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SOLVABILITY OF THE MIXED PROBLEM OF A HIGH-ORDER PDE WITH FRACTIONAL TIME DERIVATIVES, STURM-LIOUVILLE OPERATORS ON SPATIAL VARIABLES AND NON-LOCAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study the solvability of a mixed problem of a partial differential equation of high order with fractional derivatives with respect to time and with Sturm-Liouville operators with spatial variables and non-local boundary conditions; the solution is found as a series of eigenfunctions of the Sturm-Liouville operator with non-local boundary conditions.

1. Introduction. The spectral theory of operators finds numerous applications in various fields of mathematics and its applications.

An important part of the spectral theory of differential operators is the distribution of their eigenvalues. This classical question was studied for a second-order operator on a finite interval by Liouville and Sturm. Later, G.D. Birkhoff **[2, 3, 4]** studied the distribution of eigenvalues for an ordinary differential operator of arbitrary order on a finite interval with regular boundary conditions.

For quantum mechanics, it is especially interesting to distribute the eigenvalues of operators defined throughout the space and having a discrete spectrum. E.C. Titchmarsh **[16, 17, 18, 19, 20]** was the first to rigorously establish the formula for the distribution of the number of eigenvalues for a one-dimensional Sturm-Liouville operator on the whole axis with a potential growing at infinity. He also first strictly established the distribution formula for the Schrödinger operator. B.M. Levitan

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[9, 10, 11] deserves much credit for the improvement of the method of E.C. Titchmarsh.

In solving many problems of mathematical physics, there arises the need for the expansion of an arbitrary function in a Fourier series with respect to the Sturm-Liouville eigenvalues. The so-called regular case of the Sturm-Liouville problem corresponding to a finite interval and a continuous coefficient of the equation has been studied for a relatively long time and is usually described in detail in the manuals on the equations of mathematical physics and integral equations.

The Sturm-Liouville problem for the so-called singular case, as well as with non-local boundary conditions, is much less known.

As it is known, so-called fractal media are studied in solid state physics, in particular, the diffusion phenomena in them. In one of the models, the diffusion in a strongly porous medium is described by the heat conduction equation, but with a fractional derivative with respect to the time coordinate. In recent years, many authors have studied on fractional differential equations in **[5, 1, 15, 7]**

2. Formulation of the problem. In this work, we consider the equation of the form

$$D_{0t}^{\alpha}u(x,t)+\left(-\frac{\partial^2}{\partial x^2}+q(x)\right)^m u(x,t)=f(x,t), \qquad p-1<\alpha\le p, \quad p,m\in N, \tag{2.1}$$

with the initial conditions

$$\lim_{t\to 0} D_{0t}^{\alpha-k}u(x,t)=\varphi_k(x), \quad k=1,2,\dots,p \tag{2.2}$$

and the boundary conditions

$$\begin{gathered}\alpha\frac{\partial^{2i}u(0,t)}{\partial x^{2i}}+\beta\frac{\partial^{2i}u(\pi,t)}{\partial x^{2i}}=0,\\ \beta\frac{\partial^{2i+1}u(0,t)}{\partial x^{2i+1}}+\alpha\frac{\partial^{2i+1}u(\pi,t)}{\partial x^{2i+1}}=0, \quad i=0,1,\dots,m-1,\end{gathered} \tag{2.3}$$

where the functions $f(x,t)$, $\varphi_k(x)$, $k=1,2,\dots,p$ are functions that can be expanded in terms of the system of eigenfunctions $\{y_n(x), n\in\mathbb{Z}\}$ of the spectral problem

$$-y''(x)+q(x)y(x)=\lambda y(x), \tag{2.4}$$

$$\alpha y(0)+\beta y(\pi)=0, \quad \beta y'(0)+\alpha y'(\pi)=0. \tag{2.5}$$

Here for $\alpha<0$, the fractional integral D^α has the form

$$D_{at}^{\alpha}u(x,t)=\frac{\operatorname{sign}(t-a)}{\Gamma(-\alpha)}\int_a^t \frac{u(x,\tau)\cdot d\tau}{|t-\tau|^{\alpha+1}},$$

$D_{at}^{\alpha}u(x,t)=u(x,t)$ for $\alpha=0$, and for $p-1<\alpha\le p, p\in N$, the fractional derivative has the form

$$\begin{aligned} D_{at}^{\alpha}u(x,t)&=\operatorname{sign}^p(t-a)\frac{d^p}{dt^p}D_{at}^{\alpha-p}u(x,t)\\ &=\frac{\operatorname{sign}^{p+1}(t-a)}{\Gamma(p-\alpha)}\frac{d^p}{dt^p}\int_a^t\frac{u(x,\tau)\cdot d\tau}{|t-\tau|^{\alpha-p+1}}. \end{aligned}$$

We assume that $q(x)$ is sufficiently smooth on the segment $[0,\pi]$ and $q(x)\ge 0$.

In **[7]**, the problem (2.1), (2.2), (2.3) in case $m=1$ were considered.

3. Preliminaries. The spectral problem (2.4) and (2.5) was studied by many authors in the case of $|\alpha|=|\beta|$ (see, for example, **[12, 14, 13, 8]**). In order to simplify calculations, we confine ourselves to the case of $|\alpha|\ne|\beta|$, $\alpha\ne 0$, $\beta\ne 0$.

Theorem 3.1. *Let $\alpha\ne 0$, $\beta\ne 0$, $|\alpha|\ne|\beta|$ be real numbers, and*

$$\rho=\sqrt{\theta^2+2(\theta/\sqrt{2}+(\varphi+1)^s-1)^2}\cdot\sigma(s)<1,$$

where $\sigma(0)=1/\sqrt{2}$, $\sigma(s)=1$ for $s>0$,

$$\theta=\sqrt{2}\cdot\max_{x\in[0,\pi]}|e^{i\varphi x}-1|, \quad \lambda_n=s_n^2, \quad s_n=2n+\varepsilon_n\cdot\varphi,$$

$$\varphi=\frac{1}{\pi}\arccos\frac{-2\alpha\beta}{\alpha^2+\beta^2}, \quad \varepsilon_n=\varepsilon_{-n}=\pm 1,$$

for $n\in\mathbb{Z}$. Then the system of eigenfunctions

$$y_n(x)=\sqrt{\frac{2}{\pi}}\cdot\frac{\beta\cos s_nx+\varepsilon_n\cdot\operatorname{sign}(\beta^2-\alpha^2)\cdot\alpha\sin s_nx}{\sqrt{\alpha^2+\beta^2}\cdot\sqrt{1+|s_n|^{2s}}}, \quad n\in\mathbb{Z},$$

of the spectral problem (2.4) *and* (2.5) *forms at* $q(x)=0$ *the complete system in the Sobolev classes* $W_2^s(0,\pi)$.

Theorem (3.1) is proved in [**6**].

Lemma 3.2. *The operator*

$$Ly = -y'' + q(x)y$$

with the domain

$$D(L) = \big\{y(x) : y(x) \in C^2(0,\pi) \cap C^1[0,\pi], \\ y'' \in L_2[0,\pi],\ \alpha y(0) + \beta y(\pi) = 0,\ \beta y'(0) + \alpha y'(\pi) = 0\big\}$$

is a symmetric operator in the classes $L_2(0,\pi)$.

Proof. In fact, since f and $\overline{g}$ belong to $D(L)$, we obtain $Lf \in L_2(0,\pi)$, $L\overline{g} = \overline{Lg} \in L_2(0,\pi)$, and the second Green formula

$$\int_G (Lu \cdot v - u \cdot Lv)\, dx = -\int_{\partial G} \Big(\frac{\partial u}{\partial n} \cdot v - u \cdot \frac{\partial v}{\partial n}\Big)\, ds$$

at $u = f$ and $v = \overline{g}$ takes the form

$$\int_0^\pi (Lf(x)\overline{g(x)} - f(x)\overline{Lg})\, dx = -(f'(x)\overline{g(x)} - f(x)\overline{g'(x)})\big|_0^\pi.$$

Further, functions f and $\overline{g}$ satisfy the boundary conditions as follows:

$$\alpha f(0) + \beta f(\pi) = 0, \quad \beta f'(0) + \alpha f'(\pi) = 0,$$
$$\alpha\overline{g(0)} + \beta\overline{g(\pi)} = 0, \quad \beta\overline{g'(0)} + \alpha\overline{g'(\pi)} = 0.$$

By assumption, $\alpha \neq 0$ and $\beta \neq 0$. Therefore

$$f(0) \cdot \overline{g(\pi)} - f(\pi) \cdot \overline{g(0)} = 0$$

and

$$f'(0) \cdot \overline{g'(\pi)} - f'(\pi) \cdot \overline{g'(0)} = 0,$$

i.e., $f(0) \cdot \overline{g(\pi)} = f(\pi) \cdot \overline{g(0)}$ and $f'(0) \cdot \overline{g'(\pi)} = f'(\pi) \cdot \overline{g'(0)}$. Thus,

$$\frac{f(\pi)}{f(0)} = \frac{\overline{g(\pi)}}{\overline{g(0)}} = k_0 = -\frac{\alpha}{\beta}$$

and

$$\frac{f'(\pi)}{f'(0)} = \frac{\overline{g'(\pi)}}{\overline{g'(0)}} = k_1 = -\frac{\beta}{\alpha}, \quad k_0 \cdot k_1 = 1.$$

So $f(\pi) = k_0 \cdot f(0)$, $\overline{g(\pi)} = k_0 \cdot \overline{g(0)}$, and $f'(\pi) = k_1 \cdot f'(0)$, $\overline{g'(\pi)} = k_1 \cdot \overline{g'(0)}$. Thus,

$$\begin{aligned}
&\int_0^\pi (Lf(x)\overline{g(x)} - f(x)\overline{Lg})\,dx \\
&\qquad = -(f'(x) \cdot \overline{g(x)} - f(x) \cdot \overline{g'(x)})\Big|_0^\pi \\
&\qquad = -(f'(\pi) \cdot \overline{g(\pi)} - f(\pi) \cdot \overline{g'(\pi)}) + (f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}) \\
&\qquad = -(f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}) + (f'(0) \cdot \overline{g(0)} - f(0) \cdot \overline{g'(0)}) = 0.
\end{aligned}$$

So, $(Lf, g) = (f, Lg)$ for any $f, g \in D(L)$. □

Lemma 3.3. *The eigenfunctions $y(x)$ and $z(x)$ of the operator L corresponding to eigenvalues λ and μ are orthogonal if $\lambda \neq \mu$.*

Proof. Let $y(x)$ be an eigenfunction of L corresponding to λ, and $z(x)$ be an eigenfunction of L corresponding to μ. It means that

$$Ly = \lambda y, \quad Lz = \mu z.$$

We obtain from here

$$\begin{aligned}
(Ly, z) &= (\lambda y, z) = \lambda(y, z), \\
(y, Lz) &= (y, \mu z) = \mu(y, z).
\end{aligned}$$

But $(Ly, z) = (y, Lz)$. Particularly, $(Ly, y) = (y, Ly) = \overline{(Ly, y)}$. Hence, (Ly, y) is a real number. That's why all eigenvalues of the symmetric operator L are real, and subtracting term by term previous two equalities, we obtain

$$(\lambda - \mu)(y, z) = 0, \quad \lambda \neq \mu,$$

and $(y, z) = 0$. □

Rewrite the equation (2.4) in the form

$$y'' + s_n^2 y = (s_n^2 + q(x) - \lambda)y.$$

If we denote $f(x) = (s_n^2 + q(x) - \lambda)y(x)$ where

$$s_n = 2n + \varepsilon_n\varphi, \quad \varphi = \frac{1}{\pi}\arccos\frac{-2\alpha\beta}{\alpha^2+\beta^2}, \quad \varepsilon_n = \pm 1, \quad \varepsilon_{-n} = \varepsilon_n, \quad n \in \mathbb{Z},$$

$$\overline{y}_n(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\alpha^2+\beta^2}}(\beta\cos s_n x + \varepsilon_n \operatorname{sign}(\beta^2-\alpha^2)\alpha\sin s_n x),$$

we obtain the boundary value problem

$$y'' + s_n^2 y = f(x), \tag{3.1}$$

$$\begin{aligned} \alpha y(0) + \beta y(\pi) &= 0,\\ \beta y'(0) + \alpha y'(\pi) &= 0. \end{aligned} \tag{3.2}$$

The general solution of the homogeneous equation corresponding to (3.1) has the form

$$y = C_1\cos s_n x + C_2 \sin s_n x.$$

We look for a solution of (3.1) in the form

$$y = C_1\cos s_n x + C_2 \sin s_n x, \tag{3.3}$$

where $C_1 = C_1(x)$, $C_2 = C_2(x)$ are still unknown functions of x. We obtain the following system to determine them:

$$\begin{aligned} C_1'\cos s_n x + C_2'\sin s_n x &= 0,\\ C_1'\sin s_n x - C_2'\cos s_n x &= -\frac{1}{s_n}f(x). \end{aligned}$$

Solving the system with respect to $C_1'(x)$, $C_2'(x)$, we get

$$C_1' = -\frac{1}{s_n}f(x)\sin s_n x, \quad C_2' = \frac{1}{s_n}f(x)\cos s_n x;$$

what follows is

$$C_1 = \bar{A}_1 - \frac{1}{s_n}\int_0^x f(\tau)\sin s_n\tau\, d\tau, \quad C_2 = \bar{A}_2 + \frac{1}{s_n}\int_0^x f(\tau)\cos s_n\tau\, d\tau,$$

where $\bar{A}_1$, $\bar{A}_2$ are arbitrary constants. Substituting the found values of $C_1(x)$ and $C_2(x)$ in (3.3), we obtain the general solution of (3.1):

$$y(x) = \overline{y}_n(x) + A_1\cos s_n x + A_2\sin s_n x + \frac{1}{s_n}\int_0^x \sin s_n(x-\tau)f(\tau)\, d\tau,$$

where A_1, A_2 are arbitrary constants.

Since $f(x) = (s_n^2 + q(x) - \lambda)y(x)$, we obtain the integral equation

$$y(x) = \overline{y}_n(x) + A_1 \cos s_n x + A_2 \sin s_n x + \frac{1}{s_n}\int_0^x \sin s_n(x-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau. \tag{3.4}$$

We get

$$\begin{aligned}
y(0) &= \overline{y}_n(0) + A_1, \\
y(\pi) &= \overline{y}_n(\pi) + A_1 \cos \pi s_n + A_2 \sin \pi s_n \\
&\qquad + \frac{1}{s_n}\int_0^\pi \sin s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau, \\
y'(0) &= \overline{y}_n'(0) + s_n A_2, \\
y'(\pi) &= \overline{y}_n'(\pi) - s_n A_1 \sin \pi s_n + s_n A_2 \cos \pi s_n \\
&\qquad + \int_0^\pi \cos s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau.
\end{aligned}$$

Then the condition (3.2) takes the form

$$\begin{aligned}
&\alpha y(0) + \beta y(\pi) \\
&\quad = \alpha A_1 + \beta A_1 \cos \pi s_n + \beta A_2 \sin \pi s_n \\
&\qquad + \frac{\beta}{s_n}\int_0^\pi \sin s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau = 0, \\
&\beta y'(0) + \alpha y'(\pi) \\
&\quad = \beta s_n A_2 - \alpha s_n A_1 \sin \pi s_n + \alpha s_n A_2 \cos \pi s_n \\
&\qquad + \alpha \int_0^\pi \cos s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau = 0,
\end{aligned}$$

i.e., we obtain the following system of non-homogeneous linear equations for the determination of A_1 and A_2:

$$\begin{aligned}
&(\alpha + \beta \cos \pi s_n)A_1 + \beta \sin \pi s_n A_2 \\
&\qquad = -\frac{\beta}{s_n}\int_0^\pi \sin s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau, \\
&-\alpha \sin \pi s_n A_1 + (\beta + \alpha \cos \pi s_n)A_2 \\
&\qquad = -\frac{\alpha}{s_n}\int_0^\pi \cos s_n(\pi-\tau)(s_n^2 + q(\tau) - \lambda)y(\tau)\, d\tau.
\end{aligned} \tag{3.5}$$

The determinants of this system are

$$\Delta = 2\alpha\beta + (\alpha^2+\beta^2)\cos\pi s_n = 0,$$
$$\Delta_1 = \frac{\beta}{s_n}\int_0^\pi [\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau)](s_n^2+q(\tau)-\lambda)y(\tau)\,d\tau,$$
$$\Delta_2 = \frac{-\alpha}{s_n}\int_0^\pi [\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau)](s_n^2+q(\tau)-\lambda)y(\tau)\,d\tau.$$

In the case of $q(x)=0$ the system of equations is homogeneous at $\lambda_n = s_n^2$, and hence, non-trivial solutions of the problem (2.4), (2.5) are possible only at

$$\Delta = 2\alpha\beta + (\alpha^2+\beta^2)\cos\pi s_n = 0.$$

Eigenvalues and eigenfunctions in this case were studied in [**6**].

When $q(x)\neq 0$, $\Delta = 0$, the system (3.5) is compatible, and hence, non-trivial solutions of the problem (2.4), (2.5) are possible only at

$$\Delta_1 = \frac{\beta}{s_n}\int_0^\pi [\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau)](s_n^2+q(\tau)-\lambda)y(\tau)\,d\tau = 0,$$
$$\Delta_2 = \frac{-\alpha}{s_n}\int_0^\pi [\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau)](s_n^2+q(\tau)-\lambda)y(\tau)\,d\tau = 0.$$

Therefore,

$$\lambda_n = s_n^2 + \frac{\int_0^\pi (\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau))q(\tau)y_n(\tau)\,d\tau}{\int_0^\pi (\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau))y_n(\tau)\,d\tau},$$
$$\lambda_n = s_n^2 + \frac{\int_0^\pi (\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau))q(\tau)y_n(\tau)\,d\tau}{\int_0^\pi (\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau))y_n(\tau)\,d\tau}.$$

And the following relation is valid:

$$\frac{\int_0^\pi (\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau))q(\tau)y_n(\tau)\,d\tau}{\int_0^\pi (\alpha\sin s_n\tau - \beta\sin s_n(\pi-\tau))y_n(\tau)\,d\tau} = \frac{\int_0^\pi (\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau))q(\tau)y_n(\tau)\,d\tau}{\int_0^\pi (\beta\cos s_n\tau + \alpha\cos s_n(\pi-\tau))y_n(\tau)\,d\tau}.$$

Applying the Fredholm theory for integral equations with continuous kernels, we obtain that the problem on eigenvalues (2.4),(2.5) has at most a countable number of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ that do not have a finite limit point. Further, similarly to the case of local boundary conditions, one can obtain the following asymptotical formulas:

$$\sqrt{\lambda_n} = s_n + \frac{c_0}{s_n} + \frac{c_1}{s_n^3} + O\left(\frac{1}{s_n^4}\right),$$

where c_0, c_1 are constants (see, for example in **[12]**). This implies that there exists a constant $M > 0$ such that the inequalities

$$|\lambda_n - s_n^2| \leq M$$

hold. Since

$$\begin{aligned} A_2 &= \frac{1}{\beta \sin \pi s_n} \\ &\quad \times \left[-\frac{\beta}{s_n} \int_0^\pi \sin s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau - (\alpha+\beta\cos\pi s_n) A_1 \right], \\ A_2 &= \frac{1}{(\beta+\alpha\cos\pi s_n)} \\ &\quad \times \left[-\frac{\alpha}{s_n} \int_0^\pi \cos s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau + \alpha \sin\pi s_n A_1 \right], \end{aligned}$$

we obtain the equation

$$\begin{aligned} &[-(\beta+\alpha\cos\pi s_n)(\alpha+\beta\cos\pi s_n) - \alpha\beta\sin^2\pi s_n] A_1 \\ &\qquad = \beta\sin\pi s_n \left[-\frac{\alpha}{s_n} \int_0^\pi \cos s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right] \\ &\qquad\qquad + (\beta+\alpha\cos\pi s_n) \left[\frac{\beta}{s_n} \int_0^\pi \sin s_n(\pi-\tau) \right. \\ &\qquad\qquad\qquad \left. \times (s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right]. \end{aligned}$$

We have

$$\begin{aligned} A_1 &= \beta\sin\pi s_n \left[-\frac{\alpha}{s_n} \int_0^\pi \cos s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right] \\ &\qquad + (\beta+\alpha\cos\pi s_n) \left[\frac{\beta}{s_n} \int_0^\pi \sin s_n(\pi-\tau) \right. \\ &\qquad\qquad \left. \times (s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right]. \end{aligned}$$

Further, A_1 is an arbitrary constant, and we have

$$\beta \sin \pi s_n \left[-\frac{\alpha}{s_n} \int_0^{\pi} \cos s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right] \\ + (\beta + \alpha \cos \pi s_n) \left[\frac{\beta}{s_n} \int_0^{\pi} \sin s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right] = 0.$$

Hence, we obtain from solutions (3.4),

$$y_n(x) = \overline{y}_n(x) + A_1[\beta \sin s_n(\pi - x) - \alpha \sin s_n x] \frac{1}{\beta \sin \pi s_n} \\ - \frac{\sin s_n x}{\sin \pi s_n} \int_0^{\pi} \sin s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \\ + \frac{1}{s_n} \int_0^{x} \sin s_n(x-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau.$$

We take $A_1 = 0$. Then

$$y_n(x) = \overline{y}_n(x) + \frac{1}{s_n} \left[\int_0^{x} \sin s_n(x-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right. \\ \left. - \frac{\sin s_n x}{\sin \pi s_n} \int_0^{\pi} \sin s_n(\pi-\tau)(s_n^2+q(\tau)-\lambda_n) y_n(\tau)\, d\tau \right]. \tag{3.6}$$

Let $\sigma_n = \max_{0 \le x \le \pi} |y_n(x)|$. Then (3.6) implies

$$\sigma_n \le 2 + \frac{\sigma_n}{|s_n|} \cdot C \cdot \left(1 + \int_0^{\pi} |q(\tau)|\, d\tau \right),$$

where C is a positive constant. There exists a number n_0 such that

$$1 - \frac{C}{|s_n|} \left(1 + \int_0^{\pi} |q(\tau)|\, d\tau \right) > 0 \quad \text{for } n \ge n_0.$$

That's why for $n \ge n_0$, the inequalities

$$\sigma_n \le \frac{2}{1 - C/|s_n| \left(1 + \int_0^{\pi} |q(\tau)|\, d\tau \right)} \le \text{constant}$$

are valid, i.e., $|y_n(x)| \le \text{constant}$ or $y_n(x) = O(1/n)$ at $n \to \infty$. Returning again to (3.6), we obtain

$$|y_n(x) - \overline{y}_n(x)| \le \frac{M_1}{s_n},$$

i.e., $y_n(x) = \overline{y}_n(x) + O(1/n)$ at $n \to \infty$.

Theorem 3.4. *Let $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ be real numbers, and*

$$\theta = \sqrt{2} \cdot \max_{x \in [0,\pi]} |e^{i\varphi x} - 1| < 1, \quad s_n = 2n + \varepsilon_n \cdot \varphi,$$

$$\varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \qquad \varepsilon_n = \varepsilon_{-n} = \pm 1,$$

at $n \in \mathbb{Z}$. Then the system of eigenfunctions

$$y_n(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{\beta \cos s_n x + \varepsilon_n \cdot \operatorname{sign}(\beta^2 - \alpha^2) \cdot \alpha \sin s_n x}{\sqrt{\alpha^2 + \beta^2}} + O\Big(\frac{1}{n}\Big), \quad n \in \mathbb{Z},$$

of the spectral problem (2.4), (2.5) *forms the complete orthogonal system in classes $L_2(0,\pi)$.*

Proof. Since the system of functions

$$\overline{y}_n(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{\beta \cos s_n x + \varepsilon_n \cdot \operatorname{sign}(\beta^2 - \alpha^2) \cdot \alpha \sin s_n x}{\sqrt{\alpha^2 + \beta^2}}, \quad n \in \mathbb{Z},$$

forms the complete orthonormal system in the Hilbert space $L_2(0,\pi)$, and the system of orthogonal eigenfunctions

$$y_n(x) = \frac{1}{\sqrt{\pi}} \cdot \frac{\beta \cos s_n x + \varepsilon_n \cdot \operatorname{sign}(\beta^2 - \alpha^2) \cdot \alpha \sin s_n x}{\sqrt{\alpha^2 + \beta^2}} + O\Big(\frac{1}{n}\Big), \quad n \in \mathbb{Z},$$

of the spectral problem (2.4), (2.5) is quadratically close to the system $\{\overline{y}_n(x)\}_{n\in\mathbb{Z}}$ in classes $L_2(0,\pi)$, what, according to N.K. Barry theorem, implies that the system of eigenfunctions $\{y_n(x)\}_{n\in\mathbb{Z}}$ of the spectral problem (2.4), (2.5) forms the Riesz basis in the class $L_2(0,\pi)$. Since the system $\{y_n(x)\}_{n\in\mathbb{Z}}$ of the spectral problem (2.4), (2.5) is orthogonal, we obtain orthogonality of this system in classes $L_2(0,\pi)$. $\square$

4. Main results. In this section, we will give the most general case of the works done in **[7]**.

Theorem 4.1. *Let $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ be real numbers, and*

$$\theta = \sqrt{2} \cdot \max_{x \in [0,\pi]} |e^{i\varphi x} - 1| < 1, \quad s_n = 2n + \varepsilon_n \cdot \varphi,$$

$$\varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \qquad \varepsilon_n = \varepsilon_{-n} = \pm 1,$$

at $n \in \mathbb{Z}$. Then the solution of the problem (2.1), (2.2), (2.3) *exists, it is unique and is represented in the form of the series*

$$u(x,t) = \sum_{n=-\infty}^{\infty} \Big[\sum_{k=1}^{p} \varphi_{kn} t^{\alpha-k} E_{\alpha,\alpha-k+1}(-\lambda_n \cdot t^{\alpha}) + \int_0^t (t-\tau)^{\alpha-1} \cdot E_{\alpha,\alpha}[-\lambda_n (t-\tau)^{\alpha}] f_n(\tau)\, d\tau \Big] \cdot y_n(x), \tag{4.1}$$

where the coefficients are determined by

$$E_{\alpha,\alpha-k+1}(-\lambda_n \cdot t^{\alpha}) = \sum_{j=0}^{\infty} \frac{(-\lambda_n \cdot t^{\alpha})^j}{\Gamma(\alpha j + \alpha - k + 1)},$$

$$E_{\alpha,\alpha}(-\lambda_n \cdot (t-\tau)^{\alpha}) = \sum_{j=1}^{\infty} \frac{(-\lambda_n)^{j-1} \cdot (t-\tau)^{\alpha(j-1)}}{\Gamma(\alpha \cdot j)},$$

$$f(x,t) = \sum_{n=-\infty}^{\infty} f_n(t) \cdot y_n(x),$$

$$\varphi_k(x) = \sum_{n=-\infty}^{\infty} \varphi_{kn} \cdot y_n(x), k = 1, 2, \ldots, p.$$

Proof. Since the system of functions $\{y_n(x)\}_{n \in \mathbb{Z}}$ is a complete orthogonal system in classes $L_2(0,\pi)$, any function from $L_2(0,\pi)$ can be represented as a convergent Fourier series in this system. For any $t > 0$, expand the solution $u(x,t)$ of the problem (2.1), (2.2), (2.3) into the Fourier series in eigenfunctions $\{y_n(x)\}_{n \in \mathbb{Z}}$ of the spectral problem (2.4), (2.5):

$$u(x,t) = \sum_{n=-\infty}^{\infty} T_n(t) \cdot y_n(x), \quad T_n(t) = (u(x,t), y_n(x)). \tag{4.2}$$

In view of (2.1) and (2.2), unknown functions $T_n(t)$ must satisfy the equation

$$D_{0t}^{\alpha} T_n(t) + \lambda_n T_n(t) = f_n(t), \quad p-1 < \alpha \leq p, \quad p \in N, \tag{4.3}$$

with initial conditions

$$\lim_{t \to 0} D_{0t}^{\alpha-k} T_n(t) = \varphi_{kn}, \quad k = 1, 2, \ldots, p, \quad n \in \mathbb{Z}. \tag{4.4}$$

The solution of the Cauchy problem (4.3), (4.4) has the form

$$T_n(t) = \sum_{k=1}^{p} \varphi_{kn} t^{\alpha-k} E_{\alpha,\alpha-k+1}(-\lambda_n \cdot t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} \cdot E_{\alpha,\alpha}[-\lambda_n (t-\tau)^\alpha] f_n(\tau)\, d\tau, \tag{4.5}$$

where coefficients are determined by

$$E_{\alpha,\alpha-k+1}(-\lambda_n \cdot t^\alpha) = \sum_{j=0}^{\infty} \frac{(-\lambda_n \cdot t^\alpha)^j}{\Gamma(\alpha j + \alpha - k + 1)},$$
$$E_{\alpha,\alpha}(-\lambda_n \cdot (t-\tau)^\alpha) = \sum_{j=1}^{\infty} \frac{(-\lambda_n)^{j-1} \cdot (t-\tau)^{\alpha(j-1)}}{\Gamma(\alpha \cdot j)}.$$

After substituting (4.5) into (4.2), we obtain the unique solution of the problem (2.1), (2.2), (2.3) in the form of the series (4.1).

Let $m > 1$. Consider the mixed problem (2.1), (2.2), (2.3). If we look for a solution $u(x,t)$ to problem (2.1), (2.2), (2.3) in the form of a Fourier series expansion:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) \cdot v_n(x),$$

where $T_n(t) = (u(x,t), v_n(x))$ are the coefficients of the series, $\{v_n(x)\}_{n=1}^{\infty}$ is the system of eigenfunctions of the spectral problem

$$\left(-\frac{d^2}{dx^2} + q(x)\right)^m v(x) = \mu v(x), \tag{4.6}$$

$$\begin{aligned} \alpha\frac{\partial^{2i} v(0)}{\partial x^{2i}} + \beta\frac{\partial^{2i} v(\pi)}{\partial x^{2i}} &= 0, \\ \beta\frac{\partial^{2i+1} v(0)}{\partial x^{2i+1}} + \alpha\frac{\partial^{2i+1} v(\pi)}{\partial x^{2i+1}} &= 0, \quad i = 0, 1, \dots, m-1, \end{aligned} \tag{4.7}$$

where μ is a constant introduced via separation of variables.

The differential operator L^m is generated by the differential expression $l^{(m)}(v(x)) = (-d^2/dx^2 + q(x))^m v(x)$ on

$$D(L) = \Bigg\{ v(x) : v(x) \in C^{2m}(0,\pi) \cap C^{2m-1}[0,\pi], l^{(m)}(v(x)) \in L_2[0,\pi],$$
$$\alpha \frac{\partial^{2i} v(0)}{\partial x^{2i}} + \beta \frac{\partial^{2i} v(\pi)}{\partial x^{2i}} = 0,$$
$$\beta \frac{\partial^{2i+1} v(0)}{\partial x^{2i+1}} + \alpha \frac{\partial^{2i+1} v(\pi)}{\partial x^{2i+1}} = 0, \quad i = 0, 1, \ldots, m-1 \Bigg\}.$$

Similarly, as in Lemma 3.2 it can be shown that the operator L^m is a symmetric and positive operator in space $L_2(0,\pi)$.

The eigenvalues of problem (4.6), (4.7) for $\mu_n \geq 0$ and each $\mu_n = \lambda_n^m$ corresponds to one eigenvalue of problem (2.4), (2.5), and the eigenfunctions $\{v_n(x)\}_{n=1}^{\infty}$ of problem (4.6), (4.7) and eigenfunctions $\{y_n(x)\}_{n=1}^{\infty}$ of problem (2.4), (2.5) coincide, i.e., $v_n(x) \equiv y_n(x)$ for $n \in N$. □

Theorem 4.2. *Let $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \neq |\beta|$ be real numbers, and*

$$\theta = \sqrt{2} \cdot \max_{x \in [0,\pi]} |e^{i\varphi x} - 1| < 1, \quad s_n = 2n + \varepsilon_n \cdot \varphi,$$
$$\varphi = \frac{1}{\pi} \arccos \frac{-2\alpha\beta}{\alpha^2 + \beta^2}, \qquad \varepsilon_n = \varepsilon_{-n} = \pm 1,$$

at $n \in \mathbb{Z}$. Then the solution of the problem (2.1), (2.2), (2.3) *exists, it is unique and is represented in the form of the series*

$$u(x,t) = \sum_{n=-\infty}^{\infty} \Bigg[\sum_{k=1}^{p} \varphi_{kn} t^{\alpha-k} E_{\alpha,\alpha-k+1}(-\lambda_n^m \cdot t^\alpha) + \int_0^t (t-\tau)^{\alpha-1} \cdot E_{\alpha,\alpha}[-\lambda_n^m (t-\tau)^\alpha] f_n(\tau)\, d\tau \Bigg] \cdot y_n(x), \tag{4.8}$$

where the coefficients are determined by

$$E_{\alpha,\alpha-k+1}(-\lambda_n^m \cdot t^\alpha) = \sum_{j=0}^{\infty} \frac{(-\lambda_n^m \cdot t^\alpha)^j}{\Gamma(\alpha j + \alpha - k + 1)},$$
$$E_{\alpha,\alpha}(-\lambda_n^m \cdot (t-\tau)^\alpha) = \sum_{j=1}^{\infty} \frac{(-\lambda_n^m)^{j-1} \cdot (t-\tau)^{\alpha(j-1)}}{\Gamma(\alpha \cdot j)},$$

$$f(x,t) = \sum_{n=-\infty}^{\infty} f_n(t) \cdot y_n(x),$$

$$\varphi_k(x) = \sum_{n=-\infty}^{\infty} \varphi_{kn} \cdot y_n(x), \quad k = 1, 2, \dots, p.$$

5. Conclusion. We have studied the solvability of the mixed problem of a partial differential equation of high order with fractional derivatives with respect to time and with Sturm-Liouville operators with spatial variables and non-local boundary conditions. The solution is found in the form of a series of eigenfunctions of the Sturm-Liouville operator with non-local boundary conditions.

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REFERENCES

1. E. Bazhlekova, *Subordination principle for a class of fractional order differential equations*, Mathematics **3** (2015), 412–427, doi 10.3390/math3020412.

2. G.D. Birkhoff, *On the asymptotic character of the solutions of certain linear differential equations containing a parameter*, Trans. Amer. Math. Soc. **9** (1908), 219–231.

3. ______, *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Amer. Math. Soc. **9** (1908), 373–395.

4. ______, *Existence and oscillation theorem for a certain boundary value problem*, Trans. Amer. Math. Soc. **10** (1909), 259–270.

5. J. Jiang, Y. Feng, and S. Li, *Exact solutions to the fractional differential equations with mixed partial derivatives.* Axioms **7** (2018), 10 pp.

6. S.G. Kasimov, S.K. Ataev, *On completeness of the system of orthonormal vectors of a generalized spectral problem* (in Russian), Uzbek Math. J. (2009), no. 2, 101–111.

7. S.G. Kasimov, S.K. Ataev, and U.S. Madrakhimov, *On the solvability of a mixed problem for a fractional-order partial differential equation with Sturm-Liouville operators with nonlocal boundary conditions* (in Russian), Uzbek. Mat. Zh. **17** (2016), 158–169.

8. A.G. Kostychenko and I.S. Sargsyan, *Distribution of eigenvalues: self-adjoint ordinary differential operators*, Nauka, Moscow (1979).

9. B.M. Levitan, Разложение по собственным функциям дифференциалных уравнени второго порядка (*Expansion in characteristic functions of differential* *equations of the second order*), Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad (1950).

10. ______, *On expansion in eigenfunctions of the Laplace operator* (in Russian), Mat. Sb. N.S. **35** (1954), 267–316.

11. ______, *On expansion in eigenfunctions of the Schrödinger operator in the case of a potential increasing without bound* (in Russian), Dokl. Akad. Nauk SSSR N.S. **103** (1955), 191–194.

12. B.M. Levitan and I.S. Sargsyan, *Introduction to spectral theory: self-adjoint ordinary differential operators*, Nauka, Moscow (1979); English translation: Transl. Math. Monographs **39**, Amer. Math. Soc. (1975).

13. ______, *Sturm-Liouville and Dirac operators*, Nauka, Moscow (1988).

14. M.A. Naimark, *Linear differential operators*, Nauka, Moscow (1969).

15. S. Salahshour, A. Ahmadian, N. Senu, D. Baleanu, and P. Agarwal *On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem*, Entropy **17** (2015), 885–902.

16. E.C. Titchmarsh, *Eigenfunction expansions*, volumes I and II, Oxford Univ. Press (1953, 1958).

17. ______, *On the asymptotic distribution of eigenvalues*, Quart. J. Math. **5** (1954), 228–240.

18. ______, *On the eigenvalues in problems with spherical symmetry*, Proc. Roy. Soc. London. Ser. A **245** (1958), 147–155.

19. ______, *On the eigenvalues in problems with spherical symmetry* II, Proc. Roy. Soc. London Ser. A **251** (1959), 46–54.

20. ______, *On the eigenvalues in problems with spherical symmetry* III, Proc. Roy. Soc. London. Ser. A **252** (1959), 436–444.

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