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NON-LINEAR λ-JORDAN TRIPLE $*$-DERIVATION ON PRIME $*$-ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a prime $*$-algebra and Φ a λ-Jordan triple derivation on A, that is, for every $A, B, C \in \mathcal{A}$,

$$\Phi(A\diamond_\lambda B\diamond_\lambda C) = \Phi(A)\diamond_\lambda B\diamond_\lambda C+A\diamond_\lambda \Phi(B)\diamond_\lambda C+A\diamond_\lambda B\diamond_\lambda \Phi(C),$$

where $A \diamond_\lambda B = AB + \lambda BA^*$ such that a complex scalar $|\lambda| \neq 0, 1$, and Φ is additive. Moreover, if $\Phi(I)$ is self-adjoint, then Φ is a $*$-derivation.

1. Introduction. Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, we denote by $A\diamond B = AB+BA^*$ and $[A,B]_* = AB-BA^*$, the $*$-Jordan product and the $*$-Lie product, respectively. These products have recently attracted many authors' attention (for example, see **[2, 7, 10, 11]**).

In addition, some authors have considered triple $*$-products of three elements. For example, the authors in **[4]** considered two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ such that one of them has no central abelian projections. Let $\lambda \neq -1$ be a non-zero complex number, and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a, not necessarily linear, bijection with $\Phi(I) = I$. Then, Φ preserves the following condition

$$\Phi(A \diamond_\lambda B \diamond_\lambda C) = \Phi(A) \diamond_\lambda \Phi(B) \diamond_\lambda \Phi(C), \tag{1.1}$$

for $A, B, C \in \mathcal{A}$ if and only if one of the following statements holds:

- $\lambda \in \mathbb{R}$, and there exists a central projection $P \in \mathcal{A}$ such that $\Phi(P)$ is a central projection in $\mathcal{B}$, $\Phi|_{\mathcal{A}P} : \mathcal{A}P \to \mathcal{B}\Phi(P)$ is a linear $*$-isomorphism and $\Phi|_{\mathcal{A}(I-P)} : \mathcal{A}(I-P) \to \mathcal{B}(I-\Phi(P))$ is a conjugate linear $*$-isomorphism.
- $\lambda \notin \mathbb{R}$, and Φ is a linear $*$-isomorphism.

The map Φ which holds in (1.1) preserves the λ-Jordan triple product. We should note that $\diamond_\lambda$ is not necessarily associative. In order to clarify

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this, we mention that

(1.2)
$$A\diamond_\lambda B\diamond_\lambda C := (A\diamond_\lambda B)\diamond_\lambda C = ABC+\lambda(BA^*C+CB^*A^*)+|\lambda|^2CAB^*.$$

For more papers regarding maps preserving the triple product, the interested reader may refer to **[3, 5, 8, 12]**

We define λ-Jordan $*$-product by $A\diamond_\lambda B = AB+\lambda BA^*$. We say that the map Φ (not necessarily linear) with the property of $\Phi(A\diamond_\lambda B)=\Phi(A)\diamond_\lambda B + A\diamond_\lambda \Phi(B)$ is a λ-Jordan $*$-derivation map. It is clear that, for $\lambda=-1$ and $\lambda=1$, the λ-Jordan $*$-derivation map is a $*$-Lie derivation and a $*$-Jordan derivation, respectively **[1]**. We should mention here that, whenever we say Φ is a derivation, it means that $\Phi(AB)=\Phi(A)B+A\Phi(B)$.

Recently, Yu and Zhang **[14]** proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang **[6]** investigated a non-linear λ-Jordan $*$-derivation. They showed that, if $\mathcal{A}\subseteq\mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scaler, then

$$\Phi:\mathcal{A}\longrightarrow\mathcal{B}(\mathcal{H})$$

is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

In **[13]**, the authors showed that the $*$-Jordan derivation map, i.e., $\phi(A\diamond_1 B)=\phi(A)\diamond_1 B + A\diamond_1\phi(B)$, on every factor von Neumann algebra $\mathcal{A}\subseteq\mathcal{B}(\mathcal{H})$ is an additive $*$-derivation.

The authors in **[9]** introduced the concept of skew Lie triple derivations. A map

$$\Phi:\mathcal{A}\longrightarrow\mathcal{A}$$

is a nonlinear skew Lie triple derivation if

$$\Phi([[A,B]_*,C]_*)=[[\Phi(A),B]_*,C]_*+[[A,\Phi(B)]_*,C]_*+[[A,B]_*,\Phi(C)]_*$$

for all $A,B,C\in\mathcal{A}$, where $[A,B]_*=AB-BA^*$. They showed that, if Φ preserves the above characterizations on factor von Neumann algebras, then Φ is additive $*$-derivation.

In this paper, motivated by the above results, we consider a map Φ on a prime $*$-algebra $\mathcal{A}$ which holds under the following conditions

$$\Phi(A \diamond_\lambda B \diamond_\lambda C) = \Phi(A) \diamond_\lambda B \diamond_\lambda C + A \diamond_\lambda \Phi(B) \diamond_\lambda C) + A \diamond_\lambda B \diamond_\lambda \Phi(C),$$

where $A \diamond_\lambda B = AB + \lambda BA^*$ is such that a complex scalar $|\lambda| \neq 0, 1$, and Φ is additive. Also, if $\Phi(I)$ is self-adjoint, then Φ is a $*$-derivation. We say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$, if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

2. Main results.

Our first theorem is as follows.

Theorem 2.1. *Let $\mathcal{A}$ be a prime $*$-algebra with unit I and a nontrivial projection. Then, the map $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies the following condition*

$$\Phi(A \diamond_\lambda B \diamond_\lambda C) = \Phi(A) \diamond_\lambda B \diamond_\lambda C + A \diamond_\lambda \Phi(B) \diamond_\lambda C + A \diamond_\lambda B \diamond_\lambda \Phi(C), \tag{2.1}$$

where $A \diamond_\lambda B = AB + \lambda BA^*$ is such that a complex scalar, $|\lambda| \neq 0, 1$, is additive.

Proof. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j,\ i, j = 1, 2$. Then,

$$\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}.$$

For every $A \in \mathcal{A}$, we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all which follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. In order to show additivity of Φ on $\mathcal{A}$, we use the above partition of $\mathcal{A}$ and provide some claims which prove that Φ is additive on each $\mathcal{A}_{ij}$, $i, j = 1, 2$.

The above theorem is proven by several claims.

Claim 2.2. *We show that* $\Phi(0) = 0$.

Proof. If $\Phi(0) \neq 0$, then, by successively putting $A = 0$, $B = 0$, and then $C = 0$ in (1.2), we obtain a contradiction. □

Claim 2.3. *For each $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21}).$$

Proof. We show that

$$T = \Phi(A_{12} + A_{21}) - \Phi(A_{12}) - \Phi(A_{21}) = 0.$$

We can write:

$$\begin{aligned}
&\Phi(I) \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{12} + A_{21}) + I \diamond_\lambda \Phi(P_1 - P_2) \diamond_\lambda (A_{12} + A_{21}) \\
&\quad + I \diamond_\lambda (P_1 - P_2) \diamond_\lambda \Phi(A_{12} + A_{21}) \\
&= \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{12} + A_{21})) \\
&= \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda A_{12}) + \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda A_{21}) \\
&= \Phi(I) \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{12} + A_{21}) + I \diamond_\lambda \Phi(P_1 - P_2) \diamond_\lambda (A_{12} + A_{21}) \\
&\quad + I \diamond_\lambda (P_1 - P_2) \diamond_\lambda \Phi(A_{12}) + \Phi(A_{21}).
\end{aligned}$$

Thus, we have

$$I \diamond_\lambda (P_1 - P_2) \diamond_\lambda T = 0.$$

Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, then

$$(1 + 2\lambda + |\lambda|^2)T_{11} + (1 - |\lambda|^2)T_{12} + (1 - |\lambda|^2)T_{21} - (1 + 2\lambda + |\lambda|^2)T_{22} = 0.$$

We know that $|\lambda| \neq 0, 1$. Then,

$$T_{11} = T_{12} = T_{21} = T_{22} = 0. \qquad \square$$

Claim 2.4. *For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$, we have*

$$\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).$$

Proof. We show that, for T in $\mathcal{A}$, the following holds:

$$T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0. \tag{2.2}$$

We can write

$$\begin{aligned}
&\Phi(I) \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{11} + A_{12} + A_{21}) \\
&\quad + I \diamond_\lambda \Phi(P_1 - P_2) \diamond_\lambda (A_{11} + A_{12} + A_{21}) \\
&\quad + I \diamond_\lambda (P_1 - P_2) \diamond_\lambda \Phi(A_{11} + A_{12} + A_{21}) \\
&= \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{11} + A_{12} + A_{21})) \\
&= \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda A_{11}) + \Phi(I \diamond_\lambda (P_1 + P_2) \diamond_\lambda A_{12}) \\
&\quad + \Phi(I \diamond_\lambda (P_1 - P_2) \diamond_\lambda A_{21}) \\
&= \Phi(I) \diamond_\lambda (P_1 - P_2) \diamond_\lambda (A_{11} + A_{12} + A_{21})
\end{aligned}$$

$$+ I \diamond_\lambda \Phi(P_1 - P_2) \diamond_\lambda (A_{11} + A_{12} + A_{21})$$
$$+ I \diamond_\lambda (P_1 - P_2) \diamond_\lambda (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})).$$

Then, we have

$$I \diamond_\lambda (P_1 - P_2) \diamond_\lambda T = 0.$$

Since $T = T_{11} + T_{12} + T_{21} + T_{22}$, we obtain

$$(1+2\lambda+|\lambda|^2)T_{11} + (1-\lambda|^2)T_{12} + (1-|\lambda|^2)T_{21} - (1+2\lambda+|\lambda|^2)T_{22} = 0.$$

Since $|\lambda| \neq 0, 1$, we have

$$T_{11} = T_{12} = T_{21} = T_{22} = 0. \qquad \square$$

Claim 2.5. *For each* $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$, $A_{22} \in \mathcal{A}_{22}$, *we have*:

$$\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).$$

Proof. We show that, for T in $\mathcal{A}$, the following holds:

(2.3)
$$T = \Phi(A_{11}+A_{12}+A_{21}+A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0.$$

From Claim 2.4, we can write

$$\begin{aligned}
&\Phi(P_1) \diamond_\lambda I \diamond_\lambda (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + P_1 \diamond_\lambda \Phi(I) \diamond_\lambda (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + P_1 \diamond_\lambda I \diamond_\lambda \Phi(A_{11} + A_{12} + A_{21} + A_{22})\\
&= \Phi(P_1 \diamond_\lambda I \diamond_\lambda (A_{11} + A_{12} + A_{21} + A_{22})\\
&= \Phi(P_1 \diamond_\lambda I \diamond_\lambda (A_{11} + A_{12} + A_{21}) + \Phi(P_1 \diamond_\lambda I \diamond_\lambda A_{22})\\
&= \Phi(P_1) \diamond_\lambda I \diamond_\lambda A_{11}) + \Phi(P_1 \diamond_\lambda I \diamond_\lambda A_{12})\\
&\quad + \Phi(P_1 \diamond_\lambda I \diamond_\lambda A_{21}) + \Phi(P_1 \diamond_\lambda I \diamond_\lambda A_{22})\\
&= \Phi(P_1) \diamond_\lambda I \diamond_\lambda (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + P_1 \diamond_\lambda \Phi(I) \diamond_\lambda (A_{11} + A_{12} + A_{21} + A_{22})\\
&\quad + P_1 \diamond_\lambda I \diamond_\lambda (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})).
\end{aligned}$$

Then, we have

$$P_1 \diamond_\lambda I \diamond_\lambda T = 0.$$

Thus,

$$(1+2\lambda+|\lambda|^2)T_{11}+(1+\lambda)T_{12}+(\lambda+|\lambda|^2)T_{21}=0.$$

Therefore, $T_{11}=T_{21}=T_{12}=0$. Similarly, we can show that $T_{22}=0$. □

Claim 2.6. *For each $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i \neq j$, we have*

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

Proof. For $R_{ij}, S_{ij} \in \mathcal{A}_{ij}$, we have

(2.4)
$$I \diamond_\lambda (R_{ij}+P_i) \diamond_\lambda (P_j+S_{ij}) = (1+\lambda)S_{ij}+(1+\lambda)R_{ij} + (\lambda+|\lambda|^2)R_{ij}^* + (\lambda+|\lambda|^2)S_{ij}R_{ij}^*.$$

From equation (2.4), we have

$$\begin{aligned}
&\Phi((1+\lambda)(R_{ij}+S_{ij})+\Phi((\lambda+|\lambda|^2)R_{ij}^*)+\Phi((\lambda+|\lambda|^2)S_{ij}R_{ij}^*)\\
&\quad=\Phi(I \diamond_\lambda (R_{ij}+P_i) \diamond_\lambda (P_j+S_{ij}))\\
&\quad=\Phi(I) \diamond_\lambda (R_{ij}+P_i) \diamond_\lambda (P_j+S_{ij})\\
&\qquad+I \diamond_\lambda \Phi(R_{ij}+P_i) \diamond_\lambda (P_j+S_{ij}) + I \diamond_\lambda (R_{ij}+P_i) \diamond_\lambda \Phi(P_j+S_{ij})\\
&\quad=\Phi(I \diamond_\lambda R_{ij} \diamond_\lambda S_{ij})+\Phi(I \diamond_\lambda P_i \diamond_\lambda P_j)+\Phi(I \diamond_\lambda P_i \diamond_\lambda S_{ij})\\
&\qquad+\Phi(I \diamond_\lambda R_{ij} \diamond_\lambda P_j)\\
&\quad=\Phi((1+\lambda)R_{ij})+\Phi((\lambda+|\lambda|^2)R_{ij}^*)\\
&\qquad+\Phi((\lambda+|\lambda|^2)S_{ij}R_{ij}^*)+\Phi((1+\lambda)S_{ij}).
\end{aligned}$$

Hence,

$$\Phi((1+\lambda)(R_{ij}+S_{ij}))=\Phi((1+\lambda)R_{ij})+\Phi((1+\lambda)S_{ij}).$$

Let $A_{ij}=(1+\lambda)R_{ij}$ and $B_{ij}=(1+\lambda)S_{ij}$. Then, we have

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$ □

Claim 2.7. *For each $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$ such that $1 \leq i \leq 2$, we have*

$$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii}).$$

Proof. We show that

$$T = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0.$$

We can write, for $j \neq i$,

$$\begin{aligned}
&\Phi(P_j) \diamond_\lambda P_j \diamond_\lambda (A_{ii} + B_{ii}) + P_j \diamond_\lambda \Phi(P_j) \diamond_\lambda (A_{ii} + B_{ii}) \\
&\quad + P_j \diamond_\lambda P_j \diamond_\lambda \Phi(A_{ii} + B_{ii}) \\
&\quad = \Phi(P_j \diamond_\lambda P_j \diamond_\lambda (A_{ii} + B_{ii})) \\
&\quad = \Phi(P_j \diamond_\lambda P_j \diamond_\lambda A_{ii}) + \Phi(P_j \diamond_\lambda P_j \diamond_\lambda B_{ii}) \\
&\quad = \Phi(P_j) \diamond_\lambda P_j \diamond_\lambda (A_{ii} + B_{ii}) + P_j \diamond_\lambda \Phi(P_j) \diamond_\lambda (A_{ii} + B_{ii}) \\
&\qquad + P_j \diamond_\lambda P_j \diamond_\lambda (\Phi(A_{ii}) + \Phi(B_{ii})).
\end{aligned}$$

Therefore,

$$P_j \diamond_\lambda P_j \diamond_\lambda T = 0.$$

Thus,

$$(1 + 2\lambda + |\lambda|^2)T_{jj} + (1 + \lambda)T_{ji} + (\lambda + |\lambda|^2)T_{ij} = 0.$$

It follows that $T_{ij} = T_{ji} = T_{jj} = 0$. □

From Claim 2.6, for every $C_{ij} \in \mathcal{A}_{ij}$, we have

$$\begin{aligned}
&\Phi(P_i) \diamond_\lambda (A_{ii} + B_{ii}) \diamond_\lambda C_{ij} + P_i \diamond_\lambda \Phi(A_{ii} + B_{ii}) \diamond_\lambda C_{ij} \\
&\quad + P_i \diamond_\lambda (A_{ii} + B_{ii}) \diamond_\lambda \Phi(C_{ij}) \\
&= \Phi(P_i \diamond_\lambda (A_{ii} + B_{ii}) \diamond_\lambda C_{ij}) \\
&= \Phi(P_i \diamond_\lambda A_{ii} \diamond_\lambda C_{ij}) + \Phi(P_i \diamond_\lambda B_{ii} \diamond_\lambda C_{ij}) \\
&= \Phi(P_i) \diamond_\lambda (A_{ii} + B_{ii}) \diamond_\lambda C_{ij} + P_i \diamond_\lambda (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond_\lambda C_{ij} \\
&\quad + P_i \diamond_\lambda (A_{ii} + B_{ii}) \diamond_\lambda \Phi(C_{ij}).
\end{aligned}$$

Thus,

$$P_i \diamond_1 T \diamond_1 C_{ij} = 0.$$

By primeness, and since $T = T_{11} + T_{12} + T_{21} + T_{22}$, we obtain $T_{ii} = 0$. Hence, the additivity of Φ comes from Claims 2.2–2.7.

In the remainder of the paper, we show that Φ is a $*$-derivation.

Theorem 2.8. *Let $\mathcal{A}$ be a prime $*$-algebra. Let the map*

$$\Phi : \mathcal{A} \longrightarrow \mathcal{A}$$

satisfy the condition

$$\Phi(A\diamond_\lambda B\diamond_\lambda C) = \Phi(A)\diamond_\lambda B\diamond_\lambda C + A\diamond_\lambda \Phi(B)\diamond_\lambda C + A\diamond_\lambda B\diamond_\lambda \Phi(C), \tag{2.5}$$

where $A \diamond_\lambda B = AB + \lambda BA^*$ *for* $A, B \in \mathcal{A}$. *If* $\Phi(I)$ *is self-adjoint, then* Φ *is a* $*$*-derivation.*

Proof. We present the proof of the above theorem with several claims. From Theorem 2.1, we need to prove that Φ is self-adjoint and has the derivation property.

Claim 2.9. *If* $\Phi(I)$ *is self-adjoint, then* $\Phi(iI) = \Phi(I) = 0$.

Proof. It is easy to verify that

$$I \diamond_\lambda iI \diamond_\lambda iI = iI \diamond_\lambda iI \diamond_\lambda I.$$

Thus,

$$\begin{aligned}\Phi(I) \diamond_\lambda iI \diamond_\lambda iI + I \diamond_\lambda \Phi(iI) \diamond_\lambda iI + I \diamond_\lambda iI \diamond_\lambda \Phi(iI)\\ = \Phi(iI) \diamond_\lambda iI \diamond_\lambda I + iI \diamond_\lambda \Phi(iI) \diamond_\lambda I + iI \diamond_\lambda iI \diamond_\lambda \Phi(I).\end{aligned}$$

It follows that

$$\begin{aligned}&2i\Phi(iI) - |\lambda|^2 i\Phi(iI) + |\lambda|^2 i\Phi(iI)^* + \lambda i\Phi(iI) + \lambda i\Phi(iI)^* - \Phi(I) + |\lambda|^2|\Phi(I)\\ &\quad = -\Phi(I) + |\lambda|^2\Phi(I) + 2i\Phi(iI) - \lambda i\Phi(iI) - \lambda i\Phi(iI)^*\\ &\qquad - |\lambda|^2 i\Phi(iI) + |\lambda|^2 i\Phi(iI)^*.\end{aligned}$$

Then,

$$2\lambda i(\Phi(iI) + \Phi(iI)^*) = 0,$$

which gives

$$\Phi(iI)^* = -\Phi(iI). \tag{2.6}$$

On the other hand, we have

$$\Phi(iI \diamond_\lambda iI \diamond_\lambda iI) = -\Phi(I \diamond_\lambda iI \diamond_\lambda I).$$

Then, we have

$$\begin{aligned}\Phi(iI) \diamond_\lambda iI \diamond_\lambda iI + iI \diamond_\lambda \Phi(iI) \diamond_\lambda iI + iI \diamond_\lambda iI \diamond_\lambda \Phi(iI)\\ = -(\Phi(I) \diamond_\lambda iI \diamond_\lambda I + I \diamond_\lambda \Phi(iI) \diamond_\lambda I + I \diamond_\lambda iI \diamond_\lambda \Phi(I)).\end{aligned}$$

It follows that

$$\begin{aligned}-3\Phi(iI)+\lambda\Phi(iI)+\lambda\Phi(iI)^*+2|\lambda|^2\Phi(iI)-|\lambda|^2\Phi(iI)^* &= -2i\Phi(I)\\ -\lambda\Phi(iI)-\lambda\Phi(iI)^*+2|\lambda|^2 i\Phi(iI)-|\lambda|^2\Phi(iI)^*-\Phi(iI).\end{aligned}$$

From (2.6), we have

$$\begin{aligned}-3\Phi(iI)+\lambda\Phi(iI)-\lambda\Phi(iI)+2|\lambda|^2\Phi(iI)+|\lambda|^2\Phi(iI) &= -2i\Phi(I)\\ -\lambda\Phi(iI)+\lambda\Phi(iI)+2|\lambda|^2 i\Phi(iI)+|\lambda|^2\Phi(iI)-\Phi(iI).\end{aligned}$$

Equivalently, we obtain

$$-2\Phi(iI)+2|\lambda|^2\Phi(iI)+2i\Phi(I)-2|\lambda|^2 i\Phi(iI)=0. \tag{2.7}$$

By taking the adjoint of (2.7), we obtain

$$2\Phi(iI)-2|\lambda|^2\Phi(iI)-2i\Phi(I)-2|\lambda|^2 i\Phi(iI)=0. \tag{2.8}$$

From (2.7) and (2.8), we obtain

$$\Phi(iI)=0. \tag{2.9}$$

In addition, by (2.8) and (2.9), we have $\Phi(I)=0$. □

Claim 2.10. *We prove that Φ preserves the star.*

Proof. Since $\Phi(iI)=0$, we have

$$\Phi(A\diamond_\lambda iI\diamond_\lambda iI)=\Phi(A)\diamond_\lambda iI\diamond_\lambda iI.$$

It follows that

$$\Phi(-A-\lambda A^*+\lambda A^*+|\lambda|^2A)=-\Phi(A)-\lambda\Phi(A)^*+\lambda\Phi(A)^*+|\lambda|^2\Phi(A),$$

which gives

$$\Phi(|\lambda|^2A)=|\lambda|^2\Phi(A). \tag{2.10}$$

Also,

$$\Phi(A\diamond_\lambda I\diamond_\lambda I)=\Phi(A)\diamond_\lambda I\diamond_\lambda I.$$

From (2.10), we obtain

$$\Phi(A+2\lambda A^*+|\lambda|^2A)=\Phi(A)+2\lambda\Phi(A)^*+|\lambda|^2\Phi(A).$$

Hence,

$$\Phi(2\lambda A^*) = 2\lambda\Phi(A)^*. \tag{2.11}$$

In addition,

$$\Phi(I \diamond_\lambda I \diamond_\lambda A^*) = I \diamond_\lambda I \diamond_\lambda \Phi(A^*).$$

It follows that

$$\Phi(A^* + 2\lambda A^* + |\lambda|^2 A^*) = \Phi(A^*) + 2\lambda\Phi(A^*) + |\lambda|^2\Phi(A^*).$$

Thus,

$$\Phi(2\lambda A^*) = 2\lambda\Phi(A^*). \tag{2.12}$$

From (2.11) and (2.12), we obtain

$$\Phi(A^*) = \Phi(A)^*.$$ □

Claim 2.11. $\Phi(iA) = i\Phi(A)$ *for every* $A \in \mathcal{A}$.

Proof. For every $A \in \mathcal{A}$, from (2.10), we have

$$(1 - |\lambda|^2)\Phi(iA) = \Phi(I \diamond_\lambda iI \diamond_\lambda A) = I \diamond_\lambda iI \diamond_\lambda \Phi(A) = (1 - |\lambda|^2)i\Phi(A).$$

Therefore,

$$\Phi(iA) = i\Phi(A).$$ □

Claim 2.12. Φ *is a derivation.*

Proof. For every $A, B \in \mathcal{A}$, we have

$$\begin{aligned}
&\Phi(AB + \lambda AB + \lambda BA^* + |\lambda|^2 BA^*) \\
&\quad = \Phi(I \diamond_\lambda A \diamond_\lambda B) \\
&\quad = I \diamond_\lambda \Phi(A) \diamond_\lambda B + I \diamond_\lambda A \diamond_\lambda \Phi(B) \\
&\quad = \Phi(A)B + \lambda\Phi(A)B + \lambda B\Phi(A)^* + |\lambda|^2 B\Phi(A)^* + A\Phi(B) \\
&\qquad + \lambda A\Phi(B) + \lambda\Phi(B)A^* + |\lambda|^2\Phi(B)A^*.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Phi(AB + \lambda AB + \lambda BA^* + |\lambda|^2 BA^*) &= \Phi(A)B + \lambda\Phi(A)B + \lambda B\Phi(A)^* \\
&\quad + |\lambda|^2 B\Phi(A)^* A\Phi(B) + \lambda A\Phi(B) + \lambda\Phi(B)A^* + |\lambda|^2\Phi(B)A^*.
\end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned}
&\Phi(AB + \lambda AB - \lambda BA^* - |\lambda|^2 BA^*)\\
&\quad = \Phi(I \diamond_\lambda iA \diamond_\lambda (-iB))\\
&\quad = I \diamond_\lambda \Phi(iA) \diamond_\lambda (-iB) + I \diamond_\lambda iA \diamond_\lambda \Phi(-iB)\\
&\quad = \Phi(A)B + \lambda\Phi(A)B - \lambda B\Phi(A)^* - |\lambda|^2 B\Phi(A)^* + A\Phi(B)\\
&\qquad + \lambda A\Phi(B) - \lambda\Phi(B)A^* - |\lambda|^2\Phi(B)A^*.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Phi(AB + \lambda AB - \lambda BA^* - |\lambda|^2 BA^*)
&= \Phi(A)B + \lambda\Phi(A)B - \lambda B\Phi(A)^* - |\lambda|^2 B\Phi(A)^*\\
&\quad + A\Phi(B) + \lambda A\Phi(B) - \lambda\Phi(B)A^* - |\lambda|^2\Phi(B)A^*.
\end{aligned} \tag{2.14}$$

From (2.13) and (2.14), we have

$$\Phi((1+\lambda)AB) = (1+\lambda)\Phi(A)B + (1+\lambda)A\Phi(B).$$

From (2.12) and knowing that Φ preserves the star, we have

$$(1+\lambda)\Phi(AB) = (1+\lambda)(\Phi(A)B + A\Phi(B)).$$

Finally, we obtain

$$\Phi(AB) = \Phi(A)B + A\Phi(B). \quad \square$$

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