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GENERAL WEIGHTED HARDY-TYPE INEQUALITIES RELATED TO GREINER OPERATORS

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ABSTRACT. In this article, we present a general method that can be used to deduce weighted Hardy-type inequalities from a particular non-linear partial differential inequality in a relatively simple and unified way on the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, defined by the Greiner vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l},$$
$$Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l},$$

$j = 1, \dots, n$, where $z = x + iy \in \mathbb{C}^n$, $l \in \mathbb{R}$, $k \geq 1$. Our method allows us to improve, extend, and unify many previously obtained sharp weighted Hardy-type inequalities as well as to yield new ones. These cases are illustrated by giving many concrete examples, including radial, logarithmic, hyperbolic and non-radial weights. Furthermore, we introduce a new technique for constructing two-weight L^p Hardy-type inequalities with remainder terms on smooth bounded domains Ω in $\mathbb{R}^{2n+1}$. We also give several applications leading to various weighted Hardy inequalities with remainder terms.

1. Overview and generalities.

Let $n \geq 2$ and $1 \leq p < n$. The classical Hardy inequality states that

$$\int_{\mathbb{R}^n} |\nabla\phi|^p dx \geq \left(\frac{n-p}{p}\right)^p \int_{\mathbb{R}^n} \frac{|\phi|^p}{|x|^p}\, dx \tag{1.1}$$

holds for every $\phi \in C_0^\infty(\mathbb{R}^n)$. Here, the subscript zero signifies compact support and the constant $(n-p)^p/p^p$ in (1.1) is sharp, but, for $p > 1$, it is never achieved. This inequality has been intensively studied in the Euclidean framework for the last few decades, particularly, in view of its

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applications to partial differential equations motivated by physics and geometry; see, for instance, [**1, 4, 5, 8, 13, 15, 16, 17, 19, 29, 30**], and the references therein.

Considerable effort has also been devoted to extending the Hardy inequality (1.1) to subelliptic settings. For example, in the case of the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, defined by the Greiner vector fields,

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l},$$
$$Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l},$$

$j = 1, \ldots, n$, where $z = x + iy \in \mathbb{C}^n$, $l \in \mathbb{R}$, $k \geq 1$, Zhang and Niu [**33**] first proved the following Hardy inequality:

$$(1.2)\quad \int_{\mathbb{R}^{2n+1}} |\nabla_k \phi|^p dz\, dl \geq \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \left(\frac{|z|}{\rho}\right)^{p(2k-1)} \frac{|\phi|^p}{\rho^p}\, dz\, dl$$

for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{(0,0)\})$, $1 < p < Q$. In order to prove (1.2), they used a Picone-type identity for the family $\{X_j, Y_j\}$. Here, $\nabla_k = (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is the subelliptic gradient,

$$\rho = (|z|^{4k} + l^2)^{1/4k}$$

is the gauge induced by the fundamental solution for the subelliptic operator $\Delta_k = \sum_{j=1}^n (X_j^2 + Y_j^2)$, and $Q = 2n + 2k$ is the homogeneous dimension for Δ_k. If $k = 1$, then Δ_k becomes the sub-Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$ and, in this context, the inequality (1.2) was considered by Garofalo and Lanconelli in [**14**], Niu, et al., in [**27**], D'Ambrosio in [**9**] and Yener in [**32**].

On the other hand, D'Ambrosio [**10**] obtained various weighted Hardy-type inequalities related to quasilinear second-order degenerate differential operators involving the subelliptic operator Δ_k. His approach is based upon the divergence theorem and on the careful choice of a vector field. Later, Niu, et al., [**26**] used the fundamental solution of the p-degenerate subelliptic operator $\Delta_{k,p} = \nabla_k \cdot (|\nabla_k|^{p-2}\nabla_k)$ to establish a weighted version of (1.2). More precisely, they established the following inequality

$$\text{(1.3)}\qquad \int_{\mathbb{R}^{2n+1}} \rho^{\alpha p}|\nabla_k\phi|^p dz\,dl \geq \left|\frac{Q+\alpha p-p}{p}\right|^p \cdot \int_{\mathbb{R}^{2n+1}} \rho^{\alpha p}\left(\frac{|z|}{\rho}\right)^{p(2k-1)} \frac{|\phi|^p}{\rho^p}\, dz\, dl,$$

where $\phi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{(0,0)\})$, $Q \geq 3$, $Q \neq p$ and $Q+\alpha p-p > 0$. Furthermore, the constant $|(Q+\alpha p-p)/p|^p$ in (1.3) is sharp.

In the article **[24]**, Lian proved a representation formula for ∇_k and showed that, for any function $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have

$$\text{(1.4)}\qquad \int_{\mathbb{R}^{2n+1}} \frac{|z|^\beta}{\rho^{\alpha+\beta}}|\nabla_k\phi|^p dz\,dl \geq \left(\frac{Q-p-\alpha}{p}\right)^p \cdot \int_{\mathbb{R}^{2n+1}} \frac{|z|^\beta}{\rho^{\alpha+\beta}}\left(\frac{|z|}{\rho}\right)^{p(2k-1)} \frac{|\phi|^p}{\rho^p}\, dz\, dl$$

provided that $1 < p < Q-\alpha$ and $\beta+2n+(p-1)(2k-1) > 0$. Moreover, the constant $((Q-p-\alpha)/p)^p$ in (1.4) is sharp. Recently, Ahmetolan and Kombe **[3]** investigated sharp two weight Hardy-type inequalities associated with the Greiner operator Δ_k.

In view of all of these developments, it is natural to research a sufficient constructive criteria for the validity of more general weighted Hardy-type inequalities related to the Greiner operator Δ_k. In this direction, we prove that, if $a \in C^1(\mathbb{R}^{2n+1})$ and $b \in L^1_{\text{loc}}(\mathbb{R}^{2n+1})$ are nonnegative functions and $\vartheta \in C^\infty(\mathbb{R}^{2n+1})$ is a positive function such that

$$-\nabla_k \cdot (a|\nabla_k\vartheta|^{p-2}\nabla_k\vartheta) \geq b\vartheta^{p-1}$$

almost everywhere in $\mathbb{R}^{2n+1}$, then the general weighted L^p Hardy-type inequality having the form

$$\int_{\mathbb{R}^{2n+1}} a|\nabla_k\phi|^p dz\,dl \geq \int_{\mathbb{R}^{2n+1}} b|\phi|^p dz\,dl$$

is valid for every $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, $p > 1$. We would like to mention, in particular, that our approach is quite practical and constructive for obtaining both known and new weighted Hardy-type inequalities. In order to construct various weighted Hardy-type inequalities on $\mathbb{R}^{2n+1}$ or on some special domains in $\mathbb{R}^{2n+1}$, it is sufficient to determine the proper model functions a and ϑ that satisfy the above hypotheses. The flexibility in choosing the functions a and ϑ leads to the exhibition of many concrete examples of Hardy-type inequalities with radial, logarithmic, hyperbolic and non-radial weights (see subsection 3.1).

We also introduce a new method for deriving two-weight L^p Hardy-type inequalities with remainder terms on smooth bounded domains Ω in $\mathbb{R}^{2n+1}$. The main tool which is employed in deriving these types of inequalities is the nonlinear partial differential inequality

$$-\nabla_k \cdot \left(a\rho^{p-Q} \frac{|\nabla_k \vartheta|^{p-2}}{\vartheta^{p-2}} \nabla_k \vartheta \right) \geq 0, \tag{1.5}$$

where a is a nonnegative weight function, ρ is the gauge and ϑ is a positive smooth function (see Theorem 4.1). Through the careful choice of functions a and ϑ in the differential inequality (1.5), we present a variety of improved L^p Hardy-type inequalities, including radial, logarithmic and exponential weights (see subsection 4.1).

2. Preliminary results and notation. We begin by providing some notation, definitions and preliminary facts which will be necessary in the sequel. We split $\mathbb{R}^{2n+1}$ into $w = (z, l) = (x, y, l) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with $n \geq 1$. The generalized Greiner operator is of the form

$$\Delta_k = \sum_{j=1}^{n} (X_j^2 + Y_j^2), \tag{2.1}$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2k y_j |z|^{2k-2} \frac{\partial}{\partial l}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2k x_j |z|^{2k-2} \frac{\partial}{\partial l}, \tag{2.2}$$

for $j = 1, \ldots, n$, $k \geq 1$. Note that, when $k = 1$, Δ_k becomes the sub-Laplacian $\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$, see **[11]**. If $k = 2, 3, \ldots$, Δ_k is the Greiner operator, see **[18]**. The subelliptic gradient associated with Δ_k is as follows:

$$\nabla_k = (X_1, \ldots, X_n, Y_1, \ldots, Y_n).$$

A natural family of dilations is given by

$$\delta_\lambda(z, l) = (\lambda z, \lambda^{2k} l), \quad \lambda > 0. \tag{2.3}$$

The change of variables formula for the Lebesgue measure yields that

$$d\delta_\lambda(z,l) = \lambda^Q dz\, dl = \lambda^Q dw,$$

where

$$Q = 2n + 2k$$

is the homogeneous dimension with respect to the dilation δ_λ, and $dw = dz\, dl$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$.

For $w = (z,l) \in \mathbb{R}^{2n} \times \mathbb{R}$, we define the norm

$$\rho = \rho(z,l) = (|z|^{4k} + l^2)^{1/4k},$$

where we have set

$$|z| := \sqrt{|x|^2 + |y|^2}.$$

Here, $|\cdot|$ stands for the standard Euclidean norm. We remark that ρ is positive, smooth away from the origin, and symmetric. The norm function ρ is also closely related to the fundamental solution of subelliptic operator Δ_k at the origin, see **[6, 7]**, namely, if $Q > 2$, then the function $u_2 := \rho^{2-Q}$ satisfies the relation

$$-\Delta_k u_2 = \ell_2 \delta_0 \quad \text{on } \mathbb{R}^{2n+1}$$

in a weak sense, where δ_0 is the Dirac distribution at 0 and ℓ_2 is a positive constant.

For a differentiable real-valued function $u : \mathbb{R}^{2n+1} \to \mathbb{R}$, the p-degenerate, subelliptic operator $\Delta_{k,p}$ associated with the vector fields (2.2) is given by

$$\Delta_{k,p} u = \nabla_k \cdot (|\nabla_k u|^{p-2} \nabla_k u), \quad p > 1.$$

If $p = 2$, it coincides with Δ_k in (2.1). It is immediately seen that $\Delta_{k,p}$ is a homogeneous partial differential operator of degree p with respect to the anisotropic dilations (2.3), that is, $\Delta_{k,p} \circ \delta_\lambda = \lambda^p \delta_\lambda \circ \Delta_{k,p}$. Let u_p be the function, defined as

$$u_p = \begin{cases} \rho^{(p-Q)/(p-1)} & \text{if } p \neq Q, \\ -\ln \rho & \text{if } p = Q, \end{cases}$$

for $w \neq 0$. The function u_p is p-harmonic on $\mathbb{R}^{2n+1} \setminus \{0\}$, that is,

$$-\Delta_{k,p} u_p = 0 \quad \text{on } \mathbb{R}^{2n+1} \setminus \{0\}.$$

Moreover, for each $1 < p < \infty$, there exists a constant $\ell_p \neq 0$ such that

$$-\Delta_{k,p} u_p = \ell_p \delta_0$$

in the sense of distribution and $\ell_p > 0$ if and only if $Q \geq p$, see **[33]**.

Now we mention, without proofs, some useful facts which we shall use throughout the computations in this paper. An evident calculation yields

$$\nabla_k \rho = \frac{|z|^{2k-2}}{\rho^{4k-1}} (x|z|^{2k} + yl, y|z|^{2k} - xl).$$

Thus, we readily obtain

$$|\nabla_k \rho| = \frac{|z|^{2k-1}}{\rho^{2k-1}}.$$

Suppose that a smooth function u has the form $u = u(\rho)$ on $\mathbb{R}^{2n+1}$. Then, we have

$$|\nabla_k u(\rho)| = |\nabla_k \rho||u'(\rho)|$$

for all $w \in \mathbb{R}^{2n+1} - \{0\}$ and

$$\Delta_{k,p} u(\rho) = |\nabla_k \rho|^p |u'(\rho)|^{p-2} \left[(p-1)u''(\rho) + (Q-1)\frac{u'(\rho)}{\rho} \right], \quad p > 1,$$

at every point $w \in \mathbb{R}^{2n+1} - \{0\}$ with $u'(\rho(w)) \neq 0$, see **[33]**. In particular, when $u(\rho) = \rho^\alpha$, we get

$$|\nabla_k \rho^\alpha| = |\alpha||\nabla_k \rho|\rho^{\alpha-1} \tag{2.4}$$

and

$$\Delta_{k,p} \rho^\alpha = \alpha|\alpha|^{p-2}(Q + \alpha p - \alpha - p)|\nabla_k \rho|^p \rho^{\alpha p - p - \alpha}, \tag{2.5}$$

where $\alpha \in \mathbb{R}$. Observe that, in the case $p = 2$, the formula (2.5) reduces to

$$\Delta_k \rho^\alpha = \alpha(Q + \alpha - 2)|\nabla_k \rho|^2 \rho^{\alpha-2}.$$

Moreover, together with the above formulas and noting that

$$\nabla_k |z| = \frac{z}{|z|}, \qquad \Delta_k |z| = \frac{2n-1}{|z|},$$

we can easily obtain the following two identities:

$$\nabla_k \rho^\alpha \cdot \nabla_k |z|^\beta = \alpha\beta|\nabla_k \rho|^2 |z|^\beta \rho^{\alpha-2}$$

and

$$\Delta_k(\rho^\alpha|z|^\beta)=\alpha(Q+2\beta+\alpha-2)|\nabla_k\rho|^2|z|^\beta\rho^{\alpha-2}+\beta(2n+\beta-2)|z|^{\beta-2}\rho^\alpha,$$

with $\alpha,\beta\in\mathbb{R}$, $n\geq 1$. We also note that the *gauge* ρ is infinitely harmonic in $\mathbb{R}^{2n+1}-\{0\}$, that is, ρ is solution of the following equation:

$$\nabla_k(|\nabla_k\rho|^2)\cdot\nabla_k\rho=0. \tag{2.6}$$

Let $B_R=\{w\in\mathbb{R}^{2n+1}\mid\rho(w)<R\}$, $\partial B_R=\{w\in\mathbb{R}^{2n+1}\mid\rho(w)=R\}$, and call these sets, respectively, ρ-ball and ρ-sphere centered at the origin with radius R.

3. General weighted Hardy-type inequalities. There are several necessary and sufficient conditions to obtain the validity of Hardy-type inequalities in the literature. One of the most efficient ways is linking Hardy inequalities with solutions or subsolutions to differential problems. In this regard, we now provide a systematic and unified approach that includes and improves most of the Hardy- and uncertainty principle-type inequalities in a more adequate fashion on the sub-Riemannian manifold $\mathbb{R}^{2n+1}$ defined by the Greiner vector fields (2.2). Then, we illustrate these cases by giving many explicit examples, including radial, logarithmic, hyperbolic and non-radial weights. The main result of this section follows.

Theorem 3.1. *Let $a\in C^1(\mathbb{R}^{2n+1})$ and $b\in L^1_{\mathrm{loc}}(\mathbb{R}^{2n+1})$ be nonnegative functions and $\vartheta\in C^\infty(\mathbb{R}^{2n+1})$ a positive function satisfying the differential inequality*

$$-\nabla_k\cdot(a|\nabla_k\vartheta|^{p-2}\nabla_k\vartheta)\geq b\vartheta^{p-1} \tag{3.1}$$

almost everywhere in $\mathbb{R}^{2n+1}$. There exists a positive constant $c_p=c(p)$ such that, if $p\geq 2$, then

$$\int_{\mathbb{R}^{2n+1}} a|\nabla_k\phi|^p dw\geq\int_{\mathbb{R}^{2n+1}} b|\phi|^p dw+c_p\int_{\mathbb{R}^{2n+1}} a\left|\nabla_k\frac{\phi}{\vartheta}\right|^p\vartheta^p dw, \tag{3.2}$$

and, if $1 < p < 2$*, then*

(3.3)
$$\int_{\mathbb{R}^{2n+1}} a|\nabla_k\phi|^p dw \geq \int_{\mathbb{R}^{2n+1}} b|\phi|^p dw + c_p \int_{\mathbb{R}^{2n+1}} a\frac{|\nabla_k(\phi/\vartheta)|^2\vartheta^2}{(|(\phi/\vartheta)\nabla_k\vartheta| + |\nabla_k(\phi/\vartheta)|\vartheta)^{2-p}}\, dw$$

for every $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$.

Proof. Let $\varphi := \phi/\vartheta$, where $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$ and $0 < \vartheta \in C^\infty(\mathbb{R}^{2n+1})$. Thus,

$$|\nabla_k\phi|^p = |\varphi\nabla_k\vartheta + \vartheta\nabla_k\varphi|^p.$$

We now use the following convexity inequality: for any $x, y \in \mathbb{R}^n$ and $p \geq 2$,

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x\cdot y + c_p|y|^p, \tag{3.4}$$

where $c_p = c(p) > 0$, see **[25]**. From (3.4), we have

$$|\nabla_k\phi|^p \geq |\nabla_k\vartheta|^p|\varphi|^p + \vartheta|\nabla_k\vartheta|^{p-2}\nabla_k\vartheta\cdot\nabla_k(|\varphi|^p) + c_p|\nabla_k\varphi|^p\vartheta^p. \tag{3.5}$$

Multiplying the inequality (3.5) by a on both sides, and then applying integration by parts over $\mathbb{R}^{2n+1}$, yields

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} a|\nabla_k\phi|^p dw &\geq \int_{\mathbb{R}^{2n+1}} a|\nabla_k\vartheta|^p|\varphi|^p dw \\ &\quad + c_p\int_{\mathbb{R}^{2n+1}} a|\nabla_k\varphi|^p\vartheta^p dw \\ &\quad - \int_{\mathbb{R}^{2n+1}} \nabla_k\cdot(a\vartheta|\nabla_k\vartheta|^{p-2}\nabla_k\vartheta)|\varphi|^p dw \\ &= -\int_{\mathbb{R}^{2n+1}} \nabla_k\cdot(a|\nabla_k\vartheta|^{p-2}\nabla_k\vartheta)\vartheta|\varphi|^p dw \\ &\quad + c_p\int_{\mathbb{R}^{2n+1}} a|\nabla_k\varphi|^p\vartheta^p dw.\end{aligned}$$

It therefore follows from (3.1) that

$$\int_{\mathbb{R}^{2n+1}} a|\nabla_k\phi|^p dw \geq \int_{\mathbb{R}^{2n+1}} b|\varphi|^p\vartheta^p dw + c_p\int_{\mathbb{R}^{2n+1}} a|\nabla_k\varphi|^p\vartheta^p dw.$$

Making the variable change $\varphi = \phi/\vartheta$ in the above integral, we obtain the desired result (3.2). In the case of $1 < p < 2$, we apply the following convexity inequality:

$$|x+y|^p \geq |x|^p + p|x|^{p-2}x \cdot y + c_p \frac{|y|^2}{(|x|+|y|)^{2-p}}, \tag{3.6}$$

where $c_p = c(p) > 0$ and $x, y \in \mathbb{R}^n$, see **[25]**. Following a similar procedure as in the proof of inequality (3.2), the proof of Theorem 3.1 is completed. □

Remark 3.2. Note that one of the novelties of our approach is that it automatically yields a remainder term. For $p = 2$, there is an equality in (3.4) with $c_2 = 1$, and this gives the following remainder formula:

$$\int_{\mathbb{R}^{2n+1}} a|\nabla_k \phi|^2 dw = \int_{\mathbb{R}^{2n+1}} b\phi^2 dw + \int_{\mathbb{R}^{2n+1}} a\left|\nabla_k \frac{\phi}{\vartheta}\right|^2 \vartheta^2 dw.$$

3.1. Applications of Theorem 3.1. Let $\epsilon > 0$ be given. To make the following arguments rigorous, we should replace the function ρ with its regularization

$$\rho_\epsilon := (|z|_\epsilon^{4k} + l^2)^{1/4k},$$

where

$$|z|_\epsilon := \left(\epsilon^2 + \sum_{j=1}^n (x_j^2 + y_j^2)\right)^{1/2}$$

and, after calculations, take the limit as $\epsilon \to 0$. However, for the sake of simplicity, we will proceed formally.

As mentioned earlier, we now apply Theorem 3.1 to demonstrate how our approach systematically recovers, extends and improves many previously known sharp, weighted Hardy-type inequalities, such as those obtained in **[3, 10, 16, 24, 26, 29]**, and also enables us to derive new inequalities in a relatively simple and unified manner. We begin by considering the model functions

$$a = \rho^{\alpha p} \quad \text{and} \quad \vartheta = \rho^{-(Q+\alpha p - p)/p}$$

in Theorem 3.1. After some computations, the subsequent result that was established by Niu, et al., **[26]** is readily obtained.

Corollary 3.3. *Let $Q \geq 3$, $Q \neq p$ and $Q + \alpha p - p > 0$. Then, the inequality*

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha p} |\nabla_k \phi|^p dw \geq \left| \frac{Q + \alpha p - p}{p} \right|^p \int_{\mathbb{R}^{2n+1}} \rho^{\alpha p} |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p} dw$$

is valid for every $\phi \in C_0^\infty(\mathbb{R}^{2n+1} \setminus \{0\})$.

Remark 3.4. It was shown in **[26]** that the positive constant

$$\left| \frac{Q + \alpha p - p}{p} \right|^p$$

in the above inequality is sharp.

We can recapture, via our approach, most of the results of D'Ambrosio, presented in **[10]**, for the generalized Greiner operator. As a first example, if we take the following pair

$$a = \rho^{\alpha + p} \quad \text{and} \quad \vartheta = \rho^{|Q + \alpha|/p},$$

then we have the weighted L^p Hardy-type inequality in **[10]**.

Corollary 3.5. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$. If $Q + \alpha < 0$, then, for every function $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have*

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha + p} |\nabla_k \phi|^p dw \geq \left(\frac{|Q + \alpha|}{p} \right)^p \int_{\mathbb{R}^{2n+1}} \rho^{\alpha} |\nabla_k \rho|^p |\phi|^p dw.$$

Remark 3.6. Moreover, the positive constant $(|Q + \alpha|/p)^p$ in the above inequality is sharp. For the proof of sharpness, the interested reader may consult **[10]**.

On the other hand, by specializing the functions as

$$a = |z|^{\alpha + p} \quad \text{and} \quad \vartheta = |z|^{|2n + \alpha|/p},$$

we recover another weighted L^p Hardy-type inequality, due to D'Ambrosio **[10]**.

Corollary 3.7. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and let $\Omega \subset (\mathbb{R}^{2n} \setminus \{0\}) \times \mathbb{R}$ be an open set. If $2n + \alpha < 0$, then, for every $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega |z|^{\alpha+p} |\nabla_k \phi|^p dw \geq \left(\frac{|2n+\alpha|}{p} \right)^p \int_\Omega |z|^\alpha |\phi|^p \, dw.$$

Observe also that the $\alpha = -p$ case of the above choice

$$a \equiv 1 \quad \text{and} \quad \vartheta = |z|^{|2n-p|/p},$$

together with the relation

$$\frac{1}{|z|^p} \geq \frac{1}{\rho^p},$$

directly gives inequality (3.24) in **[10]**.

Corollary 3.8. *Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, and let $\Omega \subset (\mathbb{R}^{2n} \setminus \{0\}) \times \mathbb{R}$ be an open set. If $2n - p < 0$, then, for every $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega |\nabla_k \phi|^p dw \geq \left(\frac{|2n-p|}{p} \right)^p \int_\Omega \frac{|\phi|^p}{\rho^p} \, dw.$$

We now set the pair as

$$a = \left(\log \frac{R}{\rho} \right)^{\alpha+p} \quad \text{and} \quad \vartheta = \left(\log \frac{R}{\rho} \right)^{|\alpha+1|/p}.$$

Hence, we derive the power logarithmic L^p Hardy-type inequality (3.21), again, presented in **[10]**.

Corollary 3.9. *Let $Q = p > 1$ and $\alpha + 1 < 0$. Then, for any $\phi \in C_0^\infty(B_R)$, we have*

$$\int_{B_R} \left(\log \frac{R}{\rho} \right)^{\alpha+p} |\nabla_k \phi|^p dw \geq \left(\frac{|\alpha+1|}{p} \right)^p \int_{B_R} \left(\log \frac{R}{\rho} \right)^\alpha |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p} \, dw,$$

where B_R is the ρ-ball centered at the origin with radius R.

An immediate application of Theorem 3.1 with the following functions

$$a = \left(\log \frac{R}{|z|}\right)^{\alpha+p} \quad \text{and} \quad \vartheta = \left(\log \frac{R}{|z|}\right)^{|\alpha+1|/p}$$

is inequality (3.25) in **[10]**.

Corollary 3.10. *Let $p = 2n$ and $\alpha+1 < 0$. Then, for any $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \left(\log \frac{R}{|z|}\right)^{\alpha+p} |\nabla_k \phi|^p dw \geq \left(\frac{|\alpha+1|}{p}\right)^p \int_\Omega \left(\log \frac{R}{|z|}\right)^{\alpha} \frac{|\phi|^p}{|z|^p}\, dw,$$

where $\Omega = \{w = (z,l) \in \mathbb{R}^{2n} \times \mathbb{R} : |z| < R\}$ and $R > 0$.

On the other hand, by considering the units

$$a = \frac{|z|^\beta}{\rho^{\alpha+\beta}} \quad \text{and} \quad \vartheta = \rho^{-(Q-p-\alpha)/p},$$

we obtain the following two-weight L^p Hardy inequality, which was proven by Lian **[24]**.

Corollary 3.11. *Let $\alpha, \beta \in \mathbb{R}$, $1 < p < Q - \alpha$ and $\beta + 2n + (p-1)(2k-1) > 0$. Then, for any $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have*

(3.7)
$$\int_{\mathbb{R}^{2n+1}} \frac{|z|^\beta}{\rho^{\alpha+\beta}} |\nabla_k \phi|^p dw \geq \left(\frac{Q-p-\alpha}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \frac{|z|^\beta}{\rho^{\alpha+\beta}} |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw.$$

Remark 3.12. Moreover, Lian **[24]** showed that the constant $(Q-p-\alpha)^p/p^p$ in (3.7) is sharp.

As an immediate consequence of the following choice

$$a = \rho^\alpha |z|^\beta \quad \text{and} \quad \vartheta = \rho^{-(Q+\alpha+\beta-p)/p},$$

we derive another two-weight L^p Hardy inequality, due to Ahmetolan and Kombe **[3]**.

Corollary 3.13. *Let* $\alpha, \beta \in \mathbb{R}$, $Q+\alpha+\beta > p > 1$ *and* $\beta+2kp-p+2n > 0$. *Then, the inequality*

(3.8)
$$\int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^\beta |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha+\beta-p}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^\beta |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$.

Remark 3.14. Ahmetolan and Kombe [**3**] also proved that the constant $(Q+\alpha+\beta-p)^p/p^p$ in (3.8) is sharp.

We must state that Theorem 3.1 not only gives us known, weighted Hardy-type inequalities, but also gives other, new inequalities for the generalized Greiner operator Δ_k. In the literature, Hardy-type inequalities mostly involve weights of the form $|z|^\alpha \rho^\beta$ for some $\alpha, \beta \in \mathbb{R}$. We now discuss versions of Hardy inequalities with more general weights. Recall that the following, weighted Hardy-type inequalities in the Euclidean setting were proven by Ghoussoub and Moradifam [**16**]: let $s, t > 0$ and α, β, m be real numbers.

- If $\alpha\beta > 0$ and $m \leq (n-2)/2$, then, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

(3.9)
$$\int_{\mathbb{R}^n} \frac{(s+t|x|^\alpha)^\beta}{|x|^{2m}} |\nabla \phi|^2 dx \geq \left(\frac{n-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(s+t|x|^\alpha)^\beta}{|x|^{2m+2}} \phi^2 dx.$$

- If $\alpha\beta < 0$ and $2m - \alpha\beta \leq n-2$, then, for all $\phi \in C_0^\infty(\mathbb{R}^n)$,

(3.10)
$$\int_{\mathbb{R}^n} \frac{(s+t|x|^\alpha)^\beta}{|x|^{2m}} |\nabla \phi|^2 dx \geq \left(\frac{n+\alpha\beta-2m-2}{2}\right)^2 \int_{\mathbb{R}^n} \frac{(s+t|x|^\alpha)^\beta}{|x|^{2m+2}} \phi^2 dx.$$

We now extend and improve inequalities (3.9) and (3.10) to the L^p case for the generalized Greiner operator. In order to do so, we now take the pair as

$$a = \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm}} \quad \text{and} \quad \vartheta = \rho^{-(Q-pm-p)/p}$$

in Theorem 3.1. This gives the following improvement of inequality (3.9).

Corollary 3.15. *Let $s,t > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta > 0$ and $1 < p \leq Q - pm$, then, for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have*

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm}} |\nabla_k \phi|^p dw \\ \geq C_{Q,p,m}^p \int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm+p}} |\nabla_k \rho|^p |\phi|^p dw \\ + C_{Q,p,m}^{p-1} \alpha\beta t \int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^{\beta-1}}{\rho^{pm+p-\alpha}} |\nabla_k \rho|^p |\phi|^p dw,\end{aligned}$$

where $C_{Q,p,m} = (Q - pm - p)/p$.

If we consider the units

$$a = \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm}} \quad \text{and} \quad \vartheta = \rho^{-(Q+\alpha\beta-pm-p)/p},$$

then we immediately obtain the following improvement of inequality (3.10).

Corollary 3.16. *Let $s,t > 0$ and $\alpha, \beta, m \in \mathbb{R}$. If $\alpha\beta < 0$ and $1 < p \leq Q + \alpha\beta - pm$, then, for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have*

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm}} |\nabla_k \phi|^p dw \\ \geq C_{Q,p,m,\alpha,\beta}^p \int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^\beta}{\rho^{pm+p}} |\nabla_k \rho|^p |\phi|^p dw \\ - C_{Q,p,m,\alpha,\beta}^{p-1} \alpha\beta s \int_{\mathbb{R}^{2n+1}} \frac{(s+t\rho^\alpha)^{\beta-1}}{\rho^{pm+p}} |\nabla_k \rho|^p |\phi|^p dw,\end{aligned}$$

where $C_{Q,p,m,\alpha,\beta} = (Q + \alpha\beta - pm - p)/p$.

Remark 3.17. Note that, if $\alpha = 0$ or $\beta = 0$ in the above two inequalities, then they reduce to Hardy-type inequalities with the usual weights. Hence, we are interested in the case where $\alpha\beta \neq 0$.

On the other hand, by applying Theorem 3.1 with the following pair

$$a = \rho^\alpha \sinh^\beta \rho \quad \text{and} \quad \vartheta = \rho^{-(Q+\alpha+\beta-p)/p},$$

and noting that $\rho \coth \rho \geq 1$, we obtain the power hyperbolic sine L^p Hardy-type inequality.

Corollary 3.18. *Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and $Q + \alpha + \beta > p > 1$. Then, the following inequality holds*:

$$\int_{\mathbb{R}^{2n+1}} \rho^\alpha \sinh^\beta \rho |\nabla_k \phi|^p dw \geq \left(\frac{Q + \alpha + \beta - p}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \rho^\alpha \sinh^\beta \rho |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p} dw$$

for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$.

We now set the non-symmetric functions

$$a = \cosh^\alpha x_1 \quad \text{and} \quad \vartheta = \log y_1$$

in $\Omega := \{w = (x, y, l) \in \mathbb{R}^{2n+1} : y_1 > 1\}$. This immediately yields the following L^p Hardy-type inequality with non-symmetric weights.

Corollary 3.19. *For any $\phi \in C_0^\infty(\Omega)$ and $p > 1$, we have*

$$\int_\Omega \cosh^\alpha x_1 |\nabla_k \phi|^p dw \geq (p-1) \int_\Omega \frac{\cosh^\alpha x_1}{y_1^p \log^{p-1} y_1} |\phi|^p dw,$$

where $\Omega = \{w = (x, y, l) \in \mathbb{R}^{2n+1} : y_1 > 1\}$, $\alpha \in \mathbb{R}$.

When we take the pair

$$a = x_1^{p-2} \log y_1 \quad \text{and} \quad \vartheta = \log x_1$$

in $\Omega := \{w = (x, y, l) \in \mathbb{R}^{2n+1} : x_1 > 1,\ y_1 > 1\}$, we deduce another L^p Hardy-type inequality with non-symmetric weights.

Corollary 3.20. *For any $\phi \in C_0^\infty(\Omega)$ and $p > 1$, we have*

$$\int_\Omega x_1^{p-2} \log y_1 |\nabla_k \phi|^p dw \geq \int_\Omega \frac{\log y_1}{x_1^2 \log^{p-1} x_1} |\phi|^p dw,$$

where $\Omega = \{w = (x, y, l) \in \mathbb{R}^{2n+1} : x_1 > 1,\ y_1 > 1\}$.

Another application of Theorem 3.1 with the choice

$$a = \left(\frac{|z|^{2k-1}}{l}\right)^{2-p} \quad \text{and} \quad \vartheta = \log l$$

in $\Omega := \{w = (z,l) \in \mathbb{R}^{2n} \times \mathbb{R} : l > 1\}$ is the following L^p Hardy-type inequality with different non-radial weights.

Corollary 3.21. *For any $\phi \in C_0^\infty(\Omega)$ and $p > 1$, we have*

$$\int_\Omega \left(\frac{|z|^{2k-1}}{l}\right)^{2-p} |\nabla_k \phi|^p dw \geq (2k)^p \int_\Omega \frac{|z|^{4k-2}}{l^2 \log^{p-1} l} |\phi|^p dw,$$

where $\Omega = \{w = (z,l) \in \mathbb{R}^{2n} \times \mathbb{R} : l > 1\}$, $k \geq 1$.

It is worth stressing here that, by considering the model functions

$$a = (1 + \rho^{p/(p-1)})^{\alpha(p-1)} \quad \text{and} \quad \vartheta = (1 + \rho^{p/(p-1)})^{(1-\alpha)}$$

in Theorem 3.1, we extend a result of Skrzypczak [**29**] on the Euclidean space to the sub-Riemannian manifold $\mathbb{R}^{2n+1}$, defined by the Greiner vector fields (2.2).

Corollary 3.22. *Let $1 < p < Q$, $\alpha > 1$. Then, for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, the following holds*:

$$\int_{\mathbb{R}^{2n+1}} (1 + \rho^{p/(p-1)})^{\alpha(p-1)} |\nabla_k \phi|^p dw$$
$$\geq C_{Q,p,\alpha} \int_{\mathbb{R}^{2n+1}} (1 + \rho^{p/(p-1)})^{(\alpha-1)(p-1)} |\nabla_k \rho|^p |\phi|^p dw,$$

where $C_{Q,p,\alpha} = Q(p(\alpha-1)/(p-1))^{p-1}$.

Another consequence of Theorem 3.1, with the special functions

$$a = \rho^\alpha \quad \text{and} \quad \vartheta = (1 + \rho^{p/(p-1)})^{-(Q+\alpha-p)/p},$$

leads us to the subsequent weighted L^p Hardy-type inequality with different weights.

Corollary 3.23. *Let $\alpha \in \mathbb{R}$, $Q+\alpha > p > 1$. Then, the inequality:*

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha} |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha-p}{p-1}\right)^{p-1} (Q+\alpha) \cdot \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha}}{(1+\rho^{p/(p-1)})^p} |\nabla_k \rho|^p |\phi|^p dw$$

holds for every $\phi \in C_0^{\infty}(\mathbb{R}^{2n+1})$.

3.2. Uncertainty principle inequalities. The Heisenberg-Pauli-Weyl inequality [**20, 31**], a rigorous mathematical formulation of the uncertainty principle of quantum mechanics, asserts that

$$\left(\int_{\mathbb{R}^n} |\nabla \phi|^2 \, dx\right)\left(\int_{\mathbb{R}^n} |x|^2 \phi^2 dx\right) \geq \frac{n^2}{4}\left(\int_{\mathbb{R}^n} \phi^2 dx\right)^2 \tag{3.11}$$

for all $\phi \in C_0^{\infty}(\mathbb{R}^n)$, with $n^2/4$ being the sharp constant. This inequality has been exhaustively analyzed in many different settings, see [**2, 12, 21, 22, 23, 28**]. For instance, the inequality (3.11) was generalized to the Greiner operator Δ_k in the work of Ahmetolan and Kombe [**2**]. This result reads as follows:

$$\left(\int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k \phi|^2}{|\nabla_k \rho|^2} \, dw\right)\left(\int_{\mathbb{R}^{2n+1}} \rho^2 \phi^2 dw\right) \geq \frac{Q^2}{4}\left(\int_{\mathbb{R}^{2n+1}} \phi^2 dw\right)^2, \tag{3.12}$$

where $\phi \in C_0^{\infty}(\mathbb{R}^{2n+1})$, and the constant $Q^2/4$ is sharp.

It is worth mentioning here that Theorem 3.1 can, however, be applied to obtain the Heisenberg-Pauli-Weyl-type inequalities with the best constant. In order to be more precise, we now take $p = 2$ in Theorem 3.1 and choose the following functions

$$a = \frac{1}{|\nabla_k \rho|^2} \quad \text{and} \quad \vartheta = e^{-\alpha \rho^2},$$

where $\alpha > 0$. This allows the derivation of the inequality

$$\int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k \phi|^2}{|\nabla_k \rho|^2} \, dw \geq 2\alpha Q \int_{\mathbb{R}^{2n+1}} \phi^2 dw - 4\alpha^2 \int_{\mathbb{R}^{2n+1}} \rho^2 \phi^2 dw \tag{3.13}$$

with equality if and only if ϕ is proportional to ϑ. Therefore, (3.13) reads

$$A\alpha^2 + B\alpha + C \geq 0$$

for every $\alpha > 0$, where

$$A = 4\int_{\mathbb{R}^{2n+1}} \rho^2\phi^2 dw,$$
$$B = -2Q\int_{\mathbb{R}^{2n+1}} \phi^2 dw,$$
$$C = \int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k\phi|^2}{|\nabla_k\rho|^2}\, dw.$$

This is equivalent to $B^2 - 4AC \leq 0$, or the sharp Heisenberg-Pauli-Weyl-type inequality (3.12).

On the other hand, by setting the functions

$$a \equiv 1 \quad \text{and} \quad \vartheta = e^{-\alpha\rho},\ \alpha > 0,$$

we shall deduce from Theorem 3.1 that

$$\int_{\mathbb{R}^{2n+1}} |\nabla_k\phi|^2 dw \geq \alpha(Q-1)\int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k\rho|^2}{\rho}\phi^2 dw - \alpha^2\int_{\mathbb{R}^{2n+1}} |\nabla_k\rho|^2\phi^2 dw.$$

Hence, a very similar argument applies to obtain the following version of the Heisenberg-Pauli-Weyl-type inequality.

Corollary 3.24. *For every* $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$*, we have*

$$\left(\int_{\mathbb{R}^{2n+1}} |\nabla_k\phi|^2 dw\right)\left(\int_{\mathbb{R}^{2n+1}} |\nabla_k\rho|^2\phi^2 dw\right) \geq \frac{(Q-1)^2}{4}\left(\int_{\mathbb{R}^{2n+1}} \frac{|\nabla_k\rho|^2}{\rho}\phi^2 dw\right)^2.$$

We end this section by showing a further application. When we consider the pair

$$a \equiv 1 \quad \text{and} \quad \vartheta = e^{-\alpha\rho^2},\ \alpha > 0,$$

in Theorem 3.1, we readily obtain

$$\int_{\mathbb{R}^{2n+1}} |\nabla_k\phi|^2 dw \geq 2\alpha Q\int_{\mathbb{R}^{2n+1}} |\nabla_k\rho|^2\phi^2 dw - 4\alpha^2\int_{\mathbb{R}^{2n+1}} \rho^2|\nabla_k\rho|^2\phi^2 dw,$$

or the next inequality.

Corollary 3.25. *For every $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$, we have*

$$\left(\int_{\mathbb{R}^{2n+1}}|\nabla_k\phi|^2dw\right)\left(\int_{\mathbb{R}^{2n+1}}\rho^2|\nabla_k\rho|^2\phi^2dw\right) \geq \frac{Q^2}{4}\left(\int_{\mathbb{R}^{2n+1}}|\nabla_k\rho|^2\phi^2dw\right)^2.$$

4. Two-weight Hardy-type inequalities with remainders. In this section, we first prove an improved two-weight L^p Hardy-type inequality on the basis of a particular partial differential inequality and then give explicit examples to illustrate our results for different weights. Our method is inspired by the techniques from the paper by Kombe and Özaydın [**22**].

Theorem 4.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$. Let a be a nonnegative C^1-function and ϑ a positive C^∞-function such that*

$$-\nabla_k \cdot \left(a\rho^{p-Q}\frac{|\nabla_k\vartheta|^{p-2}}{\vartheta^{p-2}}\nabla_k\vartheta\right) \geq 0$$

almost everywhere in Ω. Then, for any $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}\int_\Omega a\rho^\alpha|\nabla_k\phi|^p dw \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega a\rho^\alpha|\nabla_k\rho|^p\frac{|\phi|^p}{\rho^p}\, dw \\ + \left(\frac{Q+\alpha-p}{p}\right)^{p-1}\int_\Omega \rho^{\alpha+1}|\nabla_k\rho|^{p-2}\nabla_k\rho\cdot\nabla_k a\frac{|\phi|^p}{\rho^p}\, dw \\ + \frac{c_p}{p^p}\int_\Omega a\rho^\alpha\frac{|\nabla_k\vartheta|^p}{\vartheta^p}|\phi|^p\, dw,\end{aligned} \tag{4.1}$$

provided that $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p = c(p) > 0$.

Proof. Let $\varphi := \rho^{-\beta}\phi$, where $\phi \in C_0^\infty(\Omega)$ and $\beta < 0$, which will be determined later. Note that, evidently,

$$\nabla_k(\rho^\beta\varphi) = \beta\rho^{\beta-1}\varphi\nabla_k\rho + \rho^\beta\nabla_k\varphi.$$

As a direct consequence of the convexity inequality (3.4), we obtain

$$\begin{aligned}&a\rho^\alpha|\beta\rho^{\beta-1}\varphi\nabla_k\rho + \rho^\beta\nabla_k\varphi|^p \\ &\quad\geq |\beta|^p a\rho^{\alpha+p(\beta-1)}|\nabla_k\rho|^p|\varphi|^p + c_p a\rho^{\alpha+p\beta}|\nabla_k\varphi|^p \\ &\qquad + \beta|\beta|^{p-2}a\rho^{\alpha+p(\beta-1)+1}|\nabla_k\rho|^{p-2}\nabla_k\rho\cdot\nabla_k(|\varphi|^p).\end{aligned}$$

Applying integration by parts to the third term on the right hand side of the above inequality leads to

$$\begin{aligned}\int_\Omega a\rho^\alpha|\nabla_k\phi|^p dw \geq |\beta|^p \int_\Omega a\rho^{\alpha+p(\beta-1)}|\nabla_k\rho|^p|\varphi|^p dw \\ + c_p\int_\Omega a\rho^{\alpha+p\beta}|\nabla_k\varphi|^p dw \\ - \beta|\beta|^{p-2}\int_\Omega \nabla_k\cdot(a\rho^{\alpha+p(\beta-1)+1}|\nabla_k\rho|^{p-2}\nabla_k\rho)|\varphi|^p dw.\end{aligned} \tag{4.2}$$

Standard computation yields

$$\begin{aligned}\nabla_k\cdot(a\rho^{\alpha+p(\beta-1)+1}|\nabla_k\rho|^{p-2}\nabla_k\rho) \\ = \rho^{\alpha+p(\beta-1)+1}|\nabla_k\rho|^{p-2}\nabla_k\rho\cdot\nabla_k a \\ + [Q+\alpha+p(\beta-1)]a\rho^{\alpha+p(\beta-1)}|\nabla_k\rho|^p,\end{aligned} \tag{4.3}$$

where we have used the formulas (2.4), (2.5) and (2.6). Inserting (4.3) into (4.2) and rearranging terms, we conclude that

$$\begin{aligned}\int_\Omega a\rho^\alpha|\nabla_k\phi|^p dw \geq f(\beta)\int_\Omega a\rho^{\alpha+p(\beta-1)}|\nabla_k\rho|^p|\varphi|^p dw \\ + c_p\int_\Omega a\rho^{\alpha+p\beta}|\nabla_k\varphi|^p dw \\ - \beta|\beta|^{p-2}\int_\Omega \rho^{\alpha+p(\beta-1)+1}|\nabla_k\rho|^{p-2}\nabla_k\rho \\ \cdot\nabla_k a|\varphi|^p dw,\end{aligned} \tag{4.4}$$

where $f(\beta) = |\beta|^p - \beta|\beta|^{p-2}(Q+\alpha+\beta p-p)$. Observe that $f(\beta)$ attains the maximum for

$$\beta_0 = -\frac{Q+\alpha-p}{p} < 0,$$

and this maximum value is equal to

$$f(\beta_0) = \left(\frac{Q+\alpha-p}{p}\right)^p.$$

Therefore, inequality (4.4) takes the form

$$
\begin{aligned}
(4.5)\qquad & \int_\Omega a\rho^\alpha |\nabla_k \phi|^p dw \\
& \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega a\rho^{-Q} |\nabla_k \rho|^p |\varphi|^p dw \\
& \quad + \left(\frac{Q+\alpha-p}{p}\right)^{p-1} \int_\Omega \rho^{1-Q} |\nabla_k \rho|^{p-2} \nabla_k \rho \cdot \nabla_k a |\varphi|^p dw \\
& \quad + c_p \int_\Omega a\rho^{p-Q} |\nabla_k \varphi|^p dw.
\end{aligned}
$$

We now concentrate on the integral expression $c_p \int_\Omega a\rho^{p-Q}|\nabla_k\varphi|^p dw$ on the right hand side of (4.5). Let ψ be the new function $\psi := \vartheta^{-1/p}\varphi$ with $0 < \vartheta \in C^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$. It then follows from the convexity inequality (3.4) that

$$
\begin{aligned}
(4.6)\qquad |\nabla_k \varphi|^p &= \left| \frac{1}{p} \vartheta^{(1-p)/p} \psi \nabla_k \vartheta + \vartheta^{1/p} \nabla_k \psi \right|^p \\
& \geq \frac{1}{p^p} \frac{|\nabla_k \vartheta|^p}{\vartheta^{p-1}} |\psi|^p \\
& \quad + \frac{1}{p^{p-1}} \frac{|\nabla_k \vartheta|^{p-2}}{\vartheta^{p-2}} \nabla_k \vartheta \cdot \nabla_k (|\psi|^p) + c_p \vartheta^p |\nabla_k \psi|^p.
\end{aligned}
$$

Hence, it can readily be inferred from (4.6) that

$$
\begin{aligned}
c_p \int_\Omega a\rho^{p-Q} |\nabla_k \varphi|^p dw &\geq \frac{c_p}{p^p} \int_\Omega a\rho^{p-Q} \frac{|\nabla_k \vartheta|^p}{\vartheta^{p-1}} |\psi|^p dw \\
& \quad - \frac{c_p}{p^{p-1}} \int_\Omega \nabla_k \cdot \left(a\rho^{p-Q} \frac{|\nabla_k \vartheta|^{p-2}}{\vartheta^{p-2}} \nabla_k \vartheta \right) |\psi|^p dw.
\end{aligned}
$$

Since

$$
-\nabla_k \cdot \left(a\rho^{p-Q} \frac{|\nabla_k \vartheta|^{p-2}}{\vartheta^{p-2}} \nabla_k \vartheta \right) \geq 0 \quad \text{and} \quad \psi := \vartheta^{-1/p} \rho^{(Q+\alpha-p)/p} \phi,
$$

we have

$$
(4.7)\qquad c_p \int_\Omega a\rho^{p-Q} |\nabla_k \varphi|^p dw \geq \frac{c_p}{p^p} \int_\Omega a\rho^\alpha \frac{|\nabla_k \vartheta|^p}{\vartheta^p} |\phi|^p dw.
$$

Finally, plugging (4.7) into inequality (4.5), and then taking into account that $\varphi = \rho^{(Q+\alpha-p)/p}\phi$, we deduce the claimed result (4.1). □

Remark 4.2. It is worthwhile noting that the inequality stated in Theorem 4.1 also holds for $1 < p < 2$ with different reminder terms and, in this case, we use the convexity inequality (3.6).

4.1. Applications of Theorem 4.1. We emphasize that, in our approach, the role of the differential inequality

$$-\nabla_k \cdot \left(a\rho^{p-Q} \frac{|\nabla_k \vartheta|^{p-2}}{\vartheta^{p-2}} \nabla_k \vartheta \right) \geq 0 \tag{4.8}$$

is crucial. We now present here some examples of the improved, weighted Hardy-type inequalities with several choices of a and ϑ in inequality (4.8). A first example is the following.

By applying Theorem 4.1 to the pair

$$a \equiv 1 \quad \text{and} \quad \vartheta = R - \rho$$

on the ρ-ball B_R centered at the origin with radius R, we immediately get the subsequent result.

Corollary 4.3. *Let B_R be the ρ-ball centered at the origin with radius R in $\mathbb{R}^{2n+1}$. Then, for all $\phi \in C_0^\infty(B_R)$, we have*

$$\int_{B_R} \rho^\alpha |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha-p}{p} \right)^p \int_{B_R} \rho^\alpha |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p} \, dw + \frac{c_p}{p^p} \int_{B_R} \frac{\rho^\alpha}{(R-\rho)^p} |\nabla_k \rho|^p |\phi|^p dw,$$

where $Q + \alpha > p \geq 2$, $\alpha \in \mathbb{R}$ and $c_p > 0$.

Another application of Theorem 4.1 with special functions

$$a = e^\rho \quad \text{and} \quad \vartheta = e^{-\rho}$$

is the following two-weight L^p Hardy-type inequality involving two nonnegative remainders.

Corollary 4.4. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{R}^{2n+1}$. Then, for all $\phi \in C_0^\infty(\Omega)$, we have*

$$\int_\Omega \rho^\alpha e^\rho |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega \rho^\alpha e^\rho |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw$$
$$+ \left(\frac{Q+\alpha-p}{p}\right)^{p-1} \int_\Omega \rho^{\alpha+1} e^\rho |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw$$
$$+ \frac{c_p}{p^p} \int_\Omega \rho^\alpha e^\rho |\nabla_k \rho|^p |\phi|^p dw,$$

where $Q+\alpha > p \geq 2$*,* $\alpha \in \mathbb{R}$*,* $c_p > 0$*.*

On the other hand, by setting the units

$$a \equiv 1, \qquad \vartheta = \log \frac{R}{\rho}, \quad R > \sup_{w \in \Omega} \rho,$$

on a bounded domain Ω with smooth boundary in $\mathbb{R}^{2n+1}$, we derive the weighted L^p Hardy-type inequality containing a logarithmic remainder.

Corollary 4.5. *Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$ *in* $\mathbb{R}^{2n+1}$*. Then, for all* $\phi \in C_0^\infty(\Omega)$*, we have*

$$\int_\Omega \rho^\alpha |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega \rho^\alpha |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw$$
$$+ \frac{c_p}{p^p} \int_\Omega \frac{\rho^\alpha}{(\log(R/\rho))^p} |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw,$$

where $Q+\alpha > p \geq 2$*,* $\alpha \in \mathbb{R}$*,* $c_p > 0$*,* $R > \sup_{w\in\Omega} \rho$*.*

Finally, we mention that, when considering Theorem 4.1 with the pair

$$a \equiv 1, \qquad \vartheta = \log\left(\log \frac{R}{\rho}\right), \quad R > e \sup_{w \in \Omega} \rho,$$

we can obtain the following result including a different logarithmic remainder.

Corollary 4.6. *Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$ *in* $\mathbb{R}^{2n+1}$*. Then, for all* $\phi \in C_0^\infty(\Omega)$*, we have*

$$\int_\Omega \rho^\alpha |\nabla_k \phi|^p dw \geq \left(\frac{Q+\alpha-p}{p}\right)^p \int_\Omega \rho^\alpha |\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw + \frac{c_p}{p^p}\int_\Omega \frac{\rho^\alpha}{(\log(R/\rho))^p(\log(\log(R/\rho)))^p}|\nabla_k \rho|^p \frac{|\phi|^p}{\rho^p}\, dw,$$

where $Q+\alpha > p \geq 2$, $\alpha \in \mathbb{R}$, $c_p > 0$, $R > e\sup_{w\in\Omega}\rho$.

Remark 4.7. The lack of regularities on the above choices can be readily handled by replacing the function ρ with ρ_ϵ and then passing to the limit as $\epsilon \to 0$.

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