

## ON THE ROOTS OF THE GENERALIZED ROGERS-RAMANUJAN FUNCTION

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**ABSTRACT.** We give simple proofs of the fact that, for certain parameters, the roots of the generalized Rogers-Ramanujan function are irrational numbers and, for example, that at least one of the following two numbers is irrational:  $\{\sum_{n=1}^{\infty} F_n / (m^n \prod_{i=0}^{n-1} \phi(k+i)), \sum_{n=1}^{\infty} F_n / (m^n \prod_{i=0}^{n-1} \phi(k+i+1))\}$ , where  $F_{n+2} = F_{n+1} + F_n$ ,  $F_0 = 0$ ,  $F_1 = 1$  (the Fibonacci sequence),  $m$  is a natural number  $> (1 + \sqrt{5})/2$  and  $\phi(k)$  is *any* function taking positive integer values such that  $\limsup_{k \rightarrow \infty} \phi(k) = \infty$ .

**1. Introduction and results.** The irrationality of  $\pi$  was first proven by J.H. Lambert in 1761 using the continued fraction for the function  $\tan x$ . Nowadays, proofs avoid the use of continued fractions and use a variant of Hermite's ideas; a proof of this type was given by I. Niven [3]. Laczkovich's proof of the irrationality of  $\pi$  presented in [5] is particularly simple and contains ideas from [6].

The aim of this note is to give short proofs of two irrationality theorems, both inspired by Laczkovich's proof. In fact, we use ideas that are of an elementary nature. It may be said that the crux of Laczkovich's proof is based upon the existence of a one-parameter family satisfying a certain recursion. In order to prove our theorems, we follow the same path using three one-parameter families, namely, (1.1), (1.3) and (1.4).

Our first and most important result is a general theorem which implies that, for certain parameters, the roots of the generalized Rogers-Ramanujan function are irrational numbers.

We write  $Q := a_1 a_2^2 \cdots a_r^r$ , for short, and define  $f_k = f_k(x, a_1, \dots, a_r)$  by

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$$(1.1) \quad f_k := \sum_{n=0}^{\infty} \frac{1}{a_1^{nk+(n(n-1)/2)} \dots a_r^{rnk+rn^2/2+n(1-(3r/2))}} \frac{x^n}{\prod_{i=1}^n (1 - Q^i)},$$

where, if  $n = 0$ , it is understood that  $\prod_{i=1}^n (1 - Q^i) = 1$ .

The following theorem holds.

**Theorem 1.1.** *Assume that  $a_i \in \mathbb{Z}$  and  $|a_1 \cdots a_r| \geq 2$ . If  $x \neq 0$  is a rational number and  $k = 0, 1, 2, \dots$ , then  $f_k \neq 0$  and  $f_{k+1}/f_k$  is irrational.*

**Corollary 1.2.** *Assume that  $1/q = \pm 2, \pm 3, \dots$ , and  $k = 0, 1, 2, \dots$ . If a real number  $x_0$  satisfies*

$$(1.2) \quad 1 + \sum_{n=1}^{\infty} \frac{x_0^n q^{n^2+kn}}{(1-q) \cdots (1-q^n)} = 0,$$

then  $x_0$  is irrational.

The function appearing in (1.2) is the generalized Rogers-Ramanujan function, see [1, 2, 4]. Observe that (1.2) can be written as  $f_k(-x_0, 1/q) = 0$ , and therefore, the conclusion follows from Theorem 1.1.

Note that

$$\begin{aligned} f_0(-1, 1/q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q) \cdots (1-q^n)} \\ &= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} \end{aligned}$$

and

$$\begin{aligned} f_1(-1, 1/q) &= 1 + \sum_{n=1}^{\infty} \frac{q^{n^2+n}}{(1-q) \cdots (1-q^n)} \\ &= \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} \end{aligned}$$

are the Rogers-Ramanujan functions, see [2], [4, page 78].

In order to state the next theorem, we need to define the functions  $h_k$  and  $g_k$ .

**Definition 1.3.**

(i) If  $A, B$  are real numbers, let  $F_n$  be recursively defined by

$$F_n = A(F_{n-1} - BF_{n-2}),$$

for  $2 \leq n$ , with initial values  $F_0 = 0, F_1 = 1$ .

Let  $\phi(k)$  be a function taking non-zero real values and whose domain is  $\mathbb{N} \cup \{0\}$ . If  $k = 0, 1, 2, \dots$ , we set

$$(1.3) \quad h_k := h_k(x) = \sum_{n=1}^{\infty} \frac{F_n x^n}{\prod_{i=0}^{n-1} \phi(k+i)}.$$

(ii) Let  $\eta(k), \phi(k)$  be two functions taking non-zero real values and whose domain is  $\mathbb{N} \cup \{0\}$ . If  $k = 0, 1, 2, \dots$ , we set

$$(1.4) \quad \begin{aligned} g_k &:= g_k(x) \\ &= \sum_{n=1}^{\infty} x^n \frac{\{\phi(k) + \phi(k)\phi(k+1) + \dots + \phi(k)\phi(k+1) \dots \phi(k+n-1)\}}{\prod_{i=0}^{n-1} \eta(k+i)}. \end{aligned}$$

By looking at the coefficient  $x^n$  of  $h_k$ , it can be observed that, *formally*, the following recursion holds

$$(1.5) \quad \frac{B}{\phi(k+1)} x^2 h_{k+2} = x h_{k+1} - \frac{\phi(k)}{A} h_k + \frac{x}{A}.$$

Similarly,  $g_k$  *formally* satisfies

$$(1.6) \quad \frac{\eta(k)}{\phi(k)} g_k - \left\{ 1 + \frac{1}{\phi(k+1)} \right\} x g_{k+1} + \frac{1}{\eta(k+1)} x^2 g_{k+2} = x.$$

The following theorem holds.

**Theorem 1.4.**

(i) *In the definition of  $h_k$ , let  $A, B, x$  be rational non-zero numbers such that  $1/xB$  and  $1/x^2AB$  are integers. Assume also that  $\phi(k)$  takes*

positive integer values,  $0 < x$  and  $0 \leq F_n$  for all  $n$ , and at least one of the two following conditions hold:

- (a)  $\lim_{k \rightarrow \infty} \phi(k) = \infty$ ;
- (b)  $\limsup_{k \rightarrow \infty} \phi(k) = \infty$ , and there exists an  $x_0 > x$  such that  $\sum_1^\infty F_n x_0^n$  converges.

Then, for any  $k = 1, 2, \dots$ , at least one of  $\{h_k(x), h_{k+1}(x)\}$  is an irrational number.

(ii) In the definition of  $g_k$ , assume that  $\eta(k) = P_k Q_k$ ,  $\phi(k) = P_{k-1}/R_k$ , where  $P_k, Q_k, R_k$  are positive integers for all  $k$  and  $\lim_{i \rightarrow \infty} Q_i = \infty$ . Furthermore, assume that  $\sup_k P_{k-1}/R_k P_k \leq M$  for some  $1 \leq M$ . Then, at least one of  $\{g_k(1/m), g_{k+1}(1/m)\}$  is irrational for any  $m = 1, 2, 3, \dots; k = 0, 1, 2, \dots$ .

**Remark 1.5.**

(1) As we shall see,  $g_k$  is an entire function and, in the case where condition (a) holds, the same is true for  $h_k$ .

(2) The result of the abstract follows by taking  $A = 1, B = -1$  and  $x = 1/m$  in (b) and observing that  $\sum_1^\infty F_n x_0^n$  converges if  $(1 + \sqrt{5})/2 < 1/x_0$ .

(3) The next example follows from (ii). Let  $P_k$  and  $R_k$  be two sequences of positive integers such that  $P_{k-1}/P_k$  is bounded (this is satisfied, for example, if  $P_k$  is non-decreasing) and  $Q_0 = 2, Q_1 = 3, Q_2 = 5, Q_3 = 7, \dots$  (that is,  $Q_k$  is the  $k + 1$  prime). Then, at least one of  $\{g_0(1), g_1(1)\}$  is irrational.

**2. Proof of Theorem 1.1.**

*Claim 1.* If  $f_k$  is defined by (1.1), then the following recursion holds

$$(2.1) \quad f_{k+1} - f_k = x f_{k+2} \frac{1}{a_1^{k+1} \dots a_r^{k+1}}.$$

In fact, the coefficient of  $x^n$  in the expression  $f_{k+1} - f_k$  is

$$\begin{aligned} & \frac{1}{Q_1^{nk} a_1^{n(n-1)/2} \dots a_r^{rn^2/2+n(1-(r3/2))} \prod_{i=1}^n (1 - Q^i)} \left( \frac{1}{Q^n} - 1 \right) \\ &= \frac{1}{Q_1^{nk+n} a_1^{n(n-1)/2} \dots a_r^{rn^2/2+n(1-(r3/2))} \prod_{i=1}^{n-1} (1 - Q^i)}, \end{aligned}$$

which is the coefficient of  $x^n$  of

$$xf_{k+2} \frac{1}{a_1^{k+1} \dots a_r^{k+1}} = xf_{k+2} \frac{1}{Q^k a_1 \dots a_r}.$$

*Claim 2.* We have that  $f_k \rightarrow 1$  if  $k \rightarrow \infty$ . In fact, by hypothesis,  $a_i \in \mathbb{Z}$  and  $|a_1 \dots a_r| \geq 2$ . For simplicity, assume that  $2 \leq |a_1|$ ; the other cases are similar. Then,

$$|f_k - 1| \leq \frac{|x|}{2^k} + \dots + \frac{|x|^n}{2^{nk+n(n-1)/2}} + \dots = O\left(\frac{1}{2^k}\right),$$

and the claim follows.

*Claim 3.* Let  $C \neq 0$  be a natural number such that  $C/x$  is a non-zero integer. Recall that  $|Q| \geq 2$ . Take a fixed natural number  $i \geq 1$  such that  $|C/a_1^i \dots a_r^i = C/Q^i| < 1$ , and set

$$G_n := f_{k+n} \frac{C^n}{Q^{in}}.$$

We have  $G_n \rightarrow 0$  if  $n \rightarrow \infty$  and  $G_n \neq 0$  if  $n$  is large enough; these facts follow from Claim 2.

*Claim 4.* Since  $C, C/x, a_i$  are all integers, the recursion (2.1) can be written in terms of  $G_n$  more simply as

$$G_{n+2} = S_n Q^{n-i} G_{n+1} + T_n Q^{n-2i} G_n,$$

if  $0 \leq n$ , where  $S_n$  and  $T_n$  are integers; in fact, using (2.1), we obtain

$$\begin{aligned} G_{n+2} &= f_{k+n+2} \frac{C^{n+2}}{Q^{i(n+2)}} = Q^{k+n} \frac{a_1 \dots a_r}{x} (f_{k+n+1} - f_{k+n}) \frac{C^{n+2}}{Q^{i(n+2)}} \\ &= \left\{ Q^k a_1 \dots a_r \frac{C}{x} \right\} Q^{n-i} G_{n+1} + \left\{ -Q^k a_1 \dots a_r \frac{C^2}{x} \right\} Q^{n-2i} G_n. \end{aligned}$$

Now, the proof of the theorem proceeds as follows. Assume that the conclusion of the theorem is false; then, we may write  $f_k = Ay$  and  $f_{k+1} = By$  for some real non-zero number  $y$  and integers  $A, B$ . Note that we allow  $A, B$  to be zero. This gives  $G_0 = Ay$  and  $G_1 = CBy/Q^i$ . The last recursion yields that  $G_{2i}, G_{2i+1}$  are integer multiples of  $y/Q^{j_0}$  for some  $j_0 \geq 0$ . However, for  $n \geq 2i$ , the above recursion has integer coefficients, and therefore,  $G_n$  is an integer multiple of  $y/Q^{j_0}$ . This is in contradiction with Claim 3. □

**3. Proof of Theorem 1.4.**

(i) *Claim 1.* We have that  $h_k > 0$  for all  $k$ . In the case where condition (a) holds, then  $h_k \rightarrow 0$  if  $k \rightarrow \infty$ . This follows from the fact that there exists some fixed  $\alpha > 0$  such that  $0 \leq F_n x^n \leq \alpha^n$  for all  $n$  and such that  $\phi(k) \rightarrow \infty$  as  $k \rightarrow \infty$ . This also yields that  $h_k$  is an entire function. If condition (b) holds, then  $h_{k_i} \rightarrow 0$  for some subsequence  $k_i \rightarrow \infty$ .

*Claim 2.* Set

$$G_n := \frac{h_{k+n}}{\phi(k+n-1)}.$$

In any case, we have from Claim 1 that  $G_{n_j} \rightarrow 0$  for some subsequence  $n_j \rightarrow \infty$  and  $G_n \neq 0$  for all  $n$ .

*Claim 3.* The recursion (1.5) can be written in terms of  $G_n$  more simply as

$$G_{n+2} = \frac{\phi(k+n)}{xB} G_{n+1} - \frac{\phi(k+n)\phi(k+n-1)}{x^2AB} G_n + \frac{1}{xAB},$$

if  $0 \leq n$ .

Assume that the conclusion of the theorem is false, that is, both  $h_k$  and  $h_{k+1}$ ,  $1 \leq k$ , are rational numbers. Then,  $G_n$  and  $G_{n+1}$  are both rational numbers, say, they are integer multiples of  $1/D$  with  $D \in \mathbb{N}$ . Recall that  $1/xB$  and  $1/x^2AB$  are integers and  $1/xAB$  is a rational number, say, with denominator  $K \in \mathbb{N}$ . Then, the last recursion yields that  $G_n$  is an integer multiple of  $1/KD$  for all  $n$ . This is in contradiction with Claim 2.

(ii) *Claim 1.* We show that  $g_k$ , which is a series of positive terms, is an entire function. It is sufficient to prove that the series (1.4) converges for any  $0 < x$ .

Next, observe that, if  $1 \leq i \leq n$ , then

$$\begin{aligned} 0 < \frac{\prod_{j=0}^{i-1} \phi(k+j)}{\prod_{j=0}^{n-1} \eta(k+j)} &= \left( \prod_{j=0}^{i-1} \frac{P_{k-1+j}}{R_{k+j} P_{k+j} Q_{k+j}} \right) \frac{1}{\prod_{j=i}^{n-1} \eta(k+j)} \\ &\leq \frac{M^n}{\prod_{j=0}^{n-1} Q_{k+j}}. \end{aligned}$$

Therefore, if  $0 < x$ , then

$$0 < g_k(x) \leq \frac{Mx}{Q_k} + \dots + \frac{n(Mx)^n}{\prod_{j=0}^{n-1} Q_{k+j}} + \dots,$$

where this last function converges since  $\lim_{i \rightarrow \infty} Q_i = \infty$ . Thus,  $g_k$  is an entire function. In other words, if  $0 < x$ , then  $0 \neq g_k(x) \rightarrow 0$  when  $k \rightarrow \infty$ .

*Claim 2.* Set

$$G_n := \frac{g_{k+n}}{\eta(k+n-1)} = \frac{g_{k+n}}{P_{k+n-1}Q_{k+n-1}}.$$

From Claim 1, we obtain that  $G_n \rightarrow 0$ , if  $n \rightarrow \infty$ , and  $G_n \neq 0$  for all  $n$ .

*Claim 3.* Dividing by  $x^2$  and putting  $k+n$  instead of  $k$ , the recursion (1.6) can be written in terms of  $G_n$  as

$$\frac{\eta(k+n)\eta(k+n-1)}{x^2\phi(k+n)}G_n - \left\{ 1 + \frac{1}{\phi(k+n+1)} \right\} \frac{\eta(k+n)}{x}G_{n+1} + G_{n+2} = \frac{1}{x},$$

or, putting  $x = 1/m$  and using the hypothesis, the last can be written as

$$m^2P_{k+n}Q_{k+n}Q_{k+n-1}R_{k+n}G_n - m(P_{k+n}Q_{k+n} + Q_{k+n}R_{k+n+1})G_{n+1} + G_{n+2} = m,$$

which is a recursion of the form

$$G_{n+2} = G_{n+1}A_n + G_nB_n + C_n,$$

if  $0 \leq n$ , where  $A_n, B_n, C_n$  are integers.

In order to prove the theorem, assume that both  $g_k(1/m)$  and  $g_{k+1}(1/m)$  are rational numbers. Then,  $G_0$  and  $G_1$  are both rational numbers, say, they are integer multiples of  $1/D$  with  $D \in \mathbb{N}$ . Then, the last recursion yields that  $G_n$  is an integer multiple of  $1/D$  for all  $n$ . This is in contradiction with Claim 2. □

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