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# ON VERTEX DECOMPOSABLE AND COHEN-MACAULAY REGULAR GRAPHS

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ABSTRACT. We characterize the Cohen-Macaulay property for generalized Petersen graphs and 3-regular graphs. In particular, we prove that these graphs are vertex decomposable. Also, we characterize pure vertex decomposability for 4-transitive graphs without 5-holes. Finally, we study the small cycles of well-covered and Cohen-Macaulay regular graphs.

**1. Introduction.** Let $G$ be a simple graph (without loops and multiple edges) whose vertex set is $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. A subset $F$ of $V(G)$ is a *stable set* or *independent set* if $e \nsubseteq F$ for each $e \in E(G)$. The cardinality of the maximum stable set is denoted by $\beta(G)$. The graph $G$ is called *well-covered* if every maximal stable set has the same cardinality. The *Stanley-Reisner complex* of $G$, denoted by $\Delta_G$, is the simplicial complex whose faces are the stables sets of $G$. Recall that a simplicial complex $\Delta$ is called *pure* if every facet (maximal face) has the same number of elements. Thus, $\Delta_G$ is pure if and only if $G$ is well-covered. The *deletion* of a vertex $x$ in $\Delta$ is the subcomplex $\mathrm{del}_\Delta(\{x\}) = \{F \in \Delta \mid x \notin F\}$. Furthermore, for $F \in \Delta$, the *link* of $F$ in $\Delta$ is the subcomplex, $\mathrm{link}_\Delta(F) = \{G \in \Delta \mid F \cap G = \emptyset,\ F \cup G \in \Delta\}$. A simplicial complex $\Delta$ is *vertex decomposable* if either $\{x_1, \ldots, x_n\}$ is the unique facet or there is a vertex $x$ such that:

(1) both $\mathrm{link}_\Delta(\{x\})$ and $\mathrm{del}_\Delta(\{x\})$ are vertex decomposable; and
(2) no face of $\mathrm{link}_\Delta(\{x\})$ is a facet of $\mathrm{del}_\Delta(\{x\})$.

A vertex $x$ which satisfies condition (2) is called a *shedding vertex*.

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On the other hand, $\Delta$ is *shellable* if the facets of $\Delta$ can be ordered $F_1, \dots, F_t$ such that, for all $1 \leq i < j \leq t$, there is some $x \in F_j \setminus F_i$ and $k \in \{1, \dots, j-1\}$ such that $\{x\} = F_j \setminus F_k$. A graph $G$ is called *shellable* if $\Delta_G$ is shellable. Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field $k$, the *edge ideal* of $G$, denoted $I(G)$, is the ideal of $R$ generated by all monomials $x_i x_j$ such that $\{x_i, x_j\} \in E(G)$. We say that $G$ is *Cohen-Macaulay* if $R/I(G)$ is a Cohen-Macaulay ring. In general, we have the following implications [**4, 16, 20**]:

$$\begin{matrix}\text{Pure vertex}\\ \text{decomposable}\end{matrix} \Longrightarrow \begin{matrix}\text{Pure}\\ \text{shellable}\end{matrix} \Longrightarrow \text{Cohen-Macaulay} \Longrightarrow \text{Well-covered.}$$

An *n-cycle* is a cycle with $n$ vertices with or without chords, and an *n-hole* is an $n$-cycle without chords. In [**5**], pure vertex decomposability is characterized for graphs whose 5-cycles have at least 3-chords. In this paper, we characterize the pure vertex decomposability for 4-transitive graphs without 5-holes. The equivalence between Cohen-Macaulayness and pure vertex decomposability has been studied for some families of graphs: bipartite graphs (in [**10, 17**]); very well-covered graphs (in [**13**]); theta-ring graphs (in [**6**]); graphs with girth at least 5 and block-cactus (in [**11**]); graphs without 4-cycles and 5-cycles (in [**1**]); and graphs without 3-cycles and 5-cycles (in [**4**]). In this paper, we prove that equivalence for 3-regular graphs and generalized Petersen graphs.

The paper is organized as follows. In Section 2, we study the smaller cycles of well-covered and Cohen-Macaulay regular graphs. We will use these results in the following sections. In Section 3, we prove that the pure vertex decomposability and Cohen-Macaulayness are equivalent for cubic graphs. Furthermore, we prove that the connected components of these graphs are $K_4$ or $P(3,1)$. In Section 4, we characterize 4-transitive graphs without 5-holes whose simplicial complexes are vertex decomposable. In Section 5, we prove that a generalized Petersen graph $G = P(n,r)$ is Cohen-Macaulay if and only if $(n,r) = (3,1)$ if and only if $\Delta_G$ is pure vertex decomposable.

**2. Well-covered and Cohen-Macaulay regular graphs and cycles.** Let $X$ be a subset of $V(G)$; the *subgraph induced* by $X$ in

$G$ is the graph with vertex set $X$, whose edge set is

$$\{\{x, y\} \in E(G) \mid x, y \in X\},$$

denoted by $G[X]$. Furthermore, $G \setminus X$ denotes the induced subgraph $G[V(G) \setminus X]$. The *girth* of $G$ is the length of the shortest cycle in $G$. A *matching* of $G$ is a set of pairwise non-adjacent edges. The *matching number* $\nu(G)$ of a graph $G$ is the cardinality of a maximum matching. A *perfect matching* (1-*factor*) is a matching such that each vertex in $G$ is incident to exactly one edge of the matching. The *neighbor* of a vertex $v$ is

$$N_G(v) = \{w \in V(G) \mid \{v, w\} \in E(G)\},$$

and its *closed neighborhood* is

$$N_G[v] = N_G(v) \cup \{v\}.$$

The *degree* of $v$ in $G$ is $\deg_G(v) = |N_G(v)|$. Furthermore, if $\deg_G(v) = r$ for every $v \in V(G)$, then $G$ is called *$r$-regular*. If $H$ is not an induced subgraph of $G$, then $G$ is called an *$H$-free* graph.

**Remark 2.1.** Let $G$ be a graph. We have $\mathrm{del}_{\Delta_G}(x) = \Delta_{G\setminus x}$ and $\mathrm{link}_{\Delta_G}(x) = \Delta_{G\setminus N[x]}$; hence, $x$ is a shedding vertex if and only if each stable set in $G \setminus N_G[x]$ is not a maximal stable set in $G \setminus x$.

**Definition 2.2.** An *end vertex* is a vertex of degree 1. A *pendant edge* is an edge incident with an end vertex. A 5-cycle $C$ in $G$ is called *basic* if $C$ does not contain two adjacent vertices of degree 3 or more in $G$.

Let $C(G)$ be the set of vertices contained in at least one basic 5-cycle and $P(G)$ the set of vertices contained in at least one pendant edge. We say that $G \in \mathcal{PC}$ if $\{P(G), C(G)\}$ is a partition of $V(G)$ such that the vertex sets of the basic 5-cycle is a partition of $C(G)$, and the pendant edge is a partition of $P(G)$.

**Definition 2.3.** A vertex $x$ is *simplicial* if $G[N_G[x]]$ is a complete subgraph of $G$. The graph $G$ is in the family $\mathcal{F}$ if there is a set $\{x_1, x_2, \ldots, x_k\} \subseteq V(G)$ such that $x_i$ is a simplicial vertex, $|N_G[x_i]| \leq 3$ and $\{N_G[x_1], \ldots, N_G[x_k]\}$ is a partition of $V(G)$.

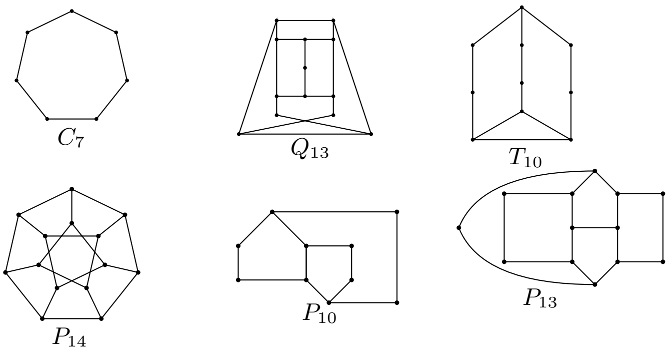



FIGURE 1. Special well-covered graphs.

**Definition 2.4.** A subset $D \subseteq V(G)$ is a *vertex cover* of $G$ if $D \cap e \neq \emptyset$ for each $e \in E(G)$. The covering number of $G$, denoted $\tau(G)$, is the cardinality of a minimum vertex cover of $G$.

**Theorem 2.5** (**[8, 9, 15]**). *Let $G$ be a connected well-covered graph.*

(i) *If the girth of $G$ is at least* 5*, then $G \in \mathcal{PC}$ or $G$ is isomorphic to one element in $\{K_1, C_7, P_{10}, P_{13}, Q_{13}, P_{14}\}$, see Figure* 1*.*

(ii) *If $G$ contains neither $C_4$ nor $C_5$, then $G \in \mathcal{F}$ or $G$ is isomorphic to one element in $\{K_1, C_7, T_{10}\}$, see Figure* 1*.*

(iii) *If $G$ is $\{C_3, C_5, C_7\}$-free, then $G$ has a perfect matching $e_1, \ldots, e_s$ with $s = \tau(G)$ such that $\{a, b\} \in E(G)$ when $e_i = \{x_i, y_i\}$ and $\{x_i, a\}$, $\{y_i, b\} \in E(G)$.*

**Theorem 2.6.** *Let $G$ be a connected regular graph. If $G$ is well-covered, then $G$ satisfies one of the following conditions*:

(i) *$G$ is isomorphic to one element in $\{K_1, K_2, C_3, C_5, C_7, P_{14}, K_{r,r}\}$.*

(ii) *$G$ has a* 4*-cycle. In addition, $G$ has an induced* 3*-,* 5*- or* 7*-cycle.*

(iii) *$G$ has a* 3*-cycle and a* 5*-hole.*

*Proof.* Suppose that the girth of $G$ is at least 5. If $G \in \mathcal{PC}$, then $V(G) = P(G) \cup C(G)$. If $P(G) \neq \emptyset$, then there is an end vertex and $G \simeq K_2$, since $G$ is regular. Now, if $C(G) \neq \emptyset$, then there is a basic 5-cycle, which implies that $G$ has a vertex of degree 2. Thus, $G \simeq C_5$, since $G$ is regular. Now, if $G \notin \mathcal{PC}$, then, from Theorem 2.5 (i),

$G \in \{K_1, C_7, P_{14}\}$ since $P_{10}$, $P_{13}$ and $Q_{13}$ are not regular. Therefore, $G$ satisfies (i).

Now, we assume that $G$ has a 3- or a 4-cycle. Suppose that $G$ has no 4-cycles. Hence, if $G$ does not satisfy (iii), then $G$ has no 5-holes. Furthermore, if $G$ has a 5-cycle, then $G$ contains a 4-cycle. Thus, $G$ has no 5-cycles. Therefore, from Theorem 2.5, $G \in \mathcal{F}$, or $G$ is isomorphic to either $K_1$, $C_7$ or $T_{10}$. Hence, $G = C_3$, since $G$ is regular and it has a 3-cycle. Therefore, $G$ satisfies (i). Now, we can assume that $G$ has a 4-cycle. If $G$ does not satisfy (ii), then $G$ is $\{C_3, C_5, C_7\}$-free. Consequently, from Theorem 2.5, there exists a perfect matching

$$e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\}, \ldots, e_g = \{x_g, y_g\}$$

with $g = \tau(G)$. We can suppose that $\{x_1, \ldots, x_g\}$ is a minimum vertex cover. Then, $\{y_1, \ldots, y_g\}$ is a stable set. Furthermore, we assume that $N_G(y_1) = \{x_1, x_2, \ldots, x_r\}$. If $\{x_j, y_i\} \in E(G)$ with $i \in \{1, \ldots, r\}$ and $j > r$, then, by Theorem 2.5, $\{x_j, y_1\} \in E(G)$, a contradiction. Thus, $N_G(y_i) \subseteq N_G(y_1)$ for $i \in \{1, \ldots, r\}$. Since $G$ is regular, $N_G(y_i) = N_G(y_1)$. Hence, $\{y_1, \ldots, y_r\} \subseteq N_G(x_l)$ for $l \in \{1, \ldots, r\}$. Therefore, $G = K_{r,r}$, since $G$ is a connected regular graph. □

**Definition 2.7.** Let $(\mathcal{G}, e)$ be a finite group, and let $S$ be an inverse closed subset of $\mathcal{G} \setminus \{e\}$. The *Cayley graph* $\mathrm{Cay}(\mathcal{G}, S)$ on $\mathcal{G}$ with respect to $S$ is the graph whose vertex set is $\mathcal{G}$ and edge set is

$$E(\mathrm{Cay}(\mathcal{G}, S)) = \{\{x, y\} \mid x, y \in \mathcal{G} \text{ such that } y = xs \text{ for some } s \in S\}.$$

**Definition 2.8.** A graph $G$ is *vertex-transitive* if, for every pair of vertices, there exists an automorphism mapping one to the other. Furthermore, if $G$ is $r$-regular, then $G$ is called *$r$-transitive.*

**Remark 2.9.** If $G = \mathrm{Cay}(\mathcal{G}, S)$ is a Cayley graph, then $G$ is a vertex-transitive graph, since $\mathcal{G}$ acts on $\mathrm{Cay}(\mathcal{G}, S)$ by left multiplication, and this action is transitive on $V(G)$.

**Lemma 2.10.** *The complete graphs and cycles are Cayley graphs.*

*Proof.* Let $G$ be a graph with $|V(G)| = n$. If $G$ is complete, then $G \simeq \mathrm{Cay}(\mathbb{Z}_n, S)$, where $S = \mathbb{Z}_n \setminus \{0\}$. Furthermore, if $G$ is a cycle, then $G \simeq \mathrm{Cay}(\mathbb{Z}_n, \{1\})$. □

**Definition 2.11.** For integers $n \geq 3$ and $1 \leq r < n/2$, the *generalized Petersen graph* $P(n,r)$ is the graph with vertex set

$$V(P(n,r)) = \{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$$

and edges $a_i b_i$, $a_i a_{i+1}$ and $b_i b_{i+r}$, for $i \in \{0, 1, \dots, n-1\}$ with arithmetic modulo $n$.

**Remark 2.12** (**[14]**)**.** With the exception of the dodecahedron $P(10,2)$, the generalized Petersen graph $P(n,r)$ is vertex-transitive, if and only if $r^2 \equiv \pm 1 \pmod{n}$. Furthermore, $P(n,r)$ is a Cayley graph if and only if $r^2 \equiv 1 \pmod{n}$.

**Corollary 2.13.** *If $G$ is a connected well-covered Cayley graph with girth at least 5, then $G$ is isomorphic to one of the elements in $\{K_1, K_2, C_5, C_7\}$.*

*Proof.* From Theorem 2.6, $G \in \{K_1, K_2, C_5, C_7, P_{14}\}$. Furthermore, by Remark 2.12, $G \neq P_{14} \simeq P(7,2)$. □

**Remark 2.14.** From Corollary 2.13, the connected Cohen-Macaulay Cayley graphs with girth at least 5 are $K_1$, $K_2$ and $C_5$, since $C_7$ is not Cohen-Macaulay.

**Definition 2.15.** A subgraph $H$ of $G$ is called a *$c$-minor* of $G$ if there exists a stable set $S$ of $G$ such that $H = G \setminus N_G[S]$.

**Remark 2.16** (**[18,** Theorems 7.4.4, 7.4.11]**).** The properties well-covered, shellable, Cohen-Macaulay and vertex decomposable are closed under $c$-minors.

**Proposition 2.17** (**[4,** Corollary 33]**).** *If $G$ is a Cohen-Macaulay graph without 3- and 5-cycles, then $G$ has an end vertex or an isolated vertex.*

**Theorem 2.18.** *If $G$ is a Cohen-Macaulay regular graph, then $G$ satisfies one of the following conditions*:

(i) *$G$ is isomorphic to one element in $\{K_1, K_2, C_3, C_5\}$.*
(ii) *$G$ has a 3-cycle. Furthermore, $G$ contains a 4-cycle or a 5-hole.*
(iii) *$G$ has a 4-hole and a 5-hole.*

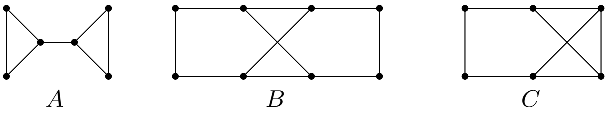


FIGURE 2.

*Proof.* Suppose that $G$ does not satisfy (ii) and (iii). First, we assume that $G$ is $C_3$-free. If $G$ has no 4-cycles, then, by Theorem 2.6, $G \in \{K_1, K_2, C_5, C_7, P_{14}\}$. However, $C_7$ is a $c$-minor of $P_{14}$ and $C_7$ is not Cohen-Macaulay; thus, $G$ satisfies (i). Now, if $G$ has a 4-cycle, then $G$ has no 5-cycles, since $G$ does not satisfy (iii), and it is $C_3$-free. Hence, by Proposition 2.17, $G \in \{K_1, K_2\}$. Now, we suppose that $G$ has a 3-cycle. Since $G$ does not satisfy (ii), $G$ has no 4-cycles and 5-holes. Consequently, by Theorem 2.6, $G = C_3$. Therefore, $G$ satisfies (i). $\square$

## 3. Cohen-Macaulay cubic graph.

In this section, we characterize which cubic graphs are Cohen-Macaulay.

**Definition 3.1.** Let $A$, $B$ and $C$ be the graphs given in Figure 2. A *terminal pair* is a pair of adjacent degree 2 vertices in $A$, $B$ or $C$. A graph $G$ is in $\mathcal{W}$ if $G$ is a collection of copies of $A$, $B$ and $C$, where every terminal pair of vertices is joined by edges to another terminal pair (possibly the same subgraphs A, B or C) such that $G$ is cubic.

**Remark 3.2.** Let $G$ be a graph in $\mathcal{W}$. $G$ is denoted by $(D_1, D_2, \ldots, D_r)$ if

$$V(G) = \bigsqcup_{i=1}^{r} V(D_i) \quad \text{with } D_i \in \{A, B, C\}$$

and a terminal pair of $D_i$ is joined by two edges to a terminal pair of $D_{i+1}$. Furthermore, $D_1 = D_r = C$ or a terminal pair of $D_1$ is joined by two edges to a terminal pair of $D_r$.

**Theorem 3.3** ([**3**]). *Let $G$ be a connected cubic graph. Then, $G$ is well-covered if and only if one of the following conditions is true*:

(i) $G \in \mathcal{W}$; *or*

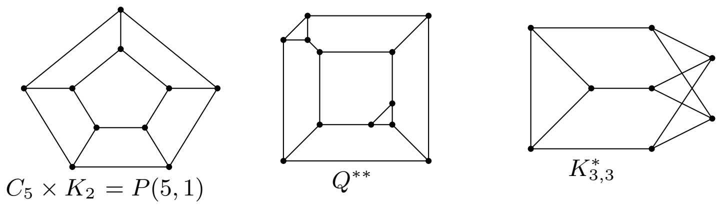



FIGURE 3. Special well-covered cubic graphs.

(ii) *$G$ is one of six exceptional graphs*: $K_4$, $K_{3,3}$, $K^*_{3,3}$, $C_5 \times K_2$, $Q^{**}$ *or* $P_{14}$.

**Definition 3.4.** Let $\Delta$ be a simplicial complex with vertex set $V$. We denote by $f_i$ the number of $i$-dimensional faces of $\Delta$. We have $f_0 = |V|$ and $f_{-1} = 1$ since $\emptyset \in \Delta$. If $\dim\Delta = d$, then the *$f$-vector* of $\Delta$ is the $(d+2)$-tuple $f(\Delta) = (f_{-1}, f_0, f_1, \ldots, f_d)$, and the *$h$-vector* of $\Delta$ is the $(d+2)$-tuple $h(\Delta) = (h_0, h_1, \ldots, h_{d+1})$, where

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1}.$$

**Proposition 3.5** ([**7**, Theorem 2.3]). *Let $\Delta$ be a simplicial complex.*

(i) *If $\Delta$ is Cohen-Macaulay, then $h(\Delta)$ has only non-negative entries.*
(ii) *If* $\dim\Delta = 1$, *then $\Delta$ is vertex decomposable/shellable/Cohen-Macaulay if and only if $\Delta$ is connected.*

In [**7**, Theorem 7.5], Earl, Vander Meulen and Van Tuyl showed that $K_4$ and $P(3,1)$ are the only cubic circulant graphs that are Cohen-Macaulay. In the following theorem, we show that there are no other Cohen-Macaulay cubic graphs.

**Theorem 3.6.** *If $G$ is a cubic graph, then the following conditions are equivalent*:

(i) *Each connected component of $G$ is $K_4$ or* $P(3,1)$;
(ii) *$G$ is Cohen-Macaulay*;
(iii) *$\Delta_G$ is pure vertex decomposable.*

*Proof.*

(ii) $\Rightarrow$ (i). We can suppose that $G$ is connected. If $G \notin \mathcal{W}$, then, by Theorem 3.3, $G \simeq K_4$ since

$$h(K_{3,3}) = (1,3,-3,1), \qquad h(K_{3,3}^*) = (1,5,3,-2),$$
$$h(C_5 \times K_2) = (1,6,6,-4,1), \quad h(Q^{**}) = (1,8,18,10,-1)$$

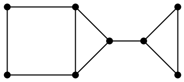
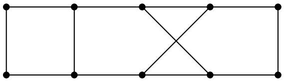
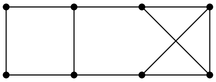
and

$$h(P_{14}) = (1,9,24,18,-2,-1).$$

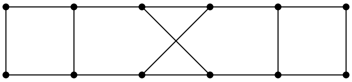
Now, we can assume that $G \in \mathcal{W}$. Hence, $G = (D_1, D_2, \ldots, D_r)$ with $D_i \in \{A,B,C\}$. If $r = 2$, then

$$G = \{(A,A),(A,B),(B,B),(C,C)\}.$$

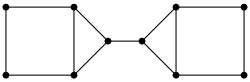
Consequently, one of the following graphs

is a $c$-minor of $G$, whose $h$-vectors are $(1,5,5,-1)$, $(1,6,8,0,-1)$ and $(1,5,4,-2)$, respectively, a contradiction, by Proposition 3.5. Now, if $r \geq 3$, then there is a $D_j \in \{A,B\}$ where $1 \leq j \leq r$. If $D_i = B$, then the following graph



is a $c$-minor of $G$, whose $h$-vector is $(1,7,12,0,-5,-3)$. Furthermore, if $D_j = A$, then the following graph is a $c$-minor of $G$:



whose $h$-vector is $(1,6,8,-2,-1)$. Thus, $r = 1$. Therefore, $G = (B)$ or $G = (A)$. However, the $h$-vector of $(B)$ is $(1,5,3,-1)$; hence, $G = (A) \simeq P(3,1)$.

(i) $\Rightarrow$ (iii). $\Delta_{K_4}$ is zero-dimensional, $\Delta_{P(3,1)}$ is one-dimensional, and they are connected. Thus, by Proposition 3.5, $K_4$ and $P(3,1)$ are pure vertex decomposable. $\square$

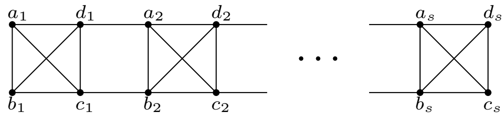



FIGURE 4. A $K_4$-chain.

## 4. 4-transitive graphs without 5-holes.

**Definition 4.1.** Let $H$ be a graph where $H_1,\dots,H_s$ is a partition of $V(H)$ such that $H[H_i]\simeq K_4$. Then, $H$ is a $K_4$-*chain* if

$$E(H)=\left(\bigcup_{i=1}^{s}E(H[H_i])\right)\bigcup\left(\bigcup_{i=1}^{s-1}\{\{d_i,a_{i+1}\},\{c_i,b_{i+1}\}\}\right),$$

where $H_i=\{a_i,b_i,c_i,d_i\}$ for $1\le i\le s$, see Figure 4. Hence, if $x\in V(H)\setminus\{a_1,b_1,c_s,d_s\}$, then $\deg_H(x)=4$. In this case, we write $H=(H_1,\dots,H_s)$. Furthermore, if $G$ is a 4-regular graph with a $K_4$-chain $H$ such that $V(H)=V(G)$, then $G$ is called $K_4$-band.

**Remark 4.2.** Let $A$ be a stable set of $G$. If $x\in V(G)\setminus A$ is such that $N_G(x)\subseteq N_G(A)$, then $x$ is not a shedding vertex.

*Proof.* We take a maximal stable set $B$ of $G$ such that $A\subseteq B$. Thus, $N_G(x)\subseteq N_G(B)$. Furthermore, $B\cap N_G(B)=\emptyset$, since $B$ is a stable set. Hence, $B\cap N_G(x)=\emptyset$. Since $B$ is maximal, we have that $x\in B$ and $B\setminus x\in \mathrm{link}_\Delta(\{x\})$, where $\Delta=\Delta_G$. Also, since $x\notin A$, then $A\subseteq B\setminus x$ and

$$N_G(x)\subseteq N_G(A)\subseteq N_G(B\setminus x).$$

If there is a $y\notin B$, then $y\in N_G(B)$, since $B$ is maximal. Hence,

$$y\in N_G(B\setminus x)\cup N_G(x)\subseteq N_G(B\setminus x).$$

This implies that $B\setminus x$ is a maximal stable set in $\mathrm{del}_\Delta(\{x\})$. Therefore, $x$ is not a shedding vertex. $\square$

**Remark 4.3.** Let $G$ be a $K_4$-chain with $x\in V(G)$. Then, $G[N_G(x)]$ has two connected components, a 3-cycle and an isolated vertex.

**Lemma 4.4.** *Let $G$ be a connected 4-transitive graph where each vertex is shedding. If $K_4 \subseteq G$, then $G$ is a $K_4$-band or $G \simeq K_5$.*

*Proof.* We can assume that $G \not\simeq K_5$. Since $K_4$ is a $K_4$-chain, there is a maximal $K_4$-chain subgraph $H = (H_1, \ldots, H_s)$ of $G$. We assume that $V(H_i) = \{a_i, b_i, c_i, d_i\}$ with $\{c_i, a_{i+1}\}, \{d_i, b_{i+1}\} \in E(G)$ for each $1 \leq i \leq s-1$.

We will prove that $N_G(a_1) \subseteq V(H)$. By contradiction, suppose that there is a $y \in N_G(a_1) \setminus V(H)$. Hence, there is a $K_4 \simeq H'_1 \subseteq G$ such that $y \in V(H'_1)$, since $G$ is vertex-transitive. Now, we consider two cases:

*Case* 1. First, we assume that $\{a_1, b_1, c_s, d_s\} \cap V(H'_1) \neq \emptyset$. Then,

$$V(H'_1) \cap \{c_1, d_1, a_s, b_s\} \neq \emptyset,$$

since $\deg_H(a_1) = \deg_H(b_1) = \deg_H(c_s) = \deg_H(d_s) = 3$. Thus, $N_G(y) \cap \{c_1, d_1, a_s, d_s\} \neq \emptyset$. However, if $s > 1$, then $\deg_H(c_1) = \deg_H(d_1) = \deg_H(a_s) = \deg_H(b_s) = 4$. Hence, $s = 1$ implies that $|V(H'_1) \cap V(H_1)| = 3$. Thus, $G[N_G(a_1)]$ is connected, since $|N_G(y) \cap V(H_1)| \geq 3$. Consequently, $G[N_G(y)]$ is connected, since $G$ is vertex-transitive. Then, $N_G(y) = V(H_1)$. Therefore, $G \simeq K_5$ since $G$ is connected and 4-regular, a contradiction.

*Case* 2. Now, we assume that $\{c_s, d_s, b_1, a_1\} \cap V(H'_1) = \emptyset$. Then, $V(H'_1) \cap V(H) = \emptyset$. We set $V(H'_1) = \{a'_1, b'_1, c'_1, y\}$. Suppose that $s \geq 2$. Then $(c_1, d_1, a_2, b_2, c_1)$ is a 4-hole. In addition, $G$ is vertex-transitive; hence, there is a 4-hole $C'$ such that $a_1 \in V(C')$. Since $C'$ does not have chords, $|V(C') \cap V(H_1)| = 2$ and $y \in V(C')$. Thus, $|V(C') \cap V(H'_1)| = 2$, since $N_G[y] = V(H'_1) \cup \{a_1\}$. Furthermore, $b_1 \in V(C')$, since $\deg_G(c_1) = \deg_G(d_1) = 4$. Thus, $C' = (y, a_1, b_1, u, y)$, where $u \in V(H'_1) \setminus \{y\}$. Hence, there is a $K_4$-chain with vertex set $V(H) \cup V(H'_1)$, a contradiction, since $H$ is maximal. This implies $s = 1$ and $H \simeq K_4$. By the maximality of $H$, we have that

$$N_G(H) \cap V(H'_1) = \{y\}.$$

For ease of exposition, we take $V(H) = \{x_1, x_2, x_3, x_4\}$. Thus, by symmetry, there are $y_1, y_2, y_3, y_4 \in V(G) \setminus V(H)$ such that

$$N_G[x_i] = V(H) \cup \{y_i\} \quad \text{for } 1 \leq i \leq 4.$$

Furthermore, there are $z_1^i, z_2^i, z_3^i \in N_G(y_i) \setminus V(H)$, such that $G_i = G[\{y_i, z_1^i, z_2^i, z_3^i\}] \simeq K_4$ and $y_j \notin V(G_i)$ for $j \neq i$. Consequently, $\{y_1, y_2, y_3, y_4\}$ is a stable set, since $y_j \notin V(G_i) = N_G[y_i] \setminus \{x_i\}$. We have

$$N_G(x_1) = \{y_1, x_2, x_3, x_4\} \subseteq N_G(\{y_2, y_3, y_4, z_j^1\}).$$

Then, $\{y_2, y_3, y_4, z_j^1\}$ is not a stable set; if it were, Remark 4.2 would then imply that $x_j^1$ is not a shedding vertex, contradicting the fact that every vertex is a shedding vertex. Thus,

$$N_G(z_j^1) \cap \{y_2, y_3, y_4\} \neq \emptyset \quad \text{for } j = 2, 3, 4.$$

In addition,

$$G[N_G(z_j^1)] \simeq G[N_G(x_1)] \simeq K_3 \sqcup K_1.$$

If $y_i \in N_G(z_j^1) \cap N_G(z_{j'}^1)$ with $j \neq j'$, then $G[N_G(z_j^1)]$ is connected, a contradiction. This implies that $|N_G(y_i) \cap \{z_1^1, z_2^1, z_3^1\}| = 1$. Hence, we can assume that $\{y_j, z_{j-1}^1\} \in E(G)$ for $j = 2, 3, 4$. Thus,

$$N_G[\{x_2, z_1^1\}] \cap N_G[y_2] = \{x_2, y_2, z_1^1\}.$$

Furthermore, there is a $K_4 \simeq K \subset G$, such that $y_2 \in V(K)$, since $H \simeq K_4$, and $G$ is vertex-transitive. However, $\deg_G(y_2) = 4$; thus, $x_2 \in V(K)$ or $z_1^1 \in V(K)$. Consequently,

$$V(K) \subset N_G[\{x_2, z_1^1\}] \cap N_G[y_2],$$

a contradiction, since $|N_G[\{x_2, z_1^1\}] \cap N_G[y_2]| = 3$. Therefore, $N_G(a_1) \subseteq V(H)$. Similarly, $N_G(\{b_1, c_s, d_s\}) \subseteq V(H)$ implies $V(G) = V(H)$. Therefore, $G$ is a $K_4$-band, since $G$ is 4-regular. $\square$

**Proposition 4.5.** *Let $G$ be a connected 4-transitive graph such that every 5-cycle of $G$ has at least two chords. If $G$ is pure vertex decomposable, then $G \simeq K_5$.*

*Proof.* If $x \in V(G)$, then $x$ is shedding, since $G$ is vertex-transitive. Consequently, by [**5,** Lemma 3.7], there exists a $y \in N_G(x)$ such that $N_G[y] \subseteq N_G[x]$. Thus, $N_G[y] = N_G[x]$, since $G$ is regular. Now, we consider two cases:

*Case* 1. First, suppose that each 5-cycle has at least four chords. Thus, $G$ has a simplicial vertex by [**5,** Theorem 3.11]. Hence, $G \simeq K_5$, since $G$ is vertex-transitive.

*Case* 2. Now, we assume that there is a 5-cycle $C = (x_1, x_2, x_3, x_4, x_5)$ with at most three chords. If $C$ has two non disjoint chords, we can suppose that $\{x_1, x_3\}$, $\{x_1, x_4\} \in E(G)$. Then, there is a $y \in N_G(x_1)$ such that

$$N_G[y] = N_G[x_1] = \{x_1, x_2, x_3, x_4, x_5\}.$$

Since $C$ has at most three chords, $y \in \{x_3, x_4\}$. Without loss of generality, we can assume that $y = x_4$. This implies that

$$\{x_2, x_4\} \in E(G) \quad \text{and} \quad G[\{x_1, x_2, x_3, x_4\}] \simeq K_4.$$

By Lemma 4.4, $G$ is a $K_4$-band, since $C$ has at most three chords, a contradiction by Remark 4.3, since $G[N_G(x_1)]$ is connected. Hence, the chords of $C$ are disjoint. We can suppose that $\{x_1, x_4\}$, $\{x_2, x_5\} \in E(G)$. Thus, there are $y_1 \neq x_5$ and $x \in N_G(x_5) \setminus V(C)$ such that

$$N_G[y_1] = N_G[x_5] = \{x, x_1, x_2, x_4, x_5\}.$$

Since the chords of $C$ are disjoint, $y_1 \in \{x, x_1\}$. Thus, $\{x, x_1\} \in E(G)$. Also, there is a $y_2 \neq x_2$ such that

$$N_G[y_2] = N_G[x_2] \supseteq \{x_1, x_2, x_3, x_5\}.$$

Since the chords of $C$ are disjoint, $y_2 \notin \{x_1, x_3, x_5\}$. Furthermore, $y_2 \in N_G(x_1) = \{x, x_2, x_4, x_5\}$. Then, $y_2 = x$ and $\{x_2, x\} \in E(G)$. This implies that $G[\{x, x_1, x_2, x_5\}] \simeq K_4$. By Lemma 4.4, $G$ is a $K_4$-band, since $|V(G)| \geq 6$, which is a contradiction by Remark 4.3, since $G[N_G(x_1)]$ is connected. $\square$

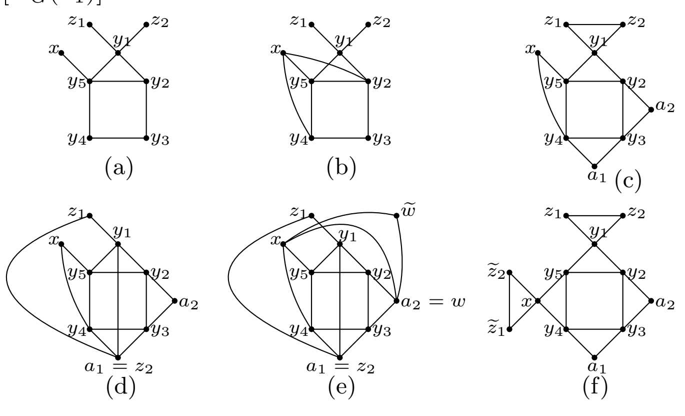



FIGURE 5.

**Lemma 4.6.** *Let $G$ be a 4-transitive vertex decomposable graph such that every 5-cycle has exactly one chord. If $C = (y_1, y_2, y_3, y_4, y_5)$ is a 5-cycle with chord $\{y_2, y_5\}$ and $x \in N_G(y_5) \setminus V(C)$, then $\{x, y_1\} \in E(G)$.*

*Proof.* By contradiction, suppose that $\{x, y_1\} \notin E(G)$. Then, there exist

$$\{z_1, z_2\} \subseteq N_G(y_1) \setminus (V(C) \cup \{x\}),$$

since $\deg_G(x_1) = 4$ and each 5-cycle has exactly one chord, see Figure 5 (a).

Now, we will prove $|\{z_1, z_2\} \cap N_G(y_3)| \leq 1$. By contradiction, assume that $\{z_1, z_2\} \subseteq N_G(y_3)$. Then,

$$N_G(y_1) = \{y_5, y_2, z_1, z_2\} \subseteq N_G(x, y_3).$$

Since $G$ is vertex-transitive, each vertex is a shedding vertex. Furthermore, $\{x, y_3\} \notin E(G)$, since $\deg_G(y_3) = 4$, a contradiction, by Remark 4.2, since $y_1$ is shedding. Therefore, $|\{z_1, z_2\} \cap N_G(y_3)| \leq 1$. Hence, we can assume that $z_1 \notin N_G(y_3)$.

If $N_G(x) \cap \{y_3, z_1\} \neq \emptyset$, then $N_G(y_5) = \{y_1, y_2, y_4, x\} \subseteq N_G(\{z_1, y_3\})$, a contradiction by Remark 4.2, since $y_5$ is shedding. This implies that $N_G(x) \cap \{y_3, z_1\} = \emptyset$.

We will prove that $y_2 \notin N_G(x)$. By contradiction, suppose that $y_2 \in N_G(x)$. If $y_4 \in N_G(x)$, then $\{y_2, y_5\}$ and $\{x, y_5\}$ are chords of $C_1 = (y_1, y_2, x, y_4, y_5, y_1)$, see Figure 5 (b), a contradiction. Therefore, $y_4 \notin N_G(x)$ and $N_G(x) \cap V(C) = \{y_2, y_5\}$. Consequently, $(x, y_2, y_1)$ is a path in $G[N_G(y_5)]$. Furthermore, $N_G(\{y_2, y_5\}) \subseteq V(C) \cup \{x\}$. Then, $\{y_2, y_5\}$ is a connected component of $G[N_G(x)]$, since $N_G(x) \cap V(C) = \{y_2, y_5\}$. Thus, $G[N_G(x)]$ does not have a path with three vertices, a contradiction, since $G$ is vertex-transitive. Therefore, $y_2 \notin N_G(x)$.

Now, we will prove $y_4 \notin N_G(x)$. By contradiction, assume that $y_4 \in N_G(x)$. Then, $N_G(x) \cap V(C) = \{y_4, y_5\}$. Thus, $G[N_G(y_5)]$ has exactly two edges. Since $G$ is vertex-transitive,

$$G[N_G(y_5)] \simeq G[N_G(y_1)] \simeq G[N_G(y_4)]$$

has exactly two edges. Then, $\{z_1, z_2\}$ and $\{a_1, y_3\} \in E(G)$ for some $a_1 \in N_G(y_4) \setminus (V(C) \cup \{x\})$. Since $G[N_G(y_3)] \simeq G[N_G(y_5)]$, $\{y_2, a_2\} \in E(G)$ for some $a_2 \in N_G(y_3) \setminus (V(C) \cup \{x, a_1\})$, see Figure 5 (c). Since

$G[N_G(z)] \simeq G[N_G(y_5)]$ for each $z \in V(G)$, we have $\{x, a_1\}$, $\{a_1, a_2\}$, $\{a_2, y_1\} \notin E(G)$.

We will prove $\{y_1, a_1\} \notin E(G)$. By contradiction, suppose that $\{y_1, a_1\} \in E(G)$. Then, $a_1 = z_2$, since $\{z_1, y_3\} \notin E(G)$, see Figure 5 (d). Since $C_2 = (y_2, y_3, y_4, y_5, y_2)$ is a 4-hole and $G$ is vertex-transitive, $x$ is in a 4-hole $C'$ such that $|V(C') \cap \{y_4, y_5\}| = 1$. Consequently, there exists a $w \in N_G(x) \setminus \{y_5, y_4\}$ such that

$$w \in N_G(N_G(\{y_5, y_4\}) \setminus \{x, y_5, y_4\}) \setminus \{x, y_5, y_4\} = \{y_1, y_2, y_3, z_1, z_2, a_2\}.$$

Thus, $w \in \{z_1, a_2\}$, since the other vertices have degree 4. If $z_1 \in N_G(x)$, then

$$N_G(y_5) = \{x, y_1, y_2, y_4\} \subseteq N_G(\{z_1, y_3\}),$$

a contradiction, by Remark 4.2. Thus, $z_1 \notin N_G(x)$ and $w = a_2$. Since $G[N_G(x)]$ has exactly two edges, there is a $\widetilde{w} \in V(G) \setminus (V(C) \cup \{x, z_1, z_2, a_2\}$ such that

$$\widetilde{w} \in N_G(x) \cap N_G(a_2),$$

see Figure 5 (e). If $\{z_1, \widetilde{w}\}$ is a stable set, then $C_2$ is a connected component of $G \setminus N_G(\{z_1, \widetilde{w}\})$, a contradiction, since a 4-hole is not vertex decomposable. Hence, $\{z_1, \widetilde{w}\} \in E(G)$. Furthermore,

$$C_3 = (y_5, y_1, z_2, y_4, y_5) \quad \text{and} \quad C_4 = (y_5, x, a_2, y_2, y_5)$$

are two 4-holes with $V(C_3) \cap V(C_4) = \{y_5\}$. Since $G$ is vertex-transitive, $z_1$ is in two 4-holes. Thus, there is a 4-hole $(z_1, \widetilde{w}, b_1, b_2, z_1)$ where $b_2 \in \{y_1, z_2\}$. Hence,

$$b_1 \in N_G(\{y_1, z_2\}) \setminus \{y_1, z_1, z_2\} = \{y_2, y_3, y_4, y_5\},$$

a contradiction, since $\widetilde{w} \notin N_G(C_2)$. Therefore, $\{y_1, a_1\} \notin E(G)$. This implies that $\{z_1, z_2\} \cap (V(C) \cup \{x, a_1, a_2\}) = \emptyset$.

Similarly, $\{x, a_2\} \notin E(G)$ (by symmetry between $x$ and $y_1$). Thus, $N_G(x) = \{y_4, y_5, \widetilde{z}_1, \widetilde{z}_2\}$ such that $\{\widetilde{z}_1, \widetilde{z}_2\} \in E(G)$ and

$$\{\widetilde{z}_1, \widetilde{z}_2\} \cap (V(C) \cup \{x, a_1, a_2\}) = \emptyset,$$

see Figure 5 (f). If $\{z_i, \widetilde{z}_j\}$ is a stable set for some $1 \leq i \leq j \leq 2$, then $N_G(y_5) \subseteq N_G(\{z_i, \widetilde{z}_j, y_3\})$, a contradiction, by Remark 4.2. Consequently, $\{z_i, \widetilde{z}_j\} \in E(G)$ for each $1 \leq i \leq j \leq 2$. Thus, $G[N_G(z_1)]$

is connected, a contradiction, since $G[N_G(z_1)] \simeq G[N_G(y_5)]$. Therefore, $y_4 \notin N_G(x)$.

Hence, $y_2, y_3, y_4, z_1 \notin N_G(y_5)$, and $G[N_G(y_5)]$ has exactly one edge. We take $a \in N_G(x) \setminus \{y_5\}$. Then, $N_G(y_5) \subseteq N_G(\{a, z_1, y_3\})$. By Remark 4.2, $a \in N_G(\{z_1, y_3\})$, since $y_5$ is a shedding vertex and $\{z_1, y_3\} \notin E(G)$. Thus,

$$N_G(x) \setminus \{y_5\} \subseteq N_G(\{z_1, y_3\}) \setminus (V(C) \cup \{z_1\}).$$

Furthermore, $|N_G(x) \setminus \{y_5\}| = 3$ and $|N_G(y_3) \setminus V(C)| = 2$; thus, there is a

$$y \in (N_G(x) \setminus \{y_5\}) \cap N_G(z_1)$$

and $C_5 = (x, y_5, y_1, z_1, y, x)$ is a 5-cycle. Since $\deg_G(y_5) = 4$, $N_G(y_5) \cap V(C_5) = \{x, y_1\}$. Furthermore $\{y, y_1\} \notin E(G)$, since $G[N_G(y_1)] \simeq G[N_G(y_5)]$ has exactly one edge. This implies that $C_5$ is induced since $N_G(x) \cap (V(C) \cup \{z_1\}) = \emptyset$. This is a contradiction, since each 5-cycle has exactly one chord. Therefore, $\{x, y_1\} \in E(G)$. $\square$

**Lemma 4.7.** *Let $G$ be a 4-transitive graph such that every 5-cycle has exactly one chord. If $G$ is vertex decomposable, then $G$ contains $K_4$.*

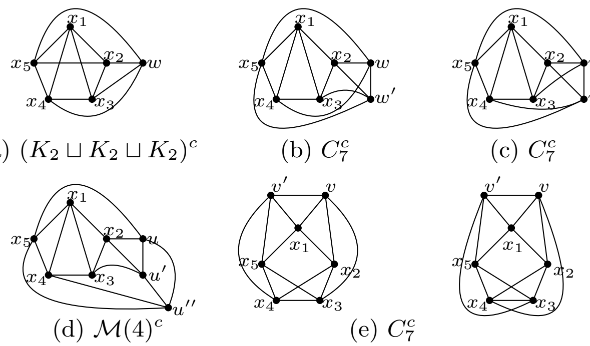
*Proof.* By Proposition 4.5, if $G$ does not have a 5-cycle, then $G \simeq K_5$, which contradicts the fact that each 5-cycle has exactly one chord. Thus, there is a 5-cycle $C = (y_1, y_2, y_3, y_4, y_5, y_1)$ of $G$. We can assume that $\{y_2, y_5\} \in E(G)$. Since $\deg_G(y_5) = \deg_G(y_2) = 4$, there are

$$x \in N_G(y_5) \setminus V(C) \quad \text{and} \quad y \in N_G(y_2) \setminus V(C).$$

By Lemma 4.6, $\{x, y_1\}, \{y, y_1\} \in E(G)$. If $x = y$, then $G[\{x, y_1, y_2, y_5\}] \simeq K_4$. Now, if $x \neq y$, then $G[N_G(y_1)]$ is connected. Since $G$ is vertex-transitive, $G[N_G(y_5)]$ is connected. Hence, $\{x, y_4\} \in E(G)$, since $\deg_G(y_1) = \deg_G(y_2) = 4$. Similarly, $\{y, y_3\} \in E(G)$, since $G[N_G(y_2)]$ is connected. Thus, $C' = (y_1, y_2, y_3, y_4, x, y_1)$ is a 5-cycle. Since $\deg_G(y_1) = \deg_G(y_2) = 4$, and since $C'$ must have a chord, we are forced to use $\{x, y_3\}$ as the chord of $C'$. Hence, $\{y_4, y_5\}$ and $\{x, y_3\}$ are chords of $(y_2, y_3, y_4, x, y_5, y_2)$, a contradiction, since each 5-cycle has exactly one chord. $\square$

**Definition 4.8.** The complement of a graph $G$, denoted $G^c$, is a graph whose vertex set is $V(G)$, and

$$E(G^c) = \{\{x,y\} : x \neq y \in V(G) \text{ and } \{x,y\} \notin E(G)\}.$$



(a) $(K_2 \sqcup K_2 \sqcup K_2)^c$ (b) $C_7^c$ (c) $C_7^c$

(d) $\mathcal{M}(4)^c$ (e) $C_7^c$

FIGURE 6.

**Proposition 4.9.** *Let $G$ be a 4-transitive connected graph such that $G$ has a 5-cycle $C$ with at least two chords and a 4-hole. Then, $G$ is isomorphic to $(K_2 \sqcup K_2 \sqcup K_2)^c$ or $C_7^c$ or $\mathcal{M}(4)^c$.*

*Proof.* We set $C = (x_1, x_2, x_3, x_4, x_5, x_1)$. First, we suppose that $C$ has two concurrent chords. Then, we assume that $\{x_1, x_3\}$ and $\{x_1, x_4\} \in E(G)$. Since $G$ has a 4-hole and is vertex-transitive, there is a 4-hole $C'$ with $x_1 \in V(C')$. Thus,

$$|\{x_2, x_5\} \cap V(C')| \geq 1,$$

since $\{x_3, x_4\} \in E(G)$. Now we will study two cases:

*Case* 1. If $|\{x_2, x_5\} \cap V(C')| = 1$, then, we can assume that $x_2 \in C'$. Thus, $C' = (x_1, x_2, w, x_4, x_1)$ with $w \in V(G) \setminus V(C)$. Since $G[N_G(x_1)]$ is connected and $G$ is vertex-transitive, we have

$$G[N_G(x_4)] = G[\{x_1, x_3, x_5, w\}]$$

is connected. Hence, $\{x_3, x_5\} \cap N_G(w) \neq \emptyset$, since $\deg_G(x_1) = 4$.

We assume $\{x_3, w\} \in E(G)$. Thus, $G[N_G(x_3)]$ has a 4-cycle. Since $G$ is vertex-transitive, $G[N_G(x_1)]$ and $G[N_G(x_4)]$ have a 4-cycle. This implies that $\{x_2, x_5\}, \{w, x_5\} \in E(G)$, since

$$\deg_G(x_4) = \deg_G(x_3) = \deg_G(x_1) = 4.$$

Therefore, $G \simeq (K_2 \sqcup K_2 \sqcup K_2)^c$, since $G$ is 4-regular, see Figure 6 (a).

Now, suppose that $\{x_3, w\} \notin E(G)$. Then,

$$\{x_5, w\} \in E(G) \quad \text{and} \quad G[N_G(x_4)]$$

have no 4-cycles. Consequently, $G[N_G(x_1)]$ has no 4-cycles, and $\{x_2, x_5\} \notin E(G)$. Since $\deg_G(x_1) = 4$ and $\{w, x_3\} \notin E(G)$, there is a $w' \in N_G(w) \setminus V(C)$. Furthermore, $G[N_G(w)] \simeq G[N_G(x_1)]$ is connected, $\deg_G(x_4) = 4$ and $\{x_2, x_5\} \notin E(G)$. Then, $x_2, x_5 \in N_G(w')$. In addition, $G[N_G(x_2)] \simeq G[N_G(x_1)]$ is connected and $\deg_G(w) = \deg_G(x_1) = 4$; thus, $\{x_3, w'\} \in E(G)$. Therefore, $G \simeq C_7^c$, see Figure 6 (b).

*Case* 2. Now, we assume that $x_2, x_5 \in V(C'')$, for each 4-hole $C''$ with $x_1 \in V(C'')$. Thus, there is a $u \in V(G) \setminus V(C)$ such that $C' = (x_1, x_5, u, x_2, x_1)$; hence, $\{x_2, x_5\} \notin E(G)$. Since $\deg_G(x_2) = 4$, there is a

$$u' \in N_G(x_2) \setminus \{x_1, x_2, x_3, x_5, u\}.$$

We shall prove $u' \neq x_4$. By contradiction, we assume that $u' = x_4$. If $P_4$ is a path with four vertices, then $P_4 \subseteq G[N_G(x_1)]$. Since $G$ is vertex-transitive, we have $P_4 \subseteq G[N_G(x_2)]$. This implies that $\{x_3, u\} \in E(G)$, since $\deg_G(x_1) = \deg_G(x_4) = 4$. Furthermore, there is a

$$v \in V(G) \setminus (V(C) \cup \{u\})$$

such that $\{u, v\} \in E(G)$, since $G$ is 4-regular and $\deg_G(x_1) = \deg_G(x_4) = 4$. Since $G$ is vertex-transitive,

$$P_4 \subseteq G[N_G(u)] \simeq G[\{x_2, x_3, x_5, v\}].$$

This is a contradiction, since $\deg_G(x_2) = \deg_G(x_3) = 4$. Therefore, $u' \neq x_4$. Hence, $u' \notin (V(C) \cup \{u\})$. Also, $P_4 \subseteq G[N_G(x_1)] \simeq G[N_G(x_2)]$. Consequently, $\{u, u'\} \in E(G)$ and $\{u, u'\} \cap N_G(x_3) \neq \emptyset$, since $\deg_G(x_1) = 4$. Thus, $\{x_3, u\} \in E(G)$ or $\{x_3, u'\} \in E(G)$.

If $\{x_3, u\} \in E(G)$, then $\{x_5, u'\} \in E(G)$, since $P_4 \subseteq G[N_G(u)]$ and $\deg_G(x_2) = \deg_G(x_3) = 4$. Similarly, $\{x_4, u'\} \in E(G)$, since $P_4 \subseteq G[N_G(x_5)]$ and $\deg_G(x_1) = \deg_G(u) = 4$. Therefore, $G \simeq C_7^c$, see Figure 6 (c). Now, if $\{x_3, u'\} \in E(G)$, then $P_4 \simeq G[N_G(x_1)]$, since $\deg_G(x_2) = \deg_G(x_3) = 4$. Since $G$ is vertex-transitive, $P_4 \simeq G[N_G(x_3)]$, thus, $\{x_4, u'\} \notin E(G)$. Hence, there is a $u'' \in N_G(u) \cap$

$N_G(u')$, since

$$G[N_G(u')] \simeq P_4, \qquad N_G(x_3) = \{x_1, x_2, x_4, u'\}$$

and $\{u', x_4\} \notin E(G)$. Furthermore, if $u'' \in V(C)$, then $u'' = x_5$. This implies that $\{u, x_4\} \in E(G)$, since $P_4 \simeq G[N_G(x_5)]$ and $\deg_G(x_1) = \deg_G(u') = 4$, a contradiction, since $C''' = (x_1, x_4, u, x_2, x_1)$ is a 4-hole with $x_1 \in V(C''')$ and $x_5 \notin V(C''')$. Consequently, $u'' \notin V(C)$. Hence, $\{u'', x_5\} \in E(G)$, since $G[N_G(u)] \simeq P_4$ and $\deg_G(x_2) = 4$. Thus, $\{x_4, u''\} \in E(G)$, since

$$G[N_G(x_5)] \simeq P_4 \quad \text{and} \quad \deg_G(u) = \deg_G(x_1) = 4.$$

Therefore, $G \simeq \mathcal{M}(4)^c$, see Figure 6 (d).

Finally, suppose that $C$ has no concurrent chords. Thus, $C$ has exactly two chords. We can assume that $\{x_2, x_4\}$ and $\{x_3, x_5\} \in E(G)$. Then, $\{x_2, x_5\} \notin E(G)$, and there exist $v, v' \in N_G(x_1) \setminus V(C)$, since $\deg_G(x_1) = 4$. Furthermore $P_3 \subseteq G[N_G(x_3)]$; thus, $P_3 \subseteq G[N_G(x_1)]$, since $G$ is vertex-transitive. Consequently, there is an edge between at least one vertex of $\{v, v'\}$ and one vertex of $\{x_2, x_5\}$. Without loss of generality, we can suppose that $\{x_2, v\} \in E(G)$. This implies that $K_2 \sqcup K_2 \subseteq G[N_G(x_2)] \simeq G[N_G(x_1)]$. Hence, $\{x_5, v'\} \in E(G)$, since $\deg_G(x_2) = 4$. In addition, $G[N_G(x_1)] \supseteq P_3$. Then, $\{v, v'\} \in E(G)$, since $\deg_G(x_2) = \deg_G(x_5) = 4$. Thus,

$$P_4 \simeq G[N_G(x_1)] \simeq G[N_G(x_2)] \simeq G[N_G(x_5)].$$

Hence, $N_G(v) \cap \{x_3, x_4\} \neq \emptyset$ and $N_G(v') \cap \{x_3, x_4\} \neq \emptyset$, since $\deg_G(x_1) = 4$. Since

$$\deg_G(v) = \deg_G(v') = \deg_G(x_3) = \deg_G(x_4) = 4,$$

we have two cases:

(i) $\{v, x_3\}, \{v', x_4\} \in E(G)$, or

(ii) $\{v, x_4\}, \{v', x_3\} \in E(G)$.

In both cases, $G \simeq C_7^c$, since $G$ is 4 regular, see Figure 6 (e). $\square$

**Definition 4.10.** Let $G$ be a graph such that $V(G) = V(F) \cup V(H)$, where $H = (H_1, \ldots, H_s)$ is a $K_4$-chain, $V(H_i) = \{a_i, b_i, c_i, d_i\}$, and its end vertices are $a_1$, $b_1$, $c_s$, $d_s$. $G$ is an edge-$K_4$-chain if $F \simeq K_2$ with

$V(F) = \{x, y\}$ and

$$E(G) = E(H) \cup E(F) \cup \{\{x, a_1\}, \{y, b_1\}\}.$$

$G$ is a triangle-$K_4$-chain if $F \simeq K_3$ with $\{x, y, z\}$ and $E(G) = E(H) \cup E(F) \cup \{\{x, a_1\}, \{y, b_1\}\}$. In both cases, we denote $G = (F, H_1, H_2, \dots, H_s)$.

**Remark 4.11.** If $S$ is a stable set of $G$, where $G = (F, H_1, \dots, H_s)$ is an (edge) triangle-$K_4$-chain, or $G = (H_1, \dots, H_s)$ is a $K_4$-chain, then $|S \cap H_i| \leq 1$ and $|S \cap F| \leq 1$. Furthermore, $\{x, c_1, \dots, c_s\}$ or $\{c_1, \dots, c_s\}$ is a stable set. Therefore, $\beta(G) = s + 1$ or $\beta(G) = s$, respectively.

**Lemma 4.12.** *If $G$ is a $K_4$-chain, triangle-$K_4$-chain or edge-$K_4$-chain, then $G$ is pure vertex decomposable.*

*Proof.* We perform the proof by induction on $|V(G)|$. First, suppose $G = (H_1, \dots, H_s)$ is a $K_4$-chain. Then, $\beta(G) = s$, by Remark 4.11. If $s = 1$, then $G \simeq K_4$, and $G$ is pure vertex decomposable. Now, assume that $s \geq 2$. Furthermore, $G_1 = G \setminus a_1$ is the triangle-$K_4$-chain $(T_1, H_2, \dots, H_s)$, where $T_1 = H_1 \setminus a_1$. In addition,

$$G_2 = G \setminus N_G[a_1]$$

is the $K_4$-chain $(H_2, \dots, H_s)$. By induction, $G_1$ and $G_2$ are pure vertex decomposable. Furthermore, by Remark 4.11, $\beta(G_1) = s$ and $\beta(G_2) = s - 1$. Hence, each face of $\Delta_{G_2}$ is not a facet of $\Delta_{G_1}$. This implies, by Remark 2.1, that $a_1$ is a shedding vertex. Therefore, $G$ is pure vertex decomposable.

Now, suppose that $G = (F_1, H_1, H_2, \dots, H_s)$ is a triangle-$K_4$-chain, with

$$V(F_1) = \{x, y, z\},$$

$$V(H_i) = \{a_i, b_i, c_i, d_i\}$$

and

$$\{x, a_1\}, \{y, b_1\} \in E(G).$$

Also, $G_3 = G \setminus z$ is the edge-$K_4$-chain $(F_1', H_1, H_2, \dots, H_s)$, where $F_1' = F_1 \setminus z \simeq K_2$. Furthermore, $G_4 = G \setminus N_G[z]$ is the $K_4$-chain $(H_1, H_2, \dots, H_s)$. Thus, by induction, $G_3$ and $G_4$ are pure vertex decomposable. From Remark 4.11, $\beta(G_3) = s + 1$ and $\beta(G_4) = s$.

Then, by Remark 2.1, $z$ is a shedding vertex. Therefore, $G$ is pure vertex decomposable.

Finally, if $G = (F_2, H_1, H_2, \ldots, H_s)$ is an edge-$K_4$-chain, $V(F_2) = \{x, y\}$ and $V(H_i) = \{a_i, b_i, c_i, d_i\}$, where $\{x, a_1\}, \{y, b_1\} \in E(G)$. Consequently,

$$G_5 = G \setminus N_G[a_1] = (H_2, \ldots, H_s) \cup \{y\}$$

if $s \geq 2$, or

$$G_5 = G \setminus N_G[a_1] = \{y\}$$

if $s = 1$. Thus, by induction, $G_5$ is pure vertex decomposable and $\beta(G_5) = s$. Furthermore,

$$G_6 = G \setminus a_1 = (F_2', H_2, \ldots, H_s) \cup \{\{x, y\},\ \{y, b_1\}\},$$

where $F_2' = G[\{b_1, c_1, d_1\}] \simeq K_3$ and $\beta(G_6) = s + 1$. Then, $a_1$ is a shedding vertex. Now, we take $G_7 = G_6 \setminus \{y\}$ and $G_8 = G_6 \setminus N_G[y]$. If $s \geq 2$, then

$$G_7 = (F_2', H_2, \ldots, H_s) \cup \{x\},$$

where $(F_2', H_2, \ldots H_s)$ is a triangle-$K_4$-chain, and $G_8$ is the edge-$K_4$-chain $(F_2'', H_2, \ldots, H_s)$, where $F_2'' = G[\{c_1, d_1\}] \simeq K_2$. Also, if $s = 1$, then

$$G_7 = F_2' \cup \{x\} \quad \text{and} \quad G_8 = G[\{c_1, d_1\}] \simeq K_2.$$

Hence, by the induction hypothesis, $G_7$ and $G_8$ are pure vertex decomposable. Furthermore, by Remark 4.11, $\beta(G_7) = s + 1$ and $\beta(G_8) = s$. Thus, $y$ is shedding. This implies that $G_6$ is pure vertex decomposable. Therefore, $G$ is pure vertex decomposable. □

**Lemma 4.13.** *If $G$ is a $K_4$-band, then $G$ is pure vertex decomposable.*

*Proof.* Since $G$ is a $K_4$-band, there is a $K_4$-chain $H = (H_1, \ldots, H_s)$ such that $V(G) = V(H)$, where $V(H_i) = \{a_i, b_i, c_i, d_i\}$ and $H_i \simeq K_4$. Furthermore, if $G$ is 4-regular, then $\{a_1, c_s\} \in E(G)$ or $\{a_1, d_s\} \in E(G)$. We take $G_1 = G \setminus N_G[a_1]$. Then, $G_1$ is a triangle if $s = 2$, or $G_1$ is a triangle-$K_4$-chain $(H_2, \ldots, H_{s-1}, F_1)$ if $s \geq 3$. In the second case, $F_1$ is a triangle. Moreover, $V(F_1) = \{a_s, b_s, d_s\}$ if $\{a_1, c_s\} \in E(G)$, or $V(F_1) = \{a_s, b_s, c_s\}$ if $\{a_1, d_s\} \in E(G)$. Thus, by Lemma 4.12 and Remark 4.11, $G_1$ is pure vertex decomposable and $\beta(G_1) = s - 1$.

Now, we take $G_2 = G \setminus a_1$. Hence, $G_2 \setminus b_1$ is the edge-$K_4$-chain $(F_2, H_2, \ldots, H_s)$, where $F_2$ is the edge $\{c_1, d_1\}$. Consequently, by Lemma 4.12 and Remark 4.11, $G_2 \setminus b_1$ is pure vertex decomposable and $\beta(G_2 \setminus b_1) = s$. Now, we take $G' = G_2 \setminus N_{G_2}[b_1]$. Then, $G'$ is a triangle if $s = 2$, or $G'$ is the triangle-$K_4$-chain $(H_2, \ldots, H_{s-1}, F_3)$ if $s \geq 3$. In the second case, $F_3$ is a triangle, where $V(F_3) = \{a_s, b_s, c_s\}$ if $\{a_1, c_s\} \in E(G)$, or $V(F_3) = \{a_s, b_s, d_s\}$ if $\{a_1, d_s\} \in E(G)$. From Remark 4.11, $\beta(G') = s - 1$. Hence, $b_1$ is a shedding vertex and $\beta(G_2) = s$. Furthermore, by Lemma 4.12, $G'$ is pure vertex decomposable. This implies that $G_2$ is pure vertex decomposable, since $G_2 \setminus b_1$ and $G'$ are pure vertex decomposable. Hence, $a_1$ is a shedding vertex, since $\beta(G_1) = s-1$ and $\beta(G_2) = s$. Therefore, $G$ is pure vertex decomposable, since $G_1$ and $G_2$ are pure vertex decomposable. □

**Theorem 4.14.** *Let $G$ be a 4-transitive graph without a 5-hole. Then, the following conditions are equivalent*:

(i) *$\Delta_G$ is pure vertex decomposable.*
(ii) *Each connected component of $G$ is isomorphic to $K_5$ or $C_7^c$ or a $K_4$-band.*

*Proof.*

(i) ⇒ (ii). Let $G'$ be a connected component of $G$. By Proposition 4.5, we can assume that there is a 5-cycle $C$ with exactly one chord in $G'$. Thus, $G'$ has a 4-hole. We consider two cases:

(a) First, suppose each 5-cycle of $G'$ has exactly one chord. Then, by Lemma 4.7, $G'$ contains a $K_4$. Hence, by Lemma 4.4, $G = K_5$ or $G$ is a $K_4$-band.

(b) Now, we assume that $G'$ has a 5-cycle with at least two chords. Thus, by Proposition 4.9,

$$G \in \{C_7^c, \mathcal{M}(4)^c, (K_2 \sqcup K_2 \sqcup K_2)^c\},$$

since $G'$ has a 4-hole. However, $\mathcal{M}(4)^c$ has a 5-hole, and

$$\Delta_{(K_2 \sqcup K_2 \sqcup K_2)^c} = K_2 \sqcup K_2 \sqcup K_2$$

has dimension 1 and is not connected. Therefore, $G \simeq C_7^c$.

(ii) ⇒ (i). By Lemma 4.13, each $K_4$-band is pure vertex decomposable. Furthermore, $\Delta_{K_5}$ has dimension 0; thus, it is pure vertex decomposable. Also, $\Delta_{C_7^c} \simeq C_7$ is a connected pure simplicial complex of dimension 1, so it is vertex decomposable by Proposition 3.5. Therefore, $G$ is vertex decomposable. $\square$

**5. Well-covered and Cohen-Macaulay generalized Petersen graphs.** In this section, we characterize the generalized Petersen graphs, see Definition 2.11, with the following properties: well-covered, Cohen-Macaulay and pure vertex decomposable.

**Lemma 5.1.** *If $G = P(n, r)$ is a generalized Petersen graph, then*:

(i) *$G$ contains a 3-cycle if and only if $n = 3$ or $n = 3r$.*
(ii) *$G$ contains a 4-cycle if and only if $n = 4$, $n = 4r$ or $r = 1$.*
(iii) *$G$ contains a 5-cycle if and only if $n = 5$, or $n = 5r$, or $r = 2$, or $n = 5(r/2)$ with $r$ even, or $r = (n-1)/2$ with $n$ odd.*

*Proof.* We assume that $V(G) = \{a_0, a_1, \dots, a_{n-1}, b_0, b_1, \dots, b_{n-1}\}$ and $E(G)$ is

$$\begin{aligned}\{a_0b_0, \dots, a_{n-1}b_{n-1}\} \cup \{a_ia_j : |i-j| \equiv 1 \pmod n\}\\ \cup \{b_ib_j, : |i-j| \equiv r \pmod n\}\end{aligned}$$

with $n \geq 3$. We take $H_1 = G[\{a_0, \dots, a_{n-1}\}]$ and $H_2 = G[\{b_0, \dots, b_{n-1}\}]$. Thus, $H_1$ is an $n$-cycle. Since $\{b_i, b_j\} \in E(G)$ if and only if $|i-j| \equiv r \pmod n$, then $i \equiv r + j$ or $i \equiv n - r + j \pmod n$. Hence, $H_2$ is 2-regular, and $C_i = (b_i, b_{i+r}, \dots, b_{i+n'r})$ is a cycle, where $n' = n/\gcd(n, r)$. Therefore, $H_2$ is the disjoint union of $n'$-cycles.

Let $C$ be an $s$-cycle in $G$, with $s \in \{3, 4, 5\}$. If $C \subseteq H_1$, then $s = n$. Now, suppose that $C \subseteq H_2$. Then, $s = n'$. This implies that $n = \gcd(n, r) \cdot s$ and $r = \gcd(n, r) \cdot r'$, where $\gcd(s, r') = 1$. Since $2r < n$, thus, $2r' < s$. If $s \leq 4$, then $r' = 1$. Hence, $r = \gcd(n, r)$ and $n = s \cdot r$.

Now, if $s = 5$, then $r' = 1$ or $r' = 2$, i.e., $n = 5 \cdot r$ or $r = 2\gcd(n, r)$ and $n = 5(r/2)$. Now, we assume that $V(C) \cap V(H_i) \neq \emptyset$ for $i = 1, 2$. Since $V(H_1) \cap V(H_2) = \emptyset$ and $C$ is connected, then there is an $e \in E(C) \setminus (E(H_1) \cup E(H_2))$. We can assume that $e = \{a_j, b_j\}$. Hence, $a_{j+1} \in V(C)$ or $a_{j-1} \in V(C)$, as well as $b_{j+r} \in V(C)$ or $b_{j-r} \in V(C)$. Thus, $s \geq 4$. If $s = 4$, then $j \pm 1 \equiv j \pm r \pmod n$. Consequently, $r = 1$, since $2r < n$. Now, we suppose that $s = 5$. If $|V(C) \cap V(H_1)| = 3$, then $i \pm 2 \equiv i \pm r \pmod{n}$. Hence, $r = 2$, since $2r < n$. Finally, if $|V(C) \cap V(H_2)| = 3$, then $i \pm 1 \equiv i \pm 2r \pmod{n}$. This implies that $2r \equiv \pm 1 \pmod{n}$. Therefore, $n$ is odd and $r = (n-1)/2$, since $2r < n$. $\square$

**Lemma 5.2.** *$K_{s,s}$ is not a generalized Petersen graph.*

*Proof.* If $x \in G = K_{s,s}$, then $\deg_G(x) = s$. If $G$ is a generalized Petersen graph, then $G$ is 3-regular. Thus, $s = 3$, and $V(G) = \{a_0, a_1, a_2, b_0, b_1, b_2\}$. Hence, $H = G[\{a_0, a_1, a_2\}]$ is a 3-cycle. This is a contradiction, since $K_{s,s}$ is bipartite. $\square$

**Theorem 5.3.** *Let $G = P(n,r)$ be a generalized Petersen graph. Then, $G$ is well-covered if and only if $(n,r) \in \{(3,1), (5,1), (6,2), (7,2)\}$.*

*Proof.*

$\Rightarrow$. $G$ satisfies Theorem 2.6 (i), (ii) or (iii). If $G$ satisfies (i), then, by Lemma 5.2, $G \cong P_{14}$, and $(n,r) = (7,2)$. Now, if $G$ satisfies (iii), then, $G$ has a 3-cycle and a 5-cycle. If $n = 3$, since $2r < n$, then, $r = 1$ and $(n,r) = (3,1)$. Now, by Lemma 5.1 (i), we can assume that $n = 3r$. Also, by Lemma 5.1 (iii), $r = 2$ or $r = (n-1)/2$. Consequently, $(n,r) = (6,2)$ or $(n,r) = (3,1)$. Now, we can assume that $G$ satisfies Theorem 2.6 (ii). Thus, $G$ has a 4-cycle, and, by Lemma 5.1, $n = 4$, $n = 4r$ or $r = 1$. Furthermore, $G$ has a 3-cycle, 5-cycle or 7-cycle. If $G$ has a 3-cycle, then, from Lemma 5.1, $(n,r) = (3,1)$.

Now, if $G$ has a 5-cycle, then $(n,r) \in \{(5,1), (3,1), (8,2)\}$ by Lemma 5.1. If $(n,r) = (8,2)$, then $H_1$ is the 8-cycle $(a_0, a_1, \ldots, a_7, a_0)$ and

$$H_2 = (b_0, b_2, b_4, b_6, b_0) \cup (b_1, b_3, b_5, b_7, b_1).$$

Hence, $\{a_0, a_2, a_5, b_1, b_4\}$ and $\{b_0, b_1, b_4, b_5, a_3, a_6\}$ are maximal stables sets, contradicting the fact that $G$ is well-covered. Thus, $(n,r) \neq (8,2)$. Finally, we assume that $G$ has a 7-cycle $C$. If $V(C) \subseteq V(H_1)$ or $V(C) \subseteq V(H_2)$, then $n = 7$ or $n = 7r$, implying $(n,r) = (7,1)$. This further implies $H_2 = (b_0, b_1, \ldots, b_6, b_0)$ and $H_1 = (a_0, a_1, \ldots, a_6, a_0)$. Consequently, $\{a_1, a_3, a_5, b_0, b_2, b_4\}$ and $\{a_2, a_5, b_0, b_3\}$ are maximal stables sets. This is a contradiction, since $G$ is well-covered. Then, $(n,r) \neq (7,1)$. Now, we can assume that $a_i b_i \in V(C)$; thus,

$$|V(C) \cap V(H_1)| \geq 2 \quad \text{and} \quad |V(C) \cap V(H_2)| \geq 2.$$

If $|V(C) \cap V(H_i)| = 2$ for some $i \in \{1,2\}$, then $4r \pm 1 \equiv 0 \pmod{n}$ or $r \pm 4 \equiv 0 \pmod{n}$. If $n = 4$ or $n = 4r$, then $r - 4 = 0$, since $1 < r < n/2$. Hence, $(n,r) = (16,4)$. However,

$$\{a_0, a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, b_1, b_3, b_9, b_{11}\}$$

and

$$\{a_0, a_3, a_6, a_8, a_{11}, a_{14}, b_1, b_2, b_4, b_7, b_9, b_{10}, b_{12}, b_{15}\}$$

are maximal stables sets of $P(16,4)$, a contradiction. Hence, $(n,r) \neq (16,4)$. If $r = 1$, then $n \in \{3,5\}$.

Now, we suppose that $|V(C) \cap V(H_i)| = 3$ for some $i \in \{1,2\}$. In this case, we have that $2r \pm 3 \equiv 0 \pmod{n}$ or $3r \pm 2 \equiv 0 \pmod{n}$. If $n = 4$ or $n = 4r$, then $n$ is even, and furthermore, $r < n$, implying $3r \pm 2 = 4rq$ for some $q \in \mathbb{Z}$. Thus, $r(4q - 3) = \pm 2$, and consequently, $r = 2$. This implies $(n,r) = (8,2)$. However, we proved that $P(8,2)$ is not well-covered. Finally, if $r = 1$, we obtain that $2 \pm 3 \equiv 0 \pmod{n}$. Therefore, $n = 5$, and $(n,r) = (5,1)$.

$\Leftarrow$. We take $G = P(n,r)$. If $(n,r) = (3,1)$, then $G \setminus N_G[x] \simeq K_2$ for any vertex $x \in V(G)$. Then, the cardinality of every maximal stable set is two. Thus, $G$ is well-covered. We take $G' = G \setminus N_G[\{x,y\}]$, where $x, y$ are non adjacent vertices. Now, if $(n,r) = (5,1)$, then $G'$ is isomorphic to $P_4$, or $K_2^c$, or $K_1 \sqcup K_2$. Hence, the cardinality of every maximal stable set is four, so $P(5,1)$ is well-covered. Now, if $G = P(6,2)$, then $G'$ is isomorphic to $P_4$, or $C_5$, or $K_2 \sqcup K_2$, or

$$K_2 \sqcup K_3 \quad \text{or} \quad K_2 \sqcup K_3,$$

with an edge of $K_2$ to $K_3$. Then, the cardinality of every maximal stable set is four. Therefore, $(n,r) = (6,2)$ is well-covered. Finally, $P(7,2) \simeq P_{14}$. Thus, $P(7,2)$ is well-covered, by Theorem 2.5. $\square$

**Theorem 5.4.** *Let $G = P(n,r)$ be a generalized Petersen graph. Then, $G$ is pure vertex decomposable* (*Cohen-Macaulay*) *if and only if* $(n,r) = (3,1)$.

*Proof.*

$\Rightarrow$. By Theorem 5.3, we have

$$(n,r) \in \{(3,1), (5,1), (6,2), (7,2)\},$$

since $G$ is well-covered. Furthermore, $h(\Delta_{P(5,1)}) = (1,6,6,-4,1)$; $h(\Delta_{P(6,2)}) = (1,8,18,10,-1)$; and $h(\Delta_{P(7,2)}) = (1,9,24,18,-2,-1)$. Therefore, $(n,r) = (3,1)$.

$\Leftarrow$. $\Delta_{P(3,1)}$ is a one-dimensional connected simplicial complex. Therefore, by Theorem 3.5, $P(3,1)$ is pure vertex decomposable (Cohen-Macaulay). $\square$

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