gamma$. An ordered monoid $(\Gamma, \leq)$ is called a *strictly ordered monoid* if for $\alpha_1, \alpha_2, \beta \in \Gamma$, $\alpha_1 < \alpha_2$ implies that $\alpha_1 + \beta < \alpha_2 + \beta$.

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Let R be a commutative ring with identity and $(\Gamma, \leq)$ a strictly ordered monoid. We denote by $[\![R^{\Gamma,\leq}]\!]$ the set of all mappings $f : \Gamma \to R$ such that $\text{supp}(f) := \{\alpha \in \Gamma | f(\alpha) \neq 0\}$ is an artinian and narrow subset of Γ. (The set $\text{supp}(f)$ is called the *support* of f.) With pointwise addition, $[\![R^{\Gamma,\leq}]\!]$ is an (additive) abelian group. Moreover, for every $\alpha \in \Gamma$ and $f, g \in [\![R^{\Gamma,\leq}]\!]$, the set $X_\alpha(f, g) := \{(\beta, \gamma) \in \Gamma \times \Gamma | \alpha = \beta + \gamma, f(\beta) \neq 0$, and $g(\gamma) \neq 0\}$ is finite [**9,** 1.16]; thus, this allows us to define the operation of *convolution*:

$$(fg)(\alpha) = \sum_{(\beta,\gamma) \in X_\alpha(f,g)} f(\beta)g(\gamma).$$

It is easy to see that $[\![R^{\Gamma,\leq}]\!]$ is a commutative ring (under these operations) with unit element $\mathbf{e}$, namely, $\mathbf{e}(0) = 1$ and $\mathbf{e}(\alpha) = 0$ for all $\alpha \in \Gamma^*$, which is called the *ring of generalized power series* of Γ over R. The elements of $[\![R^{\Gamma,\leq}]\!]$ are called *generalized power series with coefficients* in R *and exponents* in Γ. It is well known that R is canonically embedded as a subring of $[\![R^{\Gamma,\leq}]\!]$, and Γ is canonically embedded as a submonoid of $[\![R^{\Gamma,\leq}]\!] \setminus \{0\}$ by the mapping $\alpha \in \Gamma \mapsto \mathbf{e}_\alpha \in [\![R^{\Gamma,\leq}]\!]$, where $\mathbf{e}_\alpha(\alpha) = 1$ and $\mathbf{e}_\alpha(\gamma) = 0$ for every $\gamma \in \Gamma \setminus \{\alpha\}$. Also, if $f, g \in [\![R^{\Gamma,\leq}]\!]$, then $\text{supp}(f+g) \subseteq \text{supp}(f) \cup \text{supp}(g)$ and $\text{supp}(fg) \subseteq \text{supp}(f) + \text{supp}(g)$.

For $0 \neq f \in [\![R^{\Gamma,\leq}]\!]$, we denote by $\pi(f)$ the set of minimal elements in $\text{supp}(f)$. Note that $\pi(f)$ is a nonempty finite set consisting of pairwise order incomparable elements. If $\pi(f)$ consists of only one element α, then we write $\pi(f) = \alpha$ and call it the *order* of f. If $(\Gamma, \leq)$ is totally ordered and $0 \neq f \in [\![R^{\Gamma,\leq}]\!]$, then $\text{supp}(f)$ is a nonempty well-ordered subset of Γ; thus, $\pi(f)$ always consists of only one element. For the sake of convenience, we define $\pi(0) = \infty$ by adjoining an element ∞ to Γ with properties that, for all $\alpha \in \Gamma$, $\alpha < \infty$ and $\alpha + \infty = \infty = \infty + \alpha$.

Let $D \subseteq E$ be an extension of commutative rings with identity, I a nonzero proper ideal of D, $(\Gamma, \leq)$ a nonzero strictly ordered monoid and $\Gamma^* = \Gamma \setminus \{0\}$. Set

$$D + [\![E^{\Gamma^*,\leq}]\!] = \{f \in [\![E^{\Gamma,\leq}]\!] | f(0) \in D\}$$

and

$$D + [\![I^{\Gamma^*,\leq}]\!] = \{f \in [\![D^{\Gamma,\leq}]\!] | f(\alpha) \in I \text{ for all } \alpha \in \Gamma^*\}.$$

Then

$$D \subsetneq D + [\![I^{\Gamma^*,\leq}]\!] \subsetneq [\![D^{\Gamma,\leq}]\!] \subseteq D + [\![E^{\Gamma^*,\leq}]\!] \subseteq [\![E^{\Gamma,\leq}]\!];$$

thus, $D+[\![I^{\Gamma^*,\leq}]\!]$ and $D+[\![E^{\Gamma^*,\leq}]\!]$ provide information about algebraic structures of subrings of generalized power series rings and power series type rings between two generalized power series rings, respectively. We note that, if $\Gamma = \mathbb{N}_0^n$ with the product order or lexicographic order, where $\mathbb{N}_0$ is the set of nonnegative integers, then $D+[\![E^{\Gamma^*,\leq}]\!]$ and $D+[\![I^{\Gamma^*,\leq}]\!]$ are isomorphic to $D+(X_1,\ldots,X_n)E[\![X_1,\ldots,X_n]\!]$ and $D+(X_1,\ldots,X_n)I[\![X_1,\ldots,X_n]\!]$, respectively.

The notation and terminology used in this paper are standard. For more on generalized power series, the reader may refer to [**8, 10**].

0.2. Rings satisfying the ascending chain condition on principal ideals. Finiteness conditions have, for many years, been important tools in commutative algebra and algebraic geometry due to their use in producing many theorems and applications. For example, a relation between the ascending chain conditions on ideals and finite generatedness of ideals in rings permits an interesting measure of the size and behavior of such rings.

Recall that a commutative ring R satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of R. As a semigroup version, we say that a monoid Γ satisfies the *ascending chain condition on principal ideals* (ACCP) if there does not exist an infinite strictly ascending chain of principal ideals of Γ.

In [**7**], Liu showed that, for an integral domain D and a strictly totally ordered monoid $(\Gamma,\leq)$, $[\![D^{\Gamma,\leq}]\!]$ satisfies ACCP if and only if D and Γ satisfy ACCP. In [**3, 4**], Dumitrescu, et al., and Hizem investigated several chain conditions in special pullbacks of the forms $D+(X_1,\ldots,X_n)E[\![X_1,\ldots,X_n]\!]$ and $D+(X_1,\ldots,X_n)I[\![X_1,\ldots,X_n]\!]$, where $D\subseteq E$ is an extension of commutative rings (or integral domains) and I is a nonzero proper ideal of D.

The purpose of this article is to study the ACCP property on special kinds of pullbacks, which are the so-called composite generalized power series rings $D+[\![E^{\Gamma^*,\leq}]\!]$ and $D+[\![I^{\Gamma^*,\leq}]\!]$ when $(\Gamma,\leq)$ is a nonzero strictly totally ordered monoid with $0\leq\alpha$ for all $\alpha\in\Gamma$. In fact, we show that

(1) $D+[\![E^{\Gamma^*,\leq}]\!]$ is an integral domain satisfying ACCP if and only if $\bigcap_{n\geq 1} a_1\cdots a_n E=(0)$ for each sequence $(a_n)_{n\geq 1}$ of nonzero nonunits of D and Γ satisfies ACCP;

(2) if D is an integral domain satisfying ACCP and Γ satisfies ACCP, then $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfies ACCP;

(3) if E is présimplifiable, then $D+[\![E^{\Gamma^*,\leq}]\!]$ satisfies ACCP if and only if

(a) $U(E)\cap D=U(D)$,
(b) for each sequence $(e_n)_{n\geq 1}$ of E with the property that, for each $n\geq 1$, $e_n=e_{n+1}d_n$ for some $d_n\in D$, $e_1E\subseteq e_2E\subseteq\cdots$ is stationary, and
(c) Γ satisfies ACCP; and

(4) if D is a présimplifiable ring satisfying ACCP and Γ satisfies ACCP, then $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfies ACCP. As corollaries, we recover several known results [**3,** Proposition 1.2], [**4,** Propositions 4.6, 4.18, 4.21], [**7,** Theorem 3.2].

1. Main results.

We start with full descriptions of units of composite generalized power series rings. Recall that, for a commutative ring R with identity and a strictly ordered monoid $(\Gamma,\leq)$ with $0\leq\alpha$ for all $\alpha\in\Gamma$, f is a unit in $[\![R^{\Gamma,\leq}]\!]$ if and only if $f(0)$ is a unit in R [**9,** 2.3].

Lemma 1.1. *Let $D\subseteq E$ be an extension of commutative rings with identity and $(\Gamma,\leq)$ a nonzero strictly totally ordered monoid with $0\leq\alpha$ for all $\alpha\in\Gamma$. Then the following assertions hold.*

(i) *A generalized power series $f\in D+[\![E^{\Gamma^*,\leq}]\!]$ is a unit if and only if $f(0)$ is a unit in D.*
(ii) *Let I be a nonzero proper ideal of D. Then $f\in D+[\![I^{\Gamma^*,\leq}]\!]$ is a unit if and only if $f(0)$ is a unit in D.*

Proof. (i) This equivalence directly follows from an easy evaluation and [**9,** 2.3].

(ii) If f is a unit in $D+[\![I^{\Gamma^*,\leq}]\!]$, then there exists an element $g\in D+[\![I^{\Gamma^*,\leq}]\!]$ such that $fg=1$; thus, $f(0)g(0)=1$. Hence, $f(0)$ is a unit in D. Conversely, if $f(0)$ is a unit in D, then we can find a suitable $g\in[\![D^{\Gamma,\leq}]\!]$ such that $fg=1$ [**9,** 2.3]. Now, we claim that $g\in D+[\![I^{\Gamma^*,\leq}]\!]$. If $g\in D$, then we have nothing to prove; thus, we assume that $g\notin D$.

Let α be the smallest positive element in $\operatorname{supp}(g)$. Then we have

$$0 = (fg)(\alpha) = \sum_{(\alpha_1,\alpha_2)\in X_\alpha(f,g)} f(\alpha_1)g(\alpha_2) = f(0)g(\alpha) + f(\alpha)g(0);$$

thus, $g(\alpha) = -(f(\alpha)g(0))/f(0) \in I$. Let $\gamma \in \operatorname{supp}(g)\setminus\{0\}$. Note that

$$0 = (fg)(\gamma) = \sum_{(0,\gamma)\neq(\gamma_1,\gamma_2)\in X_\gamma(f,g)} f(\gamma_1)g(\gamma_2) + f(0)g(\gamma);$$

hence,

$$g(\gamma) = -\frac{\sum_{(0,\gamma)\neq(\gamma_1,\gamma_2)\in X_\gamma(f,g)} f(\gamma_1)g(\gamma_2)}{f(0)} \in I$$

since $f(\gamma_1) \in I$ for $\gamma_1 \in \Gamma^*$. Therefore, $g \in D + [\![I^{\Gamma^*,\leq}]\!]$, and thus, f is a unit in $D + [\![I^{\Gamma^*,\leq}]\!]$. □

1.1. When E is an integral domain. Recall that, if R is an integral domain, then R satisfies ACCP if and only if $\bigcap_{n\geq 1} a_1\cdots a_n R = (0)$ for each infinite sequence $(a_n)_{n\geq 1}$ of nonzero nonunits of R [**3,** Remark 1.1]. As in the integral domain case, it was shown that Γ satisfies ACCP if and only if $\bigcap_{n\geq 1}(\alpha_1 + \cdots + \alpha_n + \Gamma) = \varnothing$ for each sequence $(\alpha_n)_{n\geq 1}$ in Γ^* [**6,** Lemma 3.1] (or [**7,** Lemma 2.1]).

Theorem 1.2. *Let $D \subseteq E$ be an extension of integral domains and $(\Gamma, \leq)$ a nonzero strictly totally ordered monoid with $0 \leq \alpha$ for all $\alpha \in \Gamma$. Then $D + [\![E^{\Gamma^*,\leq}]\!]$ satisfies* ACCP *if and only if*

$$\bigcap_{n\geq 1} a_1\cdots a_n E = (0)$$

for each sequence $(a_n)_{n\geq 1}$ of nonzero nonunits of D, and Γ satisfies ACCP.

Proof. ($\Rightarrow$) Let $(a_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of D. Then by Lemma 1.1(i), $(a_n)_{n\geq 1}$ is also an infinite sequence of nonzero nonunits of $D + [\![E^{\Gamma^*,\leq}]\!]$. Since $D + [\![E^{\Gamma^*,\leq}]\!]$ satisfies ACCP,

$$\bigcap_{n\geq 1} a_1\cdots a_n(D + [\![E^{\Gamma^*,\leq}]\!]) = (0)$$

[**3,** Remark 1.1]. Let $e \in \bigcap_{n\geq 1} a_1 \cdots a_n E$ and $\alpha \in \Gamma^*$. Then

$$eX^\alpha \in \bigcap_{n\geq 1} a_1 \cdots a_n (D + [\![E^{\Gamma^*,\leq}]\!]);$$

hence, $e = 0$. Thus,

$$\bigcap_{n\geq 1} a_1 \cdots a_n E = (0).$$

If $\alpha_1 + \Gamma \subsetneq \alpha_2 + \Gamma \subsetneq \cdots$ is an infinite ascending chain of principal ideals of Γ, then $X^{\alpha_1}(D + [\![E^{\Gamma^*,\leq}]\!]) \subsetneq X^{\alpha_2}(D + [\![E^{\Gamma^*,\leq}]\!]) \subsetneq \cdots$ is also an infinite ascending chain of principal ideals of $D + [\![E^{\Gamma^*,\leq}]\!]$, which is absurd. Hence, Γ satisfies ACCP.

($\Leftarrow$) Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits of $D + [\![E^{\Gamma^*,\leq}]\!]$, set $\alpha_n = \pi(f_n)$ and let $\Lambda = \{\alpha_n | n = 1, 2, \ldots\}$.

Case 1. $\Lambda \cap \Gamma^*$ *is infinite.* By considering a subsequence of $(f_n)_{n\geq 1}$ with nonzero orders, we may assume that $\alpha_n \in \Gamma^*$ for all $n \geq 1$. If

$$0 \neq g \in \bigcap_{n\geq 1} f_1 \cdots f_n (D + [\![E^{\Gamma^*,\leq}]\!]),$$

then for each $n \geq 1$, we can find an element $\phi_n \in D + [\![E^{\Gamma^*,\leq}]\!]$ such that $g = f_1 \cdots f_n \phi_n$. Note that

$$\pi(g) = \sum_{i=1}^{n} \pi(f_i) + \pi(\phi_n) = \sum_{i=1}^{n} \alpha_i + \pi(\phi_n)$$

for all $n \geq 1$; hence, we have $\pi(g) \in \bigcap_{n\geq 1}(\alpha_1 + \cdots + \alpha_n + \Gamma)$, which is a contradiction to the fact that Γ satisfies ACCP [**6,** Lemma 3.1] (or [**7,** Lemma 2.1]).

Case 2. $\Lambda \cap \Gamma^*$ *is finite.* There exist infinitely many members in $(f_n)_{n\geq 1}$ such that $\pi(f_n) = 0$; thus, we may assume that $f_n(0) \neq 0$ for all $n \geq 1$. Since each f_n is a nonunit, $f_n(0)$ is a nonunit in D by Lemma 1.1(i). If $0 \neq g \in \bigcap_{n\geq 1} f_1 \cdots f_n (D + [\![E^{\Gamma^*,\leq}]\!])$, then for each $n \geq 1$, there exists an element $\phi_n \in D + [\![E^{\Gamma^*,\leq}]\!]$ such that $g = f_1 \cdots f_n \phi_n$. Since $\pi(g) = \sum_{i=1}^{n} \pi(f_i) + \pi(\phi_n) = \pi(\phi_n)$ for all $n \geq 1$, $g(\pi(g)) = f_1(0) \cdots f_n(0)\phi_n(\pi(\phi_n))$ for all $n \geq 1$, we therefore obtain

$$g(\pi(g)) \in \bigcap_{n\geq 1} f_1(0) \cdots f_n(0) E = (0),$$

which is a contradiction. In either case,

$$\bigcap_{n\geq 1} f_1\cdots f_n(D+[\![E^{\Gamma^*,\leq}]\!])=(0).$$

Thus we conclude that $D+[\![E^{\Gamma^*,\leq}]\!]$ satisfies ACCP [**3,** Remark 1.1]. □

The proof of the next result is almost parallel to that of Theorem 1.2. We leave it to the reader.

Theorem 1.3. *Let I be a nonzero proper ideal of an integral domain D and $(\Gamma,\leq)$ a nonzero strictly totally ordered monoid with $0\leq\alpha$ for all $\alpha\in\Gamma$. Then the following statements hold.*

(i) *If D and Γ satisfy* ACCP*, then $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfies* ACCP*.*
(ii) *If $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfies* ACCP*, then so does D.*

It is worth remarking at this point that, for any $\alpha\in\Gamma^*$ and for any unit u of E, $uX^\alpha\in D+[\![E^{\Gamma^*,\leq}]\!]$ but $uX^\alpha\notin D+[\![I^{\Gamma^*,\leq}]\!]$. This difference makes the converse of Theorem 1.3(i) not necessarily true.

The next example shows that the converse of Theorem 1.3(i) does not hold, i.e., the condition that $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfying ACCP does not imply that Γ satisfies ACCP.

Example 1.4. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Q}_0$ the set of nonnegative rational numbers.

(i) Let $(f_n)_{n\geq 1}$ be an infinite sequence of nonzero nonunits in $\mathbb{Z}+[\![(2\mathbb{Z})^{\mathbb{Q}_0^*,\leq}]\!]$. If there exists a

$$0\neq g\in\bigcap_{n\geq 1} f_1\cdots f_n(\mathbb{Z}+[\![(2\mathbb{Z})^{\mathbb{Q}_0^*,\leq}]\!]),$$

then for each $n\geq 1$, $g=f_1\cdots f_nh_n$ for some $h_n\in\mathbb{Z}+[\![(2\mathbb{Z})^{\mathbb{Q}_0^*,\leq}]\!]$, we have

$$g(\pi(g))=f_1(\pi(f_1))\cdots f_n(\pi(f_n))h_n(\pi(h_n))$$

for each $n\geq 1$. Hence, almost all $f_n(\pi(f_n))$ should be ± 1, i.e., $\pi(f_n)=0$ for almost all i. Therefore, by Lemma 1.1(ii), almost all f_n are units in $\mathbb{Z}+[\![(2\mathbb{Z})^{\mathbb{Q}_0^*,\leq}]\!]$, which contradicts the choice of $(f_n)_{n\geq 1}$. Thus, $\mathbb{Z}+[\![(2\mathbb{Z})^{\mathbb{Q}_0^*,\leq}]\!]$ satisfies ACCP.

(ii) Note that $\{1/2^n + \mathbb{Q}_0\}_{n\geq 1}$ forms an infinite strictly increasing sequence of principal ideals of $\mathbb{Q}_0$; thus, $\mathbb{Q}_0$ does not satisfy ACCP.

It is clear that the monoid $\mathbb{N}_0^n$ satisfies ACCP. By applying Theorems 1.2 and 1.3 to the case when $D=E$ or $(\Gamma,\leq)$ is isomorphic to $\mathbb{N}_0^n$ with lexicographic order, we regain the following.

Corollary 1.5. *Let $D\subseteq E$ be an extension of integral domains, I a nonzero proper ideal of D and $(\Gamma,\leq)$ a nonzero strictly totally ordered monoid with $0\leq\alpha$ for all $\alpha\in\Gamma$. Then the following assertions hold.*

- (i) (cf. [**7,** Theorem 3.2]). $[\![D^{\Gamma,\leq}]\!]$ *satisfies* ACCP *if and only if* D *and* Γ *satisfy* ACCP.
- (ii) ([**3,** Proposition 1.2]). $D+(X_1,\dots,X_n)E[\![X_1,\dots,X_n]\!]$ *satisfies* ACCP *if and only if* $\bigcap_{n\geq 1} a_1\cdots a_n E=(0)$ *for each sequence* $(a_n)_{n\geq 1}$ *of nonzero nonunits of* D.
- (iii) ([**4,** Proposition 4.6]). $D+(X_1,\dots,X_n)I[\![X_1,\dots,X_n]\!]$ *satisfies* ACCP *if and only if* D *satisfies* ACCP.

1.2. When E is présimplifiable. Let R be a commutative ring with identity and $U(R)$ the set of units in R. Following [**2**], R is *présimplifiable* if, whenever $a,b\in R$ with $ab=a$, either $a=0$ or $b\in U(R)$. It is clear that any integral domain is présimplifiable, but the converse does not hold (for example, $\mathbb{Z}_{p^n}$ for any prime p and integer $n\geq 2$). We first give an equivalent condition for the ring $D+[\![E^{\Gamma^*,\leq}]\!]$ to satisfy ACCP under the assumption that E is présimplifiable.

Theorem 1.6. *Let $(\Gamma,\leq)$ be a nonzero strictly totally ordered monoid with $0\leq\alpha$ for all $\alpha\in\Gamma$. If E is présimplifiable and D is a subring of E, then $D+[\![E^{\Gamma^*,\leq}]\!]$ satisfies* ACCP *if and only if the following three conditions hold.*

- (i) $U(E)\cap D=U(D)$;
- (ii) *for each sequence $(e_n)_{n\geq 1}$ of E with the property that for each $n\geq 1$, $e_n=e_{n+1}d_n$ for some $d_n\in D$, $e_1E\subseteq e_2E\subseteq\cdots$ is stationary*; *and*
- (iii) Γ *satisfies ACCP.*

Proof. $(\Rightarrow)$ Assume that $D+[\![E^{\Gamma^*,\leq}]\!]$ satisfies ACCP.

(i) Let $a \in U(E) \cap D$ and $\alpha \in \Gamma^*$. Then $(1/a)X^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq (1/a^2)X^\alpha(D+[\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ is a chain of principal ideals of $D+[\![E^{\Gamma^*,\leq}]\!]$; thus, it should be stationary. Therefore, there exists a positive integer m such that

$$\frac{1}{a^{m+1}}X^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]) = \frac{1}{a^m}X^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]).$$

Hence, $aD = D$. Thus, $a \in U(D)$.

(ii) Let $\alpha \in \Gamma^*$ and $(e_n)_{n\geq 1}$ be a sequence of E such that, for each $n \geq 1$, $e_n = e_{n+1}d_n$ for some $d_n \in D$. Then $e_1X^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq e_2X^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ is stationary; thus, we can find a positive integer m such that $e_nX^\alpha(D + [\![E^{\Gamma^*,\leq}]\!]) = e_mX^\alpha(D + [\![E^{\Gamma^*,\leq}]\!])$ for all $n \geq m$. Hence, $e_nE = e_mE$ for all $n \geq m$. Therefore, the chain $e_1E \subseteq e_2E \subseteq \cdots$ stops.

(iii) If $\alpha_1 + \Gamma \subsetneq \alpha_2 + \Gamma \subsetneq \cdots$ is an infinite ascending chain of principal ideals of Γ, then $X^{\alpha_1}(D + [\![E^{\Gamma^*,\leq}]\!]) \subsetneq X^{\alpha_2}(D + [\![E^{\Gamma^*,\leq}]\!]) \subsetneq \cdots$ is also an infinite ascending chain of principal ideals of $D + [\![E^{\Gamma^*,\leq}]\!]$, which is impossible. Thus, Γ satisfies ACCP.

($\Leftarrow$) Let $f_1(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ be an ascending chain of nonzero principal ideals of $D + [\![E^{\Gamma^*,\leq}]\!]$. Then for each $n \geq 1$, $f_n = f_{n+1}g_n$ for some $g_n \in D + [\![E^{\Gamma^*,\leq}]\!]$. If f_n is a unit for some $n \geq 1$, then there is nothing to prove; thus, we assume that f_n is a nonunit for all $n \geq 1$. Note that $\pi(f_1) \geq \pi(f_2) \geq \cdots \geq 0$; hence, we have only two cases: either $\pi(f_n) \neq 0$ for all $n \geq 1$ or $\pi(f_n) \neq 0$ only for finitely many indices n.

Case 1. $\pi(f_n) \neq 0$ *for all* $n \geq 1$. Let β_2 be the smallest member in $\mathrm{supp}(f_2)$ such that $(\beta_2, \gamma_1) \in X_{\pi(f_1)}(f_2, g_1)$ and $f_2(\beta_2)g_1(\gamma_1) \neq 0$ for some $\gamma_1 \in \mathrm{supp}(g_1)$. For each $n \geq 2$, we denote by β_{n+1} the smallest element in $\mathrm{supp}(f_{n+1})$ such that $(\beta_{n+1}, \gamma_n) \in X_{\beta_n}(f_{n+1}, g_n)$ and $f_{n+1}(\beta_{n+1})g_n(\gamma_n) \neq 0$ for some $\gamma_n \in \mathrm{supp}(g_n)$. Then

$$\pi(f_1) = \gamma_1 + \cdots + \gamma_n + \beta_{n+1} \quad \text{for all } n \geq 1;$$

hence, $\pi(f_1) \in \bigcap_{n\geq 1}(\gamma_1 + \cdots + \gamma_n + \Gamma)$. If $\gamma_n \neq 0$ for infinitely many n, then it contradicts the fact that Γ satisfies ACCP [**6,** Lemma 3.1] (or [**7,** Lemma 2.1]). Hence, $\gamma_n \neq 0$ for finitely many n. Let m be the largest positive integer such that $\gamma_m \neq 0$. (If there is no such m, then we take $m = 0$ and $\beta_1 = \pi(f_1)$.) Then $\gamma_k = 0$ for all $k \geq m + 1$; thus,

$\beta_k = \beta_{m+1}$ for all $k \geq m+1$, which implies that, for all $k \geq m+1$, $f_k(\beta_k) = f_{k+1}(\beta_{k+1})g_k(0)$ by the minimality of β_{k+1}. Note that by (ii), $f_{m+1}(\beta_{m+1})E \subseteq f_{m+2}(\beta_{m+2})E \subseteq \cdots$ is stationary. Hence, there exists a large enough integer $t \geq m+1$ such that $f_k(\beta_k)E = f_t(\beta_t)E$ for all $k \geq t$. Since E is présimplifiable, $g_k(0)$ is a unit in E for all $k \geq t$; thus, by (i), $g_k(0) \in U(E) \cap D = U(D)$ for all $k \geq t$. By Lemma 1.1(i), g_k is a unit in $D + [\![E^{\Gamma^*,\leq}]\!]$ for all $k \geq t$, which indicates that

$$f_t(D + [\![E^{\Gamma^*,\leq}]\!]) = f_{t+1}(D + [\![E^{\Gamma^*,\leq}]\!]) = \cdots .$$

Thus, the chain $f_1(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ stops.

Case 2. *$\pi(f_n) \neq 0$ for finitely many indices n.* By eliminating f_n of nonzero orders, we may assume that $\pi(f_n) = 0$ for all $n \geq 1$. Then $f_n(0) = f_{n+1}(0)g_n(0)$ for all $n \geq 1$; thus, by (ii), $f_1(0)E \subseteq f_2(0)E \subseteq \cdots$ is stationary. Therefore, we can find a positive integer m such that $f_n(0)E = f_m(0)E$ for all $n \geq m$. Since E is présimplifiable, $g_n(0)$ is a unit in E for all $n \geq m$, which indicates that for all $n \geq m$, $g_n(0) \in U(E) \cap D = U(D)$ by (i). Thus for all $n \geq m$, g_n is a unit in $D + [\![E^{\Gamma^*,\leq}]\!]$ by Lemma 1.1(i), which states that the chain $f_1(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ stops. In either case, the chain $f_1(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![E^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ is stationary, and thus, $D + [\![E^{\Gamma^*,\leq}]\!]$ satisfies ACCP. □

Theorem 1.7. *Let I be a nonzero proper ideal of a présimplifiable ring D and $(\Gamma, \leq)$ a nonzero strictly totally ordered monoid with $0 \leq \alpha$ for all $\alpha \in \Gamma$. Then the following statements hold.*

(i) *If D and Γ satisfy* ACCP, *then $D + [\![I^{\Gamma^*,\leq}]\!]$ satisfies* ACCP.
(ii) *If I contains a nonzero idempotent element, then the converse of* (i) *is also true.*

Proof. (i) Let $f_1(D+[\![I^{\Gamma^*,\leq}]\!]) \subseteq f_2(D+[\![I^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ be an ascending chain of nonzero principal ideals of $D + [\![I^{\Gamma^*,\leq}]\!]$. Then for each $n \geq 1$, there exists a $g_n \in D + [\![I^{\Gamma^*,\leq}]\!]$ such that $f_n = f_{n+1}g_n$. If f_n is a unit for some $n \geq 1$, then the result is obvious.

Assume that f_n is a nonunit for all $n \geq 1$. Note that $\pi(f_1) \geq \pi(f_2) \geq \cdots \geq 0$; thus, only two cases are possible: either $\pi(f_n) \neq 0$ for all $n \geq 1$ or $\pi(f_n) \neq 0$ for finitely many indices n.

We first consider the case $\pi(f_n) \neq 0$ for all $n \geq 1$. Let β_2 be the smallest member in $\mathrm{supp}(f_2)$ such that $\pi(f_1) = \beta_2 + \gamma_1$ and $f_2(\beta_2)g_1(\gamma_1) \neq 0$ for some $\gamma_1 \in \mathrm{supp}(g_1)$, and for each $n \geq 2$, we denote by β_{n+1} the smallest element in $\mathrm{supp}(f_{n+1})$ such that $\beta_n = \beta_{n+1} + \gamma_n$ and $f_{n+1}(\beta_{n+1})g_n(\gamma_n) \neq 0$ for some $\gamma_n \in \mathrm{supp}(g_n)$. Then $\pi(f_1) = \gamma_1 + \cdots + \gamma_n + \beta_{n+1}$ for all $n \geq 1$; thus, $\pi(f_1) \in \bigcap_{n\geq 1}(\gamma_1 + \cdots + \gamma_n + \Gamma)$. Since Γ satisfies ACCP, $\gamma_n \neq 0$ for finitely many n [**6,** Lemma 3.1] (or [**7,** Lemma 2.1]). Let m be the largest positive integer such that $\gamma_m \neq 0$. (If there is no such m, then we take $m = 0$ and $\beta_1 = \pi(f_1)$.) Then $\beta_k = \beta_{m+1}$ for all $k \geq m+1$. Since $\gamma_k = 0$ for all $k \geq m+1$, $f_k(\beta_k) = f_{k+1}(\beta_{k+1})g_k(0)$ by the minimality of β_k. Since D satisfies ACCP, $f_{m+1}(\beta_{m+1})D \subseteq f_{m+2}(\beta_{m+2})D \subseteq \cdots$ is stationary, and hence, there exists an integer $t \geq m+1$ such that $f_t(\beta_t)D = f_{t+1}(\beta_{t+1})D = \cdots$. Since D is présimplifiable, for all $n \geq t$, $g_n(0)$ is a unit in D; thus, by Lemma 1.1(ii), g_n is a unit in $D + [\![I^{\Gamma^*,\leq}]\!]$, which indicates that the chain $f_{m+1}(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq f_{m+2}(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ stops. Thus,

$$f_1(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq \cdots$$

stops.

Next, we assume that $\pi(f_n) \neq 0$ for finitely many indices n. By getting rid of f_n of nonzero order, we may assume that $\pi(f_n) = 0$ for all $n \geq 1$. Then $f_n(0) = f_{n+1}(0)g_n(0)$ for all $n \geq 1$; thus, $f_1(0)D \subseteq f_2(0)D \subseteq \cdots$ is an ascending chain of principal ideals of D. Since D satisfies ACCP, we can find an integer $t \geq 1$ such that $f_m(0)D = f_t(0)D$ for all $m \geq t$. Since D is présimplifiable, for all $n \geq t$, $g_n(0)$ is a unit in D. Thus, by Lemma 1.1(ii), g_n is a unit in $D + [\![I^{\Gamma^*,\leq}]\!]$ for all $n \geq t$, which means that the chain $f_1(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq f_2(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ stops.

(ii) We assume that $D + [\![I^{\Gamma^*,\leq}]\!]$ satisfies ACCP and let $d_1D \subseteq d_2D \subseteq \cdots$ be an ascending chain of nonzero principal ideals of D. Then $d_1(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq d_2(D + [\![I^{\Gamma^*,\leq}]\!]) \subseteq \cdots$ is an ascending chain of nonzero principal ideals of $D + [\![I^{\Gamma^*,\leq}]\!]$; thus, there exists a positive integer $n \geq 1$ such that $d_m(D + [\![I^{\Gamma^*,\leq}]\!]) = d_n(D + [\![I^{\Gamma^*,\leq}]\!])$ for all $m \geq n$. Hence, $d_mD = d_nD$ for all $m \geq n$, and thus, D satisfies ACCP. Let $\alpha_1 + \Gamma \subsetneq \alpha_2 + \Gamma \subsetneq \cdots$ be an infinite strictly ascending chain of principal ideals of Γ, and let a be a nonzero idempotent element in I. Note that, for each $n \geq 1$, $\alpha_n = \alpha_{n+1} + \gamma_n$ for some $\gamma_n \in \Gamma^*$; thus, we have $aX^{\alpha_n} = (aX^{\alpha_{n+1}})(aX^{\gamma_n}) \in aX^{\alpha_{n+1}}(D + [\![I^{\Gamma^*,\leq}]\!])$. Therefore, $aX^{\alpha_1}(D + [\![I^{\Gamma^*,\leq}]\!]) \subsetneq aX^{\alpha_2}(D + [\![I^{\Gamma^*,\leq}]\!]) \subsetneq \cdots$ is also an infinite strictly ascending chain of principal ideals of $D+[\![I^{\Gamma^*,\leq}]\!]$, which is absurd. Thus, Γ satisfies ACCP. $\square$

Note that if $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfies ACCP, then D satisfies ACCP without the condition on an ideal I. Hence, by applying Theorems 1.6 and 1.7 to the case when $(\Gamma,\leq)$ is the monoid $\mathbb{N}_0^n$ with lexicographic order, we recover the following.

Corollary 1.8 ([**4,** Propositions 4.18, 4.21]). *Let $D\subseteq E$ be an extension of commutative rings with identity and I a nonzero proper ideal of D.*

(i) *If E is présimplifiable, then $D+(X_1,\dots,X_n)E[\![X_1,\dots,X_n]\!]$ satisfies* ACCP *if and only if $U(E)\cap D=U(D)$, and, for each sequence $(e_n)_{n\geq 1}$ of nonzero nonunits of E with $e_n/e_{n+1}\in D$, the chain $e_1E\subseteq e_2E\subseteq\cdots$ stops.*

(ii) *If D is a présimplifiable ring, then $D+(X_1,\dots,X_n)I[\![X_1,\dots,X_n]\!]$ and D satisfy* ACCP *simultaneously.*

Let R be a commutative ring and M an R-module. The *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M:=\{(r,m)\mid r\in R \text{ and } m\in M\}$$

under usual addition and multiplication, defined as $(r_1,m_1)(r_2,m_2)=(r_1r_2,r_1m_2+r_2m_1)$ for all $(r_1,m_1),(r_2,m_2)\in R(+)M$. It is well known that (a,m) is a unit in $R(+)M$ if and only if a is a unit in R [**1,** Theorem 3.7] (or [**5,** Theorem 25.1(6)]).

We close this paper with an example which shows that the assumption on I in the second part of Theorem 1.7 is essential, which means that there is a ring $D+[\![I^{\Gamma^*,\leq}]\!]$ satisfying ACCP, even though Γ does not satisfy ACCP.

Example 1.9. Let R be any unique factorization domain which is not a field.

(1) Since R is an integral domain, $R(+)R$ is présimplifiable [**1,** Theorem 5.1(1)].

(2) Note that $(0)(+)R$ is a nilpotent ideal of $R(+)R$ of index 2; thus, $(0,0)$ is the only idempotent element in $(0)(+)R$.

(3) By Example 1.4(ii), $\mathbb{Q}_0$ does not satisfy ACCP.

(4) Let $D = R(+)R$ and $I = (0)(+)R$. Suppose that

$$f_1(D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]) \subseteq f_2(D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]) \subseteq \cdots$$

is an ascending chain of principal ideals of $D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]$. Then $(\pi(f_n))_{n\geq 1}$ is a decreasing sequence in Γ, and for each $n \geq 1$, $f_n = f_{n+1}g_n$ for some $g_n \in D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]$. Therefore, we have only two cases: either $\pi(f_n) \neq 0$ only for finitely many n or $\pi(f_n) \neq 0$ for all $n \geq 1$.

Case 1. *$\pi(f_n) \neq 0$ for finitely many n.* We may assume that $\pi(f_n) = 0$ for all $n \geq 1$. Let $f_n(0) = (a_n, b_n)$ and $g_n(0) = (c_n, d_n)$ for each $n \geq 1$. Since $f_n(0) = f_{n+1}(0)g_n(0)$ for all $n \geq 1$, we have $(a_1, b_1) = (a_{n+1}, b_{n+1})(c_1, d_1) \cdots (c_n, d_n)$ for all $n \geq 1$. Note that if $a_k = 0$ for some $k \geq 1$, then $a_i = 0$ for all $1 \leq i \leq k$; thus, we have two possibilities: either $a_n = 0$ for all $n \geq 1$ or there exists an integer $m \geq 1$ such that $a_n \neq 0$ for all $n \geq m$. If $a_n = 0$ for all $n \geq 1$, then the c_is are units, except for finitely many i, since R is a UFD and $b_1 = c_1 \cdots c_n b_{n+1}$ for all $n \geq 1$. Hence, almost all g_i are units by Lemma 1.1(ii), and thus, our chain should be stationary. If we have an integer $m \geq 1$ such that $a_n \neq 0$ for all $n \geq m$, then we have the same conclusion by applying the similar argument as the previous case to a_ns for $n \geq m$.

Case 2. *$\pi(f_n) \neq 0$ for all $n \geq 1$.* Since $f_n = f_{n+1}g_n$ for each $n \geq 1$, $f_1 = f_{n+1}h_n$, where $h_n = g_1 \cdots g_n \in D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]$. Note that

$$f_1(\pi(f_1)) = \sum_{(\alpha,\beta)\in X_{\pi(f_1)}(f_{n+1},h_n)} f_{n+1}(\alpha)h_n(\beta).$$

For $(\alpha, \beta) \in X_{\pi(f_1)}(f_{n+1}, h_n)$ with $\alpha \neq \pi(f_1)$, we have $f_{n+1}(\alpha)h_n(\beta) = 0$ since $\pi(f_1) > \alpha \geq \pi(f_{n+1}) > 0$ and $\beta > 0$. Therefore, $f_n(\pi(f_1)) \neq (0, 0)$. Let $f_n(\pi(f_1)) = (0, x_n)$ for each $n \geq 1$. If $(\pi(f_n))_{n\geq 1}$ contains an infinite strictly decreasing subsequence, then we may assume that $\pi(f_n) > \pi(f_{n+1})$ for all $n \geq 1$. Note that for all $n \geq 1$,

$$f_n(\pi(f_1)) = \sum_{(\alpha,\beta)\in X_{\pi(f_1)}(f_{n+1},g_n)} f_{n+1}(\alpha)g_n(\beta).$$

Hence $f_n(\pi(f_1)) = f_{n+1}(\pi(f_1))g_n(0)$ for all $n \geq 1$; thus, $x_1 = x_{n+1}c_1 \cdots c_n$, which implies that almost all c_i are units. Hence, by Lemma 1.1(ii), almost all g_i are units, and therefore, the chain $f_1(D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]) \subseteq f_2(D + [\![I^{\mathbb{Q}_0^*,\leq}]\!]) \subseteq \cdots$ stops. If we have a fixed integer $m \geq 1$ such that

$\pi(f_n) = \pi(f_m)$ for all $n \geq m$, then $x_n = x_{t+1}c_n \cdots c_t$ for all $t \geq n$. Hence, almost all of the c_i are units, and therefore, again by Lemma 1.1(ii), almost all g_i are units. Thus, $R(+)R + [\![((0)(+)R)^{\mathbb{Q}_0^*, \leq}]\!]$ satisfies ACCP.

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