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INFLUENCE OF BOUNDED STATES IN THE NEUMANN LAPLACIAN IN A THIN WAVEGUIDE

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ABSTRACT. Let $-\Delta_\Omega^N$ be the Neumann Laplacian operator restricted to a twisted waveguide Ω. Our first goal is to find the effective operator when Ω is "squeezed." However, since, in this process, there are divergent eigenvalues, we consider $-\Delta_\Omega^N$ acting in specific subspaces of the initial Hilbert space. The strategy is interesting since we find different effective operators in each situation. In the case where Ω is periodic and sufficiently thin, we also obtain information regarding the absolutely continuous spectrum of $-\Delta_\Omega^N$ (restricted to such subspaces) and the existence and location of band gaps in its structure.

1. Introduction and main results. The Laplacian operator in a set with Neumann boundary conditions has been studied in various situations **[11, 12, 14, 15, 16, 18, 19, 20, 22]**–**[27]**. In particular, let $-\Delta_\Omega^N$ be the Neumann Laplacian operator restricted to a thin waveguide Ω in $\mathbb{R}^3$. An interesting question is to study the behavior of $-\Delta_\Omega^N$ when the diameter of Ω tends to zero and to find the effective operator T in this process. Since Ω shrinks to a spatial curve, it is natural to associate T with a one-dimensional operator. In fact, it is known that T is the one-dimensional Neumann Laplacian operator; in this case, its action is given by $w \mapsto -w''$, see, for example, **[25]**. This result holds even if Ω is a twisted or a bent waveguide, i.e., the geometry of Ω does not influence the action of the effective operator.

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In this work, we study $-\Delta_\Omega^N$ in the case where Ω is a twisted waveguide. When Ω is "squeezed," there are divergent eigenvalues due to transverse oscillations. Then, we consider $-\Delta_\Omega^N$ restricted to specific subspaces of the initial Hilbert space. The interesting point is that, when the diameter of Ω tends to zero, we find different effective operators in each situation, namely, these operators depend on the geometry of Ω, see (1.9), (1.10), (1.11) and (1.12). The second goal of this work is to consider the case where Ω is periodic in the sense that the twisted effect varies periodically. In the case that Ω is sufficiently thin, we find information regarding the absolutely continuous spectrum of $-\Delta_\Omega^N$ (restricted to the chosen subspaces) and the existence and location of band gaps in its structure. In the next paragraphs, we explain the model and provide details of our main results.

Let $I = \mathbb{R}$ or $I = (a,b)$ be a bounded interval in $\mathbb{R}$. Choose $S \neq \emptyset$ as an open, bounded, smooth, connected subset of $\mathbb{R}^2$; denote by $y := (y_1, y_2)$ an element of S. Let $\alpha : \overline{I} \to \mathbb{R}$ be a C^2 function. We suppose that $\alpha', \alpha'' \in L^\infty(I)$ and $\alpha(0) = 0$ if $I = \mathbb{R}$, or $\alpha(a) = 0$ if $I = (a,b)$. For each $\varepsilon > 0$ small enough, we define the thin, twisted waveguide

$$\Omega_\varepsilon^\alpha := \{\Gamma_\varepsilon^\alpha(s)\mathbf{x}^t,\ \mathbf{x} = (s,y) \in I \times S\},$$

where

$$\Gamma_\varepsilon^\alpha(s) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon\cos\alpha(s) & -\varepsilon\sin\alpha(s) \\ 0 & \varepsilon\sin\alpha(s) & \varepsilon\cos\alpha(s) \end{pmatrix}. \tag{1.1}$$

Let $-\Delta_{\Omega_\varepsilon^\alpha}^N$ be the Neumann Laplacian operator on $L^2(\Omega_\varepsilon^\alpha)$, i.e., the self-adjoint operator associated with the quadratic form

$$\widetilde{b}_\varepsilon(\psi) = \int_{\Omega_\varepsilon^\alpha} |\nabla\psi|^2 d\vec{x}, \quad \operatorname{dom}\widetilde{b}_\varepsilon = H^1(\Omega_\varepsilon^\alpha). \tag{1.2}$$

Since we shall use the Γ-convergence technique (see Appendix A.2 and [**7**]), our analysis is based on the study of the sequence $(\widetilde{b}_\varepsilon)_\varepsilon$. In order to simplify the calculations, it is convenient to change the variables. Using the change of coordinates described in Section 2, the quadratic form $\widetilde{b}_\varepsilon$ becomes

$$\widehat{b}_\varepsilon(\psi) = \int_Q \left(|\psi' + \langle \nabla_y\psi, Ry\rangle\alpha'(s)|^2 + \frac{|\nabla_y\psi|^2}{\varepsilon^2} \right) \mathrm{d}s\, \mathrm{d}y, \tag{1.3}$$

$\operatorname{dom}\widehat{b}_\varepsilon = H^1(Q)$, $Q := I \times S$. Here, $\psi' := \partial\psi/\partial s$, $\nabla_y\psi := (\partial\psi/\partial y_1, \partial\psi/\partial y_2)$, and R is the rotation matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote by $-\widehat{\Delta}^\varepsilon$ the self-adjoint operator associated with $\widehat{b}^\varepsilon$.

When the waveguide is "squeezed," i.e., $\varepsilon \to 0$, $-\Delta^N_{\Omega^\alpha_\varepsilon}$ presents divergent eigenvalues due to the transverse oscillations in $\Omega^\alpha_\varepsilon$; this can easily be seen by the presence of the term $(1/\varepsilon^2)\int_Q |\nabla_y\psi|^2 \mathrm{d}s\,\mathrm{d}y$ in (1.3). In order to control this divergent energy, we take the following strategy. Let $-\Delta^N_S$ be the Neumann Laplacian operator restricted to S, i.e., the self-adjoint operator associated with the quadratic form

$$u \longmapsto \int_S |\nabla_y u|^2 \mathrm{d}y, \quad u \in H^1(S).$$

Denote by λ_n the nth eigenvalue of $-\Delta^N_S$ and by u_n the corresponding normalized eigenfunction, i.e.,

$$\begin{aligned} 0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \\ -\Delta^N_S u_n = \lambda_n u_n, \quad n = 1, 2, 3, \ldots. \end{aligned}$$

We assume that each eigenvalue λ_n is simple; note that u_1 is a constant function.

With $n \in \mathbb{N}$ fixed, our strategy is to study the sequence

$$\widehat{b}^\varepsilon_n(\psi) := \widehat{b}_\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2}\|\psi\|^2_{L^2(Q)},$$

$\operatorname{dom}\widehat{b}^\varepsilon_n := H^1(Q)$. Denote by $\widehat{T}^\varepsilon_n$ the self-adjoint operator associated with $\widehat{b}^\varepsilon_n$; this can be done since each quadratic form $\widehat{b}^\varepsilon_n$ is closed and lower bounded in $L^2(Q)$, namely, $\widehat{T}^\varepsilon_n = -\widehat{\Delta}^\varepsilon - (\lambda_n/\varepsilon^2)\mathbf{1}$, where $\mathbf{1}$ denotes the identity operator.

It is standard in the literature to consider only the case $n = 1$, i.e., since $\lambda_1 = 0$, the sequence of quadratic forms $\widehat{b}_\varepsilon(\psi)$ may be directly studied. The idea is to consider $n \neq 1$, based on [**8**]; the author considered the Dirichlet Laplacian operator restricted to a thin waveguide with the goal of finding the effective operator. In that case, the action of the effective operator is the same for $n = 1$ or $n \neq 1$ and depends upon the geometry of the waveguide.

Now, for each $n \in \mathbb{N}$, consider the closed subspaces

$$\mathcal{L}_n := \{w(s)u_n(y) : w \in L^2(I)\}$$

and

$$\mathcal{K}_n := \{w(s)u_n(y) : w \in H^1(I)\}$$

of $L^2(Q)$ and $H^1(Q)$, respectively. We have the decompositions

$$L^2(Q) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \oplus \cdots ,$$

$$H^1(Q) = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \oplus \cdots ,$$

and each $\mathcal{K}_n$ is a dense subspace of $\mathcal{L}_n$.

Let $T_1 w := -w''$ be the one-dimensional Laplacian operator with domain $\operatorname{dom} T_1 = H^2(\mathbb{R})$ if $I = \mathbb{R}$, or $\operatorname{dom} T_1 = \{w \in H^2(I) : w'(a) = w'(b) = 0\}$ if $I = (a, b)$. Denote by $\mathbf{0}$ the null operator on the subspace $\mathcal{L}_1^\perp$. In the particular case $n = 1$, it is known that $\widehat{T}_1^\varepsilon \approx T_1 \oplus \mathbf{0}$, as $\varepsilon \to 0$, see **[25]**. As already noted, we can see that the effective operator in this situation does not depend upon the geometry of the waveguide.

The main goal of this work is to study the sequence $(\widehat{T}_n^\varepsilon)_\varepsilon$, for each $n > 1$ fixed, and to characterize the effective operator in the limit $\varepsilon \to 0$. However, some adjustments will be necessary so that the limit exists in some sense. The interesting point in this situation is that we find an effective operator that depends on the geometry of the waveguide. To our knowledge, this fact remains unknown.

In order to study the sequence $(\widehat{T}_n^\varepsilon)_\varepsilon$, some considerations will be necessary. If $v(s, y) = w(s)u_j(y)$ with $w \in H^1(I)$, some calculations show that

$$\begin{aligned}\widehat{b}_n^\varepsilon(v) &= \int_Q |w' u_j + \langle \nabla_y u_j, Ry\rangle \alpha'(s) w|^2 \mathrm{d}s\,\mathrm{d}y \\ &\quad + \frac{1}{\varepsilon^2}\int_Q (|\nabla_y u_j|^2 - \lambda_n |u_j|^2)|w|^2 \mathrm{d}s\,\mathrm{d}y \\ &= \int_Q |w' u_j + \langle \nabla_y u_j, Ry\rangle \alpha'(s) w|^2 \mathrm{d}s\,\mathrm{d}y + \frac{(\lambda_j - \lambda_n)}{\varepsilon^2}\|w\|^2_{L^2(I)},\end{aligned}$$

i.e., for $j < n$,

$$\lim_{\varepsilon \to 0} \widehat{b}_n^\varepsilon(v) = -\infty. \tag{1.4}$$

Then, the sequence $(\widehat{b}_n^\varepsilon(v))_\varepsilon$ is not bounded from below. Therefore, to study $(\widehat{b}_n^\varepsilon)_\varepsilon$, it will be necessary to exclude some vectors of the domains $\operatorname{dom} \widehat{b}_n^\varepsilon$. Based upon (1.4), the procedure for this problem is as follows. We define the Hilbert spaces

$$\mathcal{H}_n := \begin{cases} L^2(Q) & n = 1, \\ (\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-1})^\perp & n = 2, 3, \ldots, \end{cases} \tag{1.5}$$

equipped with the norm of $L^2(Q)$. Then, we consider the sequence of quadratic forms acting in $\mathcal{H}_n$:

$$\overline{b}_n^\varepsilon(\psi) = \int_Q \left(|\psi' + \langle \nabla_y \psi, Ry \rangle \alpha'(s)|^2 + \frac{1}{\varepsilon^2} |\nabla_y \psi|^2 \right) \mathrm{d}s \, \mathrm{d}y, \tag{1.6}$$

$\operatorname{dom} \overline{b}_n^\varepsilon = H^1(Q) \cap \mathcal{H}_n$, and we denote by $-\Delta_n^\varepsilon$ the self-adjoint operator on $\mathcal{H}_n$ associated with it. Finally, define

$$b_n^\varepsilon(\psi) := \overline{b}_n^\varepsilon(\psi) - \frac{\lambda_n}{\varepsilon^2} \|\psi\|_{\mathcal{H}_n}^2, \tag{1.7}$$

$\operatorname{dom} b_n^\varepsilon := H^1(Q) \cap \mathcal{H}_n$. Denote by T_n^ε the self-adjoint operator associated with b_n^ε which is a positive and closed quadratic form; T_n^ε acts in the Hilbert space $\mathcal{H}_n$, namely, $T_n^\varepsilon = -\Delta_n^\varepsilon - (\lambda_n/\varepsilon^2)\mathbf{1}$.

Next, we study the sequence $(T_n^\varepsilon)_\varepsilon$ instead of $(\widehat{T}_n^\varepsilon)_\varepsilon$. As a note, let U_ε be the unitary operator defined by (2.1). The operator $-\Delta_n^\varepsilon = T_n^\varepsilon + (\lambda_n/\varepsilon^2)\mathbf{1}$ is unitarily equivalent to the Laplacian operator with domain $U_\varepsilon^{-1}(\operatorname{dom} T_n^\varepsilon)$ in the Hilbert space $U_\varepsilon^{-1}(\mathcal{H}_n)$. In this sense, we say that we work with the Neumann Laplacian operator restricted to specific subspaces of the initial Hilbert space $L^2(\Omega_\varepsilon^\alpha)$.

Let b_n be the one-dimensional quadratic form

$$b_n(w) := b_n^\varepsilon(wu_n) = \int_Q |w'u_n + \langle \nabla_y u_n, Ry \rangle \alpha'(s) w|^2 \, \mathrm{d}s \, \mathrm{d}y, \tag{1.8}$$

$\operatorname{dom} b_n = H^1(I)$. In fact, b_n is obtained by the restriction of b_n^ε to the space $\mathcal{K}_n$. Denote by T_n the self-adjoint operator associated with b_n.

For each $n \in \mathbb{N}$, define the constants

$$\begin{aligned} C_n^1(S) &:= \int_S |\langle \nabla_y u_n, Ry\rangle|^2 \mathrm{d}y, \\ C_n^2(S) &:= \int_S u_n \langle \nabla_y u_n, Ry\rangle \,\mathrm{d}y, \end{aligned} \tag{1.9}$$

and the real potential

$$V_n(s) := C_n^1(S)(\alpha'(s))^2 - C_n^2(S)\alpha''(s); \tag{1.10}$$

note that $C_n^1(S)$ and $C_n^2(S)$ depend only upon the tube cross section S. From Appendix A.1,

$$T_n w = -w'' + V_n(s)w, \tag{1.11}$$

where $\operatorname{dom} T_n = H^2(\mathbb{R})$ if $I = \mathbb{R}$, and

$$\operatorname{dom} T_n = \left\{ w \in H^2(I) : \begin{matrix} w'(a) = -C_n^2(S)\alpha'(a)w(a) \\ w'(b) = -C_n^2(S)\alpha'(b)w(b) \end{matrix} \right\} \tag{1.12}$$

if $I = (a, b)$. In the latter, for $n > 1$, we have the Robin conditions in $\operatorname{dom} T_n$. On the other hand, for $n = 1$, $C_1^1(S) = C_1^2(S) = 0$ and $\operatorname{dom} T_1 = \{w \in H^2(I) : w'(a) = w'(b) = 0\}$, i.e., T_1 is the one-dimensional Neumann Laplacian operator. We emphasize that, for $n > 1$, in both cases, $I = \mathbb{R}$ or $I = (a, b)$, and the constants $C_n^1(S)$ and $C_n^2(S)$ govern the effects of $\alpha(s)$ in T_n.

Now, we present the first result of this work.

Theorem 1.1.

(A) *For each $n \in \mathbb{N}$ fixed, the sequence of self-adjoint operators $(T_n^\varepsilon)_\varepsilon$ converges in the strong resolvent sense to T_n in $\mathcal{L}_n$, as $\varepsilon \to 0$, that is,*

$$\lim_{\varepsilon \to 0} R_{-\lambda}(T_n^\varepsilon)\zeta = R_{-\lambda}(T_n)P\zeta \quad \textit{for all } \zeta \in \mathcal{H}_n,\ \lambda > 0,$$

where P is the orthogonal projection onto $\mathcal{L}_n$.

(B) *In addition, suppose that $I = (a, b)$ is a bounded interval. Denote by $\mu_j(\varepsilon)$, respectively μ_j, the jth eigenvalue of $-\Delta_n^\varepsilon$, respectively T_n, counted according to its multiplicity. Then, for each $j \in \mathbb{N}$,*

$$\mu_j = \lim_{\varepsilon \to 0} \left(\mu_j(\varepsilon) - \frac{\lambda_n}{\varepsilon^2} \right). \tag{1.13}$$

The proof of Theorem 1.1 is based on arguments of [**4, 8**]. In those works, the authors considered the Dirichlet Laplacian restricted to a thin waveguide. We perform the necessary adjustments in order to work in the Sobolev space $H^1(Q)$.

In the case $n = 1$, the operator $T_1^\varepsilon = -\Delta_1^\varepsilon$ is unitarily equivalent to the Neumann Laplacian operator $-\Delta_{\Omega_\varepsilon^\alpha}^N$ on $L^2(\Omega_\varepsilon^\alpha)$. Theorem 1.1 shows that the effective operator in this case is the one-dimensional Neumann Laplacian operator. This problem was studied in [**25**]. Therefore, our main contribution is the case $n > 1$.

Here, we treat the periodic case. Consider the twisted waveguide $\Omega_\varepsilon^\alpha$ in the particular case where $I = \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is a C^2, periodic function, i.e., there exists an $L > 0$ so that $\alpha(s + L) = \alpha(s)$ for all $s \in \mathbb{R}$. One of the goals of this work is to find spectral properties of $-\Delta_n^\varepsilon$ in this situation. We have:

Theorem 1.2. *For each $n \in \mathbb{N}$ and $E > 0$, there exists an $\varepsilon_E > 0$ such that the spectrum of $-\Delta_n^\varepsilon$ is absolutely continuous in the interval $[0, E + \lambda_n/\varepsilon^2]$ for all $\varepsilon \in (0, \varepsilon_E)$.*

Theorem 1.3. *Suppose that $V_n(s)$ is not a constant function in $[0, L]$. For each $n \in \mathbb{N} \setminus \{1\}$, there exist $j \in \mathbb{N}$ and $\varepsilon_j > 0$ so that, for all $\varepsilon \in (0, \varepsilon_j)$, the spectrum of the operator $-\Delta_n^\varepsilon$ has at least one gap.*

Furthermore, in Theorem 4.7 in subsection 4.4, we find a location in $\sigma(-\Delta_n^\varepsilon)$ where Theorem 1.3 holds true.

The proof of Theorem 1.3 is based on the fact that $V_n(s)$ is not a constant function in $[0, L]$. For this reason, we eliminate the case $n = 1$ since $V_1(s) \equiv 0$. As already noted, this latter situation refers to the Neumann Laplacian operator $-\Delta_{\Omega_\varepsilon^\alpha}^N$ on $L^2(\Omega_\varepsilon^\alpha)$. Our strategy for study does not provide conditions to guarantee the existence of gaps in the spectrum $\sigma(-\Delta_{\Omega_\varepsilon^\alpha}^N)$. Therefore, in this case, we only obtain information regarding the absolutely continuous spectrum of $-\Delta_{\Omega_\varepsilon^\alpha}^N$ (Theorem 1.2).

The reader is referred to [**12, 19**] for spectral properties of the Neumann Laplacian operator in periodic domains. In [**12**], under the condition of symmetry $s \mapsto -s$ in the waveguide build, the absolute continuity of $\sigma(-\Delta_{\Omega_\varepsilon^\alpha}^N)$ is proven. In [**19**], the author discussed the existence of gaps in $\sigma(-\Delta_{\Omega_\varepsilon^\alpha}^N)$. For other situations, such as twisted waveguides with Dirichlet boundary conditions, we recommend [**5, 6, 9, 10**].

This work is organized as follows. In Section 2, we perform the change of variables to obtain (1.3), and, in Section 3, we prove Theorem 1.1. Section 4 is dedicated to the periodic case and is separated into subsections. In subsection 4.1, we present some preliminary results, and, in subsection 4.2, we prove Theorem 1.2. Subsection 4.3 is dedicated to proving Theorem 1.3, and, in subsection 4.4, we study the location of band gaps. Throughout, the symbol K is used to denote different constants, and it never depends upon θ.

2. Geometry of the domain. Recall that the quadratic form $\widetilde{b}_\varepsilon$ is defined by (1.2). In this section, we perform a standard change of variables so that the domain $\operatorname{dom} \widetilde{b}_\varepsilon$ becomes independent of ε. Here, we consider the mapping

$$\begin{aligned} F_\varepsilon : Q &\longrightarrow \Omega_\varepsilon^\alpha \\ (s, y_1, y_2) &\longmapsto \Gamma_\varepsilon^\alpha(s)(s, y_1, y_2)^t, \end{aligned}$$

where $\Gamma_\varepsilon^\alpha(s)$ is given by (1.1); F_ε will be a (global) diffeomorphism for $\varepsilon > 0$ small enough.

In the new variables, the domain $\operatorname{dom} \widetilde{b}_\varepsilon$ turns out to be $H^1(Q)$. On the other hand, the price to be paid is a nontrivial Riemannian metric $G = G_\varepsilon^\alpha$, which is induced by F_ε i.e., $G = (G_{ij})$, $G_{ij} = \langle e_i, e_j \rangle$, $1 \leq i,\ j \leq 3$, where $e_1 = \partial F_\varepsilon / \partial s$, $e_2 = \partial F_\varepsilon / \partial y_1$ and $e_3 = \partial F_\varepsilon / \partial y_2$. Calculation shows that

$$J = \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 1 & -\varepsilon \alpha'(s) \langle z_\alpha^\perp(s), y \rangle & \varepsilon \alpha'(s) \langle z_\alpha(s), y \rangle, \\ 0 & \varepsilon \cos \alpha(s) & \varepsilon \sin \alpha(s) \\ 0 & -\varepsilon \sin \alpha(s) & \varepsilon \cos \alpha(s) \end{pmatrix},$$

where

$$z_\alpha(s) := (\cos \alpha(s), -\sin \alpha(s)), \qquad z_\alpha^\perp(s) := (\sin \alpha(s), \cos \alpha(s)).$$

The inverse matrix of J is given by

$$J^{-1} = \begin{pmatrix} 1 & \alpha'(s) y_2 & -\alpha'(s) y_1 \\ 0 & (\cos \alpha(s))/\varepsilon & -(\sin \alpha(s))/\varepsilon \\ 0 & (\sin \alpha(s))/\varepsilon & (\cos \alpha(s))/\varepsilon \end{pmatrix}.$$

Note that $JJ^t = G$ and $\det J = |\det G|^{1/2} = \varepsilon^2 > 0$. Thus, F_ε is a local diffeomorphism. In the case that F_ε is injective (for this, merely consider $\varepsilon > 0$ small enough), a global diffeomorphism is obtained.

Introducing the unitary transformation

$$\begin{aligned} U_\varepsilon : L^2(\Omega_\varepsilon^\alpha) &\longrightarrow L^2(Q) \\ \phi &\longmapsto \varepsilon\phi \circ F_\varepsilon, \end{aligned} \tag{2.1}$$

we obtain the quadratic form

$$\begin{aligned} \widehat{b}_\varepsilon(\psi) := \widetilde{b}_\varepsilon(U_\varepsilon^{-1}\psi) &= \|J^{-1}\nabla\psi\|^2_{L^2(Q)} \\ &= \int_Q \left(|\psi' + \langle \nabla_y \psi, Ry\rangle \alpha'(s)|^2 + \frac{|\nabla_y\psi|^2}{\varepsilon^2} \right) \mathrm{d}s\,\mathrm{d}y, \end{aligned}$$

$\operatorname{dom} \widehat{b}_\varepsilon = H^1(Q)$. Recall that R is the rotation matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $-\widehat{\Delta}_\varepsilon$ denotes the self-adjoint operator associated with $\widehat{b}_\varepsilon$. We have $-\widehat{\Delta}_\varepsilon\psi = U_\varepsilon(-\Delta^N_{\Omega_\varepsilon^\alpha})U_\varepsilon^{-1}\psi$, where $\operatorname{dom}(-\widehat{\Delta}_\varepsilon) = U_\varepsilon(\operatorname{dom}(-\Delta^N_{\Omega_\varepsilon^\alpha}))$.

3. Preliminary results and proof of Theorem 1.1. This section is dedicated to the proof of Theorem 1.1. The strategy is based on the study of the sequence $(b_n^\varepsilon)_\varepsilon$, see (1.7). Some preliminary results will be necessary. We begin with the following considerations. Denote by $[u_1, u_2, \ldots, u_k]$ the subspace of $L^2(S)$ generated by $\{u_1, u_2, \ldots, u_k\}$. Since the subspace $\mathcal{W}_k := [u_1, u_2, \ldots, u_k]^\perp$ is invariant under the operator $-\Delta_S^N$, the restriction $-\Delta_S^N|_{\mathcal{W}_k}$ is well defined, and its first eigenvalue is λ_{k+1}. Denote by q_k the quadratic form associated with $-\Delta_S^N|_{\mathcal{W}_k}$. We have

$$q_k(v) \geq \lambda_{k+1}\|v\|^2_{L^2(S)} \quad \text{for all } v \in \mathcal{W}_k \cap H^1(S). \tag{3.1}$$

In order to study the sequence $(b_n^\varepsilon)_\varepsilon$, we shall use the Γ-convergence technique, see Appendix A.2. It is necessary to extend each b_n^ε on $\mathcal{H}_n$ by setting (we denote by the same symbol)

$$b_n^\varepsilon(v) = \begin{cases} b_n^\varepsilon(v) & \text{if } v \in \operatorname{dom} b_n^\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

In a similar manner, we extend b_n on $\mathcal{H}_n$:

$$b_n(v) = \begin{cases} b_n(w) & \text{if } v = wu_n \text{ with } w \in \operatorname{dom} b_n, \\ +\infty & \text{otherwise;} \end{cases}$$

recall the definition of b_n by (1.8) in Section 1.

Lemma 3.1. *If $v_\varepsilon \rightharpoonup v$ in $\mathcal{H}_n$ and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ is a bounded sequence, then $(v'_\varepsilon)_\varepsilon$ and $(\nabla_y v_\varepsilon)_\varepsilon$ are bounded sequences in $\mathcal{H}_n$. Furthermore, $v \in H^1(Q)$ and there exists a subsequence of $(v_\varepsilon)_\varepsilon$, denoted by the same symbol $(v_\varepsilon)_\varepsilon$, so that $v'_\varepsilon \rightharpoonup v'$ and $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$.*

Proof. Since $(v_\varepsilon)_\varepsilon$ and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ are bounded sequences, there exists a number $K > 0$ so that

$$\limsup_{\varepsilon\to 0} \int_Q |v'_\varepsilon + \langle \nabla_y v_\varepsilon, Ry\rangle \alpha'(s)|^2 \mathrm{d}s\,\mathrm{d}y \le \limsup_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < K,$$

and

$$\begin{aligned} &\limsup_{\varepsilon\to 0} \int_Q |\nabla_y v_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y \\ &= \limsup_{\varepsilon\to 0} \left(\int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2)\,\mathrm{d}s\,\mathrm{d}y + \int_Q \lambda_n |v_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y \right) \\ &\le \limsup_{\varepsilon\to 0} K\varepsilon^2 + \limsup_{\varepsilon\to 0} \int_Q \lambda_n |v_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y < K. \end{aligned} \tag{3.2}$$

These estimates along with the fact that α' and Ry are bounded functions show that $(v'_\varepsilon)_\varepsilon$ and $(\nabla_y v_\varepsilon)_\varepsilon$ are bounded sequences in $L^2(Q)$. Therefore, $(v_\varepsilon)_\varepsilon$ is a bounded sequence in $H^1(Q)$. Thus, there exists $\psi \in H^1(Q)$ and a subsequence of $(v_\varepsilon)_\varepsilon$, also denoted by $(v_\varepsilon)_\varepsilon$, so that $v_\varepsilon \rightharpoonup \psi$ in $H^1(Q)$ (recall that this Hilbert space is reflexive). Since $v_\varepsilon \rightharpoonup v$ in $\mathcal{H}_n$, it follows that $v = \psi$, $v'_\varepsilon \rightharpoonup v'$, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in $\mathcal{H}_n$, $v \in H^1(Q)$. □

Lemma 3.2. *If $v_\varepsilon \to v$ in $\mathcal{H}_n$ and the limit $\lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$ exists, then $v(s,y) = w(s)u_n(y)$ with $w \in H^1(I)$, i.e., $v \in \mathcal{K}_n$.*

Proof. From Lemma 3.1, passing to a subsequence if necessary, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in $L^2(Q)$. By weak lower semi-continuity of the L^2-norm, inequality (3.2) and the strong convergence of $(v_\varepsilon)_\varepsilon$, we have

$$\int_Q |\nabla_y v|^2 \mathrm{d}s\,\mathrm{d}y \leq \liminf_{\varepsilon\to 0} \int_Q |\nabla_y v_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y$$
$$\leq \limsup_{\varepsilon\to 0} \lambda_n \int_Q |v_\varepsilon|^2 \mathrm{d}s\,\mathrm{d}y = \lambda_n \int_Q |v|^2 \mathrm{d}s\,\mathrm{d}y.$$

Now, define the function $f_n(s) := \int_S (|\nabla_y v(s,y)|^2 - \lambda_n |v(s,y)|^2)\,\mathrm{d}y$. The latter inequalities show that $f_n(s) \leq 0$. However, (3.1) ensures that $f_n(s) \geq 0$. Then, $f_n = 0$ almost everywhere. We conclude that $v(s,\cdot) \in \mathcal{W}_{n-1} \cap H^1(S)$, and $v(s,\cdot)$ is an eigenfunction of the operator $-\Delta_S^N|_{\mathcal{W}_{n-1}}$ whose associated eigenvalue is λ_n. Since λ_n is simple, $v(s,\cdot)$ is proportional to u_n. Thus, we can write $v(s,y) = w(s)u_n(y)$ with $w \in H^1(I)$, as $v \in H^1(Q)$. □

Proposition 3.3. *For each $n \in \mathbb{N}$, the sequence of quadratic forms $(b_n^\varepsilon)_\varepsilon$ strongly Γ-converges to b_n, as $\varepsilon \to 0$.*

Proof. We must prove items (i) and (ii) according to the definition of strong Γ-convergence in Appendix A.2.

Let $v \in \mathcal{H}_n$ and $v_\varepsilon \to v$ in $\mathcal{H}_n$. If $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = +\infty$, then $b_n(v) \leq \liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon)$. Now, assume that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. Passing to a subsequence, if necessary, we can suppose that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$.

Lemma 3.1 ensures that $v_\varepsilon' \rightharpoonup v'$, $\nabla_y v_\varepsilon \rightharpoonup \nabla_y v$ in $L^2(Q)$, $v \in H^1(Q)$. Since α' is a bounded function,

$$v_\varepsilon' + \langle \nabla_y v_\varepsilon, Ry\rangle \alpha' \rightharpoonup v' + \langle \nabla_y v, Ry\rangle \alpha'$$

in $L^2(Q)$. Then,

$$\int_Q |v' + \langle \nabla_y v, Ry\rangle \alpha'(s)|^2 \mathrm{d}s\,\mathrm{d}y \leq \liminf_{\varepsilon\to 0} \int_Q |v_\varepsilon' + v\langle \nabla_y v_\varepsilon, Ry\rangle \alpha'(s)|^2 \mathrm{d}s\,\mathrm{d}y$$
$$\leq \liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon).$$

By Lemma 3.2, we can write $v(s,y) = w(s)u_n(y)$ with $w \in H^1(I)$. Thus,

$$b_n(w) = b_n(v) \leq \liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon),$$

and item (i) is proven.

In order to prove (ii), we shall show that, for each $v \in \mathcal{H}_n$, there exists a sequence $(v_\varepsilon)_\varepsilon$ in $\mathcal{H}_n$ such that $v_\varepsilon \to v$ in $\mathcal{H}_n$ and

$\lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = b_n(v)$. First, consider the particular case $v(s,y) = w(s)u_n(y)$ with $w \in H^1(I)$. Take $v_\varepsilon := v$ for all $\varepsilon > 0$. Note that $b_n^\varepsilon(v) = b_n(w)$, for all $\varepsilon > 0$, and

$$\lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = b_n(v).$$

On the other hand, if $v \in \mathcal{H}_n \setminus \{w(s)u_n(y) : w \in H^1(I)\}$, we have $b_n(v) = +\infty$. Let $(v_\varepsilon)_\varepsilon$ be an arbitrary sequence so that $v_\varepsilon \to v$ in $\mathcal{H}_n$. In this case, $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = +\infty$. In fact, if we suppose that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$, by Lemmas 3.1 and 3.2, we should have $v = wu_n$, with $w \in H^1(I)$; however, this is not true. Therefore, $+\infty = \liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = b_n(v)$. Hence, item (ii) is satisfied. □

Proposition 3.4. *For each $n \in \mathbb{N}$, the sequence of quadratic forms $(b_n^\varepsilon)_\varepsilon$ weakly Γ-converges to b_n as $\varepsilon \to 0$.*

Proof. First, we shall show the condition (i) of the definition of weak Γ-convergence, i.e., $b_n(v) \le \liminf_{\varepsilon\to 0} b_n(v_\varepsilon)$, for the sequence $v_\varepsilon \rightharpoonup v$ in $\mathcal{H}_n$. Thus, assume the weak convergence $v_\varepsilon \rightharpoonup v$. Consider the case where $(v_\varepsilon)_\varepsilon$ does not belong to $\mathcal{H}_n \cap H^1(Q)$. Then, $b_n^\varepsilon(v_\varepsilon) = +\infty$, for all $\varepsilon > 0$, and the inequality is proven. Now, assume that $(v_\varepsilon)_\varepsilon \subset \mathcal{H}_n \cap H^1(Q)$. Suppose that $v = wu_n$ with $w \in H^1(I)$. By definition, $b_n(v) < +\infty$. If $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = +\infty$, the inequality is proven. Now, suppose that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. Passing to a subsequence, if necessary, we can suppose that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = \lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) < +\infty$. As in the proof of Proposition 3.3,

$$\lim_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) \ge \int_Q |v' + \langle \nabla_y v, Ry\rangle \alpha'(s)|^2 \mathrm{d}s\, \mathrm{d}y = b_n(w).$$

Now, suppose that v does not belong to the subspace $\{wu_n : w \in H^1(I)\}$. We show, necessarily, that $\liminf_{\varepsilon\to 0} b_n^\varepsilon(v_\varepsilon) = +\infty$. In fact, let P_{n+1} be the orthogonal projection onto $\mathcal{H}_{n+1}$. We have $\|P_{n+1}v\| > 0$. Since $v_\varepsilon \rightharpoonup v$ in $\mathcal{H}_n \cap H^1(Q)$, it holds that $P_{n+1}v_\varepsilon \rightharpoonup P_{n+1}v$ and

$$\liminf_{\varepsilon\to 0} \|P_{n+1}v_\varepsilon\| \ge \|P_{n+1}v\| > 0. \tag{3.3}$$

Due to the inequality

$$b_n^\varepsilon(v_\varepsilon) \ge \frac{1}{\varepsilon^2} \int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2)\, \mathrm{d}s\, \mathrm{d}y, \tag{3.4}$$

the strategy is to estimate its term on the right side.

For $\psi \in H^1(S) \cap \mathcal{W}_{n-1}$, denote by ψ^n the component of ψ in $[u_n]$ and by Q_{n+1} the orthogonal projection onto $\mathcal{W}_n$ in $H^1(S)$. Thus,

$$
\begin{aligned}
&\frac{1}{\varepsilon^2}\int_Q (|\nabla_y v_\varepsilon|^2 - \lambda_n |v_\varepsilon|^2)\, \mathrm{d}s\, \mathrm{d}y \\
&= \frac{1}{\varepsilon^2}\int_I (\|\nabla_y v_\varepsilon(s,\cdot)\|^2_{L^2(S)} - \lambda_n \|v_\varepsilon(s,\cdot)\|^2_{L^2(S)})\, \mathrm{d}s \\
&= \frac{1}{\varepsilon^2}\int_I (\|v_\varepsilon(s,\cdot)\|^2_{H^1(S)} - (\lambda_n+1) \|v_\varepsilon(s,\cdot)\|^2_{L^2(S)})\, \mathrm{d}s \\
&= \frac{1}{\varepsilon^2}\int_I (\|Q_{n+1} v_\varepsilon(s,\cdot)\|^2_{H^1(S)} + \|v^n_\varepsilon(s,\cdot)\|^2_{H^1(S)} - (\lambda_n+1) \|v_\varepsilon(s,\cdot)\|^2_{L^2(S)})\, \mathrm{d}s \\
&= \frac{1}{\varepsilon^2}\int_I (\|\nabla_y Q_{n+1} v_\varepsilon(s,\cdot)\|^2_{L^2(S)} + \|Q_{n+1} v_\varepsilon(s,\cdot)\|^2_{L^2(S)} \\
&\quad + \|\nabla_n v^n_\varepsilon(s,\cdot)\|^2_{L^2(S)} + \|v^n_\varepsilon(s,\cdot)\|^2_{L^2(S)} - (\lambda_n+1) \|v_\varepsilon(s,\cdot)\|^2_{L^2(S)})\, \mathrm{d}s \\
&\geq \frac{1}{\varepsilon^2}\int_I (\lambda_{n+1} \|Q_{n+1} v_\varepsilon(s,\cdot)\|^2_{L^2(S)} + \lambda_n \|v^n_\varepsilon(s,\cdot)\|^2_{H^1(S)} - \lambda_n \|v_\varepsilon(s,\cdot)\|^2_{L^2(S)})\, \mathrm{d}s \\
&= \frac{1}{\varepsilon^2}\int_I (\lambda_{n+1} - \lambda_n) |Q_{n+1} v_\varepsilon|^2 \mathrm{d}s\, \mathrm{d}y \\
&= \frac{(\lambda_{n+1} - \lambda_n)}{\varepsilon^2} \|P_{n+1} v_\varepsilon\|^2 \geq \frac{(\lambda_{n+1} - \lambda_n)}{\varepsilon^2} \|P_{n+1} v\|^2.
\end{aligned}
$$

This estimate, (3.3), (3.4) and the fact that $\lambda_{n+1} > \lambda_n$ imply that lim $\inf_{\varepsilon\to 0} b^\varepsilon_n(v_\varepsilon) = +\infty$.

Finally, condition (ii) of the definition of weak Γ-convergence can be proven in a similar manner to the proof of Proposition 3.3. □

Proof of Theorem 1.1.

(A) This item follows from Propositions 3.3 and 3.4 of this section and Proposition A.1 in Appendix A.2.

(B) We must verify Proposition A.2 (a), (b), (c). Item (a) follows by Propositions 3.3 and 3.4. It is known that the operator T_n has a compact resolvent. Thus, (b) is satisfied. It remains to find (c). Consider the subspace $\mathcal{K} := \{\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{n-1}\}^{\perp}$. By the Rellich-Kondrachov theorem, $\mathcal{K}$ is compactly embedded in $\mathcal{H}_n$. Thus, if $(v_\varepsilon)_\varepsilon$ is a bounded sequence in $\mathcal{H}_n$ and $(b_n^\varepsilon(v_\varepsilon))_\varepsilon$ is also bounded, a similar proof to Lemma 3.1 shows that $(v_\varepsilon)_\varepsilon$ is a bounded sequence in $\mathcal{K}$. Thus, item (c) is satisfied. From Proposition A.2, T_n^ε converges in the norm resolvent sense to T_n in $\mathcal{L}_n$. By [**13,** Corollary 2.3], we have the asymptotic behavior of the eigenvalues given by (1.13). □

4. Spectral properties in the case of the periodic waveguide. Consider $\Omega_\varepsilon^\alpha$ as in Section 1 in the particular case where $I = \mathbb{R}$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is both C^2 and a periodic function, i.e., there exists an $L > 0$ so that $\alpha(s+L) = \alpha(s)$ for all $s \in \mathbb{R}$. In this context, the goal of this section is to find spectral information about the spectrum of $-\Delta_n^\varepsilon$ for each $n \in \mathbb{N}$, namely, we study the absolutely continuous spectrum $\sigma_{\rm ac}(-\Delta_n^\varepsilon)$ and the existence and location of band gaps in $\sigma(-\Delta_n^\varepsilon)$.

4.1. Preliminary results. Due to the periodic characteristics of $-\Delta_n^\varepsilon$, to prove Theorems 1.2 and 1.3, we shall use the Floquet-Bloch reduction under the Brillouin zone $\mathcal{C} = [-\pi/L, \pi/L)$. More precisely, define $Q_L := (0, L) \times S$, $\mathcal{L}_n^L := \{w(s)u_n(y) : w \in L^2(0, L)\}$, $n \in \mathbb{N}$,

$$\mathcal{H}_n^L := \begin{cases} L^2(Q_L) & n = 1, \\ (\mathcal{L}_1^L \oplus \mathcal{L}_2^L \oplus \cdots \oplus \mathcal{L}_{n-1}^L)^{\perp} & n = 2, 3, \dots. \end{cases}$$

Consider the family of quadratic forms acting in $\mathcal{H}_n^L$:
(4.1)
$$\widehat{b}_n^\varepsilon(\theta)(\varphi) = \int_{Q_L} \left(|\varphi' + i\theta\varphi + \langle \nabla_y \varphi, Ry \rangle \alpha'(s)|^2 + \frac{|\nabla_y \varphi|^2}{\varepsilon^2} \right) \mathrm{d}s\, \mathrm{d}y, \quad \theta \in \mathcal{C},$$

$\mathrm{dom}\, \widehat{b}_n^\varepsilon(\theta) = \{\varphi \in H^1(Q_L) \cap \mathcal{H}_n^L; \varphi(0, \cdot) = \varphi(L, \cdot) \text{ in } L^2(S)\}$. Denote by $-\Delta_n^\varepsilon(\theta)$ the self-adjoint operator associated with $\widehat{b}_n^\varepsilon(\theta)$.

Lemma 4.1. *For each $n \in \mathbb{N}$, $\{-\Delta_n^\varepsilon(\theta), \theta \in \mathcal{C}\}$ is an analytic family of type* (B).

Proof. First, note that $\operatorname{dom} \widehat{b}_n^\varepsilon(\theta)$ does not depend upon θ. For each $\theta \in \mathcal{C}$, write $\widehat{b}_n^\varepsilon(\theta) = \widehat{b}_n^\varepsilon(0) + c_n^\varepsilon(\theta)$, where, for $\varphi \in \operatorname{dom} \widehat{b}_n^\varepsilon(0)$,

$$\begin{aligned} c_n^\varepsilon(\theta)(\varphi) &:= \widehat{b}_n^\varepsilon(\theta)(\varphi) - \widehat{b}_n^\varepsilon(0)(\varphi) \\ &= 2\operatorname{Re}\left(\int_{Q_L} \overline{(\varphi' + \langle \nabla_y \varphi, R_y\rangle \alpha'(s))}(i\theta\varphi)\, \mathrm{d}s\, \mathrm{d}y\right) \\ &\quad + \theta^2 \int_{Q_L} |\varphi|^2 \mathrm{d}s\, \mathrm{d}y. \end{aligned}$$

We affirm that $c_n^\varepsilon(\theta)$ is $\widehat{b}_n^\varepsilon(0)$-bounded with zero relative bound. In fact, given $\delta > 0$,

$$\begin{aligned} |c_n^\varepsilon(\theta)(\varphi)| &\le 2\int_{Q_L} |\varphi' + \langle \nabla_y \varphi, Ry\rangle \alpha'(s)|\, |i\theta\varphi|\, \mathrm{d}s\, \mathrm{d}y + \theta^2 \int_{Q_L} |\varphi|^2 \mathrm{d}s\, \mathrm{d}y \\ &\le \delta \int_{Q_L} |\varphi' + \langle \nabla_y \varphi, Ry\rangle \alpha'(s)|^2 \mathrm{d}s\, \mathrm{d}y + \theta^2(1/\delta + 1)\int_{Q_L} |\varphi|^2 \mathrm{d}s\, \mathrm{d}y \\ &\le \delta\, \widehat{b}_n^\varepsilon(0)(\varphi) + (\pi/L)^2(1/\delta + 1)\|\varphi\|^2_{\mathcal{H}_n^L}, \end{aligned}$$

for all $\varphi \in \operatorname{dom} \widehat{b}_n^\varepsilon(0)$, for all $\theta \in \mathcal{C}$. Since $\delta > 0$ is arbitrary, the affirmation is proven. By **[17,** Chapter 7, Theorem 4.8], $\{\widehat{b}_n^\varepsilon(\theta) : \theta \in \mathcal{C}\}$ is an analytic family of type (A). Consequently, $\{-\Delta_n^\varepsilon(\theta), \theta \in \mathcal{C}\}$ is an analytic family of type (B). $\square$

Lemma 4.2. *There exists a unitary operator $\mathcal{U}_n : \mathcal{H}_n \to \int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L \, \mathrm{d}\theta$ such that*:

$$\mathcal{U}_n(-\Delta_n^\varepsilon)\,\mathcal{U}_n^{-1} = \int_{\mathcal{C}}^{\oplus} -\Delta_n^\varepsilon(\theta)\, \mathrm{d}\theta.$$

Proof. For $(\theta, s, y) \in \mathcal{C} \times Q_L$, define

$$(\mathcal{U}_n f)(\theta, s, y) := \sum_{k\in\mathbb{Z}} \sqrt{\frac{L}{2\pi}} e^{-ikL\theta - i\theta s} f(s + Lk, y), \quad \operatorname{dom} \mathcal{U}_n = \mathcal{H}_n,$$

which is a unitary operator onto $\int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L \, \mathrm{d}\theta$; the definition of $\mathcal{U}_n$ is based on **[2, 21]** .

Recall the quadratic form $\overline{b}_n^\varepsilon$, see (1.6). Consider

$$q_n^\varepsilon(\varphi) := \overline{b}_n^\varepsilon(\mathcal{U}_n^{-1}\varphi), \qquad \operatorname{dom} q_n^\varepsilon := \mathcal{U}_n(\operatorname{dom} \overline{b}_n^\varepsilon).$$

Note that q_n^ε is a closed, bounded from below, quadratic form in the Hilbert space $\int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L \, \mathrm{d}\theta$, and $\mathcal{U}_n(-\Delta_n^\varepsilon)\mathcal{U}_n^{-1}$ is the self-adjoint operator associated with it.

For $(s,y) \in Q_L$ and $k \in \mathbb{Z}$,

$$(\mathcal{U}_n^{-1}\varphi)(s+Lk,y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} \varphi(\theta,s,y)\, \mathrm{d}\theta,$$

$$(\mathcal{U}_n^{-1}\varphi)'(s+Lk,y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} (\varphi'(\theta,s,y) + i\theta\varphi(\theta,s,y))\, \mathrm{d}\theta,$$

and

$$\nabla_y(\mathcal{U}_n^{-1}\varphi)(s+Lk,y) = \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta + is\theta} \nabla_y\varphi(\theta,s,y)\, \mathrm{d}\theta.$$

Since α' is an L-periodic function, by Parseval's identity and Fubini's theorem, we have:

$$\begin{aligned}
q_n^\varepsilon(\varphi) &= \overline{b}_n^\varepsilon(\mathcal{U}_n^{-1}\varphi) \\
&= \int_Q \left(|(\mathcal{U}_n^{-1}\varphi)' + \langle \nabla_y(\mathcal{U}_n^{-1}\varphi), Ry\rangle \alpha'(s)|^2 + \frac{|\nabla_y(\mathcal{U}_n^{-1}\varphi)|^2}{\varepsilon^2} \right) \mathrm{d}s\, \mathrm{d}y \\
&= \sum_{k\in\mathbb{Z}} \int_{Q_L} |(\mathcal{U}_n^{-1}\varphi)'(s+Lk,y) + \langle \nabla_y(\mathcal{U}_n^{-1}\varphi)(s+Lk,y), Ry\rangle \alpha'(s)|^2 \mathrm{d}s\, \mathrm{d}y \\
&\quad + \sum_{k\in\mathbb{Z}} \int_{Q_L} \frac{1}{\varepsilon^2} |\nabla_y(\mathcal{U}_n^{-1}\varphi)(s+Lk,y)|^2 \mathrm{d}s\, \mathrm{d}y \\
&= \int_{Q_L} \sum_{k\in\mathbb{Z}} \left| \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta+is\theta} (\varphi'(\theta,s,y) + i\theta\varphi(\theta,s,y) \right. \\
&\qquad\qquad \left. + \langle \nabla_y\varphi(\theta,s,y), Ry\rangle \alpha'(s))\, \mathrm{d}\theta \right|^2 \mathrm{d}s\, \mathrm{d}y \\
&\quad + \int_{Q_L} \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon^2} \left| \int_{\mathcal{C}} \sqrt{\frac{L}{2\pi}} e^{ikL\theta+is\theta} \nabla_y\varphi(\theta,s,y)\, \mathrm{d}\theta \right|^2 \mathrm{d}s\, \mathrm{d}y
\end{aligned}$$

$$= \int_{Q_L}\left(\int_{\mathcal{C}} |(\varphi'(\theta,s,y)+i\theta\varphi(\theta,s,y)+\langle\nabla_y\varphi(\theta,s,y), Ry\rangle\alpha'(s))|^2 \mathrm{d}\theta\right) \mathrm{d}s\, \mathrm{d}y$$

$$+ \int_{Q_L}\left(\int_{\mathcal{C}} \frac{1}{\varepsilon^2}|\nabla_y\varphi(\theta, s, y)\, d\theta|^2 \mathrm{d}\theta\right) \mathrm{d}s\, \mathrm{d}y$$

$$= \int_{\mathcal{C}}\left(\int_{Q_L} |(\varphi'(\theta,s,y)+ i\theta\varphi(\theta,s,y)+\langle\nabla_y\varphi(\theta,s,y), Ry\rangle\alpha'(s))|^2 \mathrm{d}s\mathrm{d}y\right) \mathrm{d}\theta$$

$$+ \int_{\mathcal{C}}\left(\int_{Q_L} \frac{1}{\varepsilon^2}|\nabla_y\varphi(\theta, s, y)\, d\theta|^2 \mathrm{d}s\, \mathrm{d}y\right) \mathrm{d}\theta$$

$$=: \int_{\mathcal{C}} \widehat{b}_n^\varepsilon(\theta)(\varphi(\theta))\, \mathrm{d}\theta.$$

Then, $\varphi \in \operatorname{dom} q_n^\varepsilon$ if, and only if, $\varphi \in \int_{\mathcal{C}}^{\oplus} \mathcal{H}_n^L \mathrm{d}\theta$ and $\varphi(\theta) \in \operatorname{dom} \widehat{b}_n^\varepsilon(\theta)$, almost everywhere θ.

Now, consider the self-adjoint operator

$$Q_n^\varepsilon := \int_{\mathcal{C}}^{\oplus} -\Delta_n^\varepsilon(\theta)\, d\theta,$$

where

$$\operatorname{dom} Q_n^\varepsilon := \Big\{ \varphi : \varphi(\theta) \in \operatorname{dom}(-\Delta_n^\varepsilon(\theta)) \text{ almost everywhere } \theta; \\ \int_{\mathcal{C}} \| -\Delta_n^\varepsilon(\theta)\varphi(\theta)\|^2_{\mathcal{H}_n^L} \mathrm{d}\theta < +\infty \Big\}.$$

For each $\varphi \in \operatorname{dom} q_n^\varepsilon$ and $\eta \in \operatorname{dom} Q_n^\varepsilon$,

$$q_n^\varepsilon(\varphi,\eta) = \int_{\mathcal{C}} \widehat{b}_n^\varepsilon(\theta)(\varphi(\theta),\eta(\theta))\, d\theta = \int_{\mathcal{C}} \langle\varphi(\theta), -\Delta_n^\varepsilon(\theta)\eta(\theta)\rangle_{\mathcal{H}_n^L} d\theta$$
$$= \int_{\mathcal{C}} \langle\varphi(\theta), (Q_n^\varepsilon\eta)(\theta)\rangle_{\mathcal{H}_n^L} d\theta = \langle\varphi, Q_n^\varepsilon\eta\rangle .$$

Therefore, Q_n^ε is the self-adjoint operator associated with q_n^ε and, by uniqueness, $Q_n^\varepsilon = \mathcal{U}_n(-\Delta_n^\varepsilon)\,\mathcal{U}_n^{-1}$. □

4.2. Proof of Theorem 1.2. Since each $-\Delta_n^\varepsilon(\theta)$ is compact resolvent and lower bounded, its spectrum is discrete. We denote by $E_{n,j}(\varepsilon,\theta)$ the jth eigenvalue of $-\Delta_n^\varepsilon(\theta)$, counted with multiplicity, and by $\psi_{n,j}(\varepsilon,\theta)$ the corresponding normalized eigenfunction, i.e.,

$$-\Delta_n^\varepsilon(\theta)\psi_{n,j}(\varepsilon,\theta) = E_{n,j}(\varepsilon,\theta)\psi_{n,j}(\varepsilon,\theta), \quad j = 1,2,3,\ldots, \ \theta \in \mathcal{C}.$$

We have

$$E_{n,1}(\varepsilon,\theta) \le E_{n,2}(\varepsilon,\theta) \le \cdots \le E_{n,j}(\varepsilon,\theta)\ldots, \quad \theta \in \mathcal{C},$$

$$\sigma(-\Delta_n^\varepsilon) = \bigcup_{j=1}^{\infty}\{E_{n,j}(\varepsilon,\mathcal{C})\},$$

where

$$E_{n,j}(\varepsilon,\mathcal{C}) := \{E_{n,j}(\varepsilon,\theta) : \theta \in \mathcal{C}\};$$

each $E_{n,j}(\varepsilon,\mathcal{C})$ is called the jth band of $\sigma(-\Delta_n^\varepsilon)$.

Lemma 4.1 ensures that the functions $E_{n,j}(\varepsilon,\theta)$ are continuous and piecewise analytic in $\mathcal{C}$; consequently, each $E_{n,j}(\varepsilon,\mathcal{C})$ is either a closed interval or a one point set. The goal is to find an asymptotic behavior for the eigenvalues $E_{n,j}(\varepsilon,\theta)$, as $\varepsilon \to 0$.

Based on the discussion in Section 1, now we study the sequence:

$$b_n^\varepsilon(\theta)(\psi) := \widehat{b}_n^\varepsilon(\theta)(\psi) - \frac{\lambda_n}{\varepsilon^2}\|\psi\|^2_{\mathcal{H}_n^L}, \tag{4.2}$$

$\operatorname{dom} b_n^\varepsilon(\theta) := \operatorname{dom} \widehat{b}_n^\varepsilon(\theta)$. The self-adjoint operator associated with $b_n^\varepsilon(\theta)$ is $T_n^\varepsilon(\theta) := -\Delta_n^\varepsilon(\theta) - (\lambda_n/\varepsilon^2)\mathbf{1}$. We define the one-dimensional quadratic form

$$\begin{aligned} b_n(\theta)(w) &:= b_n^\varepsilon(\theta)(wu_n) \\ &= \int_{Q_L} |w'u_n + i\theta w u_n + \langle \nabla_y u_n, Ry\rangle \alpha'(s) w|^2 \mathrm{d}s\,\mathrm{d}y, \end{aligned}$$

$\operatorname{dom} b_n(\theta) := \{w \in H^1(0,L) : w(0) = w(L)\}$ and denote by $T_n(\theta)$ the self-adjoint operator associated with it, namely,

$$T_n(\theta)w := (-i\partial_s + \theta)^2 w + V_n w,$$

$\operatorname{dom} T_n(\theta) = \{w \in H^2(0,L) : w(0) = w(L), w'(0) = w'(L)\}$, where V_n is defined by (1.10).

Theorem 4.3. *For each $n \in \mathbb{N}$ and each $\theta \in \mathcal{C}$ fixed, the sequence of self-adjoint operators $(T_n^\varepsilon(\theta))_\varepsilon$ converges in the norm resolvent sense to $T_n(\theta)$ in $\mathcal{L}_n^L$, as $\varepsilon \to 0$. Furthermore, for $n \in \mathbb{N}$, $j \in \mathbb{N}$ and $\theta \in \mathcal{C}$ fixed, we have*

$$\lim_{\varepsilon \to 0} \left(E_{n,j}(\varepsilon, \theta) - \frac{\lambda_n}{\varepsilon^2} \right) = k_{n,j}(\theta).$$

The proof of Theorem 4.3 is very similar to the proof of Theorem 1.1; it will be omitted here.

Denote by $k_{n,j}(\theta)$ the jth eigenvalue (counted with multiplicity) of $T_n(\theta)$. As a consequence of Theorem 4.3, we have

Corollary 4.4. *For each $n \in \mathbb{N}$ and each $j \in \mathbb{N}$ fixed, we have*

$$\lim_{\varepsilon \to 0} \left(E_{n,j}(\varepsilon, \theta) - \frac{\lambda_n}{\varepsilon^2} \right) = k_{n,j}(\theta) \tag{4.3}$$

uniformly in $\mathcal{C}$.

Proof. For $n \in \mathbb{N}$ fixed, extend $b_n^\varepsilon(\theta)$ by formulas (4.1) and (4.2) for all $\theta \in \overline{\mathcal{C}}$. Theorem 4.3 holds if we consider $\overline{\mathcal{C}}$ instead of $\mathcal{C}$. Then, (4.3) holds for each $j \in \mathbb{N}$ and $\theta \in \overline{\mathcal{C}}$. On the other hand, if $\varepsilon_1 < \varepsilon_2$, then $b_n^{\varepsilon_2}(\theta)(\psi) \le b_n^{\varepsilon_1}(\theta)(\psi)$ for all $\psi \in \operatorname{dom} b_n^\varepsilon(\theta)$, and for all $\theta \in \overline{\mathcal{C}}$. Thus, for each $j \in \mathbb{N}$ and $\theta \in \overline{\mathcal{C}}$, the sequence $(E_{n,j}(\varepsilon, \theta) - \lambda_n/\varepsilon^2)$ decreases in ε. Now, the result follows by Dini's theorem. $\square$

Proof of Theorem 1.2. Let $E > 0$ and, without loss of generality, we can suppose that, for all $\theta \in \mathcal{C}$, the spectrum of $-\Delta_n^\varepsilon(\theta)$ below $E + \lambda_n/\varepsilon^2$ consists of exactly j_0 eigenvalues $\{E_{n,j}(\varepsilon, \theta)\}_{j=1}^{j_0}$. Lemma 4.1 ensures that there exists a finite partition $\mathcal{P}$ of $\mathcal{C}$ so that the functions $\{E_{n,j}(\varepsilon, \theta)\}_{j=1}^{j_0}$ are analytic in each of its intervals.

The functions $k_{n,j}(\theta)$ are nonconstant by **[21,** Theorem 13]. By Corollary 4.4, there exist $\varepsilon_E > 0$, $K(\varepsilon) > 0$, so that $|E_{n,j}(\varepsilon, \theta) - (\lambda_n/\varepsilon^2) - \kappa_{n,j}(\theta)| < K(\varepsilon)$ for all $\theta \in \mathcal{C}$, $\varepsilon \in (0, \varepsilon_E)$, $j = 1, 2, \ldots, j_0$, and $K(\varepsilon) \to 0$ as $\varepsilon \to 0$. Consequently, the functions $E_{n,j}(\varepsilon, \theta)$ are nonconstant in each interval of partition $\mathcal{P}$. Note that $\varepsilon_E > 0$ depends on j_0, i.e., the thickness of the tube depends upon the length of the energies to be covered. Now, by **[21,** Section 13.16], the conclusion follows. $\square$

4.3. Existence of band gaps. In this section, we shall prove Theorem 1.3. Consider the one-dimensional operator

$$\widetilde{T}_n w := -w'' + V_n w, \qquad \operatorname{dom} \widetilde{T}_n = H^2(\mathbb{R}).$$

We have denoted by $k_{n,j}(\theta)$ the jth eigenvalue (counted with multiplicity) of the operator $T_n(\theta)$. For each $j \in \mathbb{N}$, $k_{n,j}(\theta)$ is a continuous and piecewise analytic function in $\mathcal{C}$. From [**21,** Chapter 13.16], we have the following properties:

(a) $k_{n,j}(\theta) = k_{n,j}(-\theta)$ for all $\theta \in \mathcal{C}$, $j = 1, 2, 3, \ldots$.

(b) For j odd, respectively even, $k_{n,j}(\theta)$ is strictly monotone increasing, respectively decreasing, as θ increases from 0 to π/L. In particular,

$$\begin{aligned} k_{n,1}(0) < k_{n,1}(\pi/L) \le k_{n,2}(\pi/L) < k_{n,2}(0) \le \cdots \\ \le k_{n,2j-1}(0) < k_{n,2j-1}(\pi/L) \\ \le k_{n,2j}(\pi/L) < k_{n,2j}(0) \le \cdots . \end{aligned}$$

For each $j \in \mathbb{N}$, define

$$B_{n,j} := \begin{cases} [k_{n,j}(0), k_{n,j}(\pi/L)] & \text{for } j \text{ odd}, \\ [k_{n,j}(\pi/L), k_{n,j}(0)] & \text{for } j \text{ even}, \end{cases}$$

and

$$G_{n,j} := \begin{cases} (k_{n,j}(\pi/L), k_{n,j+1}(\pi/L)) & \text{for } j \text{ odd} \\ & \text{so that } k_{n,j}(\pi/L) \neq k_{n,j+1}(\pi/L), \\ (k_{n,j}(0), k_{n,j+1}(0)) & \text{for } j \text{ even} \\ & \text{so that } k_{n,j}(0) \neq k_{n,j+1}(0), \\ \emptyset & \text{otherwise}. \end{cases}$$

Then, by [**21,** Theorem 13.90], we have $\sigma(\widetilde{T}_n) = \cup_{j=1}^{\infty} B_{n,j}$, where $B_{n,j}$ is called the jth band of $\sigma(\widetilde{T}_n)$. If $G_{n,j} \neq \emptyset$, $G_{n,j}$ is called the gap of $\sigma(\widetilde{T}_n)$.

By Corollary 4.4 and, since $E_{n,j}(\varepsilon, \theta)$ is a decreasing sequence, for each $j \in \mathbb{N}$, $\varepsilon > 0$,

$$\max_{\theta \in \mathcal{C}} E_{n,j}(\varepsilon, \theta) \le \begin{cases} \lambda_n/\varepsilon^2 + k_{n,j}(\pi/L) & \text{for } j \text{ odd}, \\ \lambda_n/\varepsilon^2 + k_{n,j}(0) & \text{for } j \text{ even}. \end{cases}$$

If $G_{n,j} \neq \emptyset$, again by Corollary 4.4, there exists an $\varepsilon_j > 0$ so that, for all $\varepsilon \in (0, \varepsilon_j)$,

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\varepsilon, \theta) \geq \begin{cases} \lambda_n/\varepsilon^2 + k_{n,j+1}(\pi/L) - |G_{n,j}|/2 & \text{for } j \text{ odd}, \\ \lambda_n/\varepsilon^2 + k_{n,j+1}(0) - |G_{n,j}|/2 & \text{for } j \text{ even}, \end{cases}$$

where $|\cdot|$ denotes the Lebesgue measure. Thus, we have:

Corollary 4.5. *If $G_{n,j} \neq \emptyset$, there exists an $\varepsilon_j > 0$ so that, for all $\varepsilon \in (0, \varepsilon_j)$,*

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\varepsilon, \theta) \geq \frac{1}{2}|G_{n,j}|.$$

Another important tool used to prove Theorem 1.3 is the following result due to Borg [**3**].

Theorem 4.6 ([**3**]). *Suppose that W is a real-valued, piecewise, continuous function on $[0, L]$. Let $\mu_j^{\pm}$ be the jth eigenvalue of the following multiplicity counted operator*

$$T^{\pm} := -\frac{d^2}{ds^2} + W(s) \quad \text{in } L^2(0, L),$$

with domain

$$\{w \in H^2(0, L) : w(0) = \pm w(L), w'(0) = \pm w'(L)\}.$$

We suppose that

$$\mu_j^+ = \mu_{j+1}^+ \quad \text{for all even } j,$$

and

$$\mu_j^- = \mu_{j+1}^- \quad \text{for all odd } j.$$

Then, W is constant on $[0, L]$.

Proof of Theorem 1.3. Take $W(s) = V_n(s)$ in Theorem 4.6. The operator $T_n(0)$, respectively $T_n(\pi/L)$, is unitarily equivalent to T^+, respectively T^-; in fact, merely consider the unitary operator $(u_\theta w)(s) := e^{-i\theta s} w(s)$ with $\theta = 0$, respectively $\theta = \pi/L$. Recall that $\{k_{n,j}(0)\}_{j \in \mathbb{N}}$, respectively $\{k_{n,j}(\pi/L)\}_{j \in \mathbb{N}}$, are the eigenvalues of $T_n(0)$, respectively $T_n(\pi/L)$.

Since $V_n(s)$ is not a constant function in $[0, L]$, by Borg's theorem, without loss of generality, we affirm that there exists a $j \in \mathbb{N}$ so that $k_{n,j}(0) \neq k_{n,j+1}(0)$. Now, the result follows from Corollary 4.5. $\square$

4.4. Location of band gaps. In this section, we find a location in $\sigma(-\Delta_n^\varepsilon)$ where Theorem 1.3 holds. For this purpose, we use the scaling

$$\alpha \longmapsto \gamma\alpha, \tag{4.4}$$

where $\gamma > 0$ is a small parameter. Thus, we obtain the waveguide $\Omega^\alpha_{\varepsilon,\gamma} := \Omega^{\gamma\alpha}_\varepsilon$. Consider $-\Delta^N_{\Omega^\alpha_{\varepsilon,\gamma}}$ instead of $-\Delta^N_{\Omega^\alpha_\varepsilon}$. Denote by $\overline{b}_n^{\varepsilon,\gamma}$ and $\widehat{b}_n^{\varepsilon,\gamma}(\theta)$, the quadratic forms obtained by replacing (4.4) in (1.6) and (4.1), respectively. The self-adjoint operators associated with these quadratic forms are denoted by $-\Delta_n^{\varepsilon,\gamma}$ and $-\Delta_n^{\varepsilon,\gamma}(\theta)$, respectively. Denote by $E_{n,j}(\gamma, \varepsilon, \theta)$ the jth eigenvalue of $-\Delta_n^{\varepsilon,\gamma}(\theta)$ counted with multiplicity. Define $W_n(s) := C_n^1(S)(\alpha'(s))^2$. Write $W_n(s)$ as a Fourier series, i.e.,

$$W_n(s) = \sum_{j=-\infty}^{+\infty} \frac{1}{\sqrt{L}} w_n^j e^{2\pi j i s/L} \quad \text{in } L^2[0, L].$$

The sequence $\{w_n^j\}_{j=-\infty}^{\infty}$ is called the Fourier coefficients of W_n. Since W_n is a real function, $w_n^j = \overline{w_n^{-j}}$ for all $j \in \mathbb{Z}$. We have:

Theorem 4.7. *Suppose that $V_n(s)$ is not a constant function in $[0, L]$ and $W_n(s)$ is non null. Let $j \in \mathbb{N}$ so that $w_n^j \neq 0$. Then, there exists a $\gamma > 0$ small enough, $\varepsilon_{n,j+1} > 0$ and $C_{n,j}(\gamma) > 0$ so that, for all $\varepsilon \in (0, \varepsilon_{n,j+1})$,*

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\gamma, \varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\gamma, \varepsilon, \theta) \geq C_{n,j}(\gamma).$$

In order to prove Theorem 4.7, we shall use a strategy adopted in **[28]**. Some steps will be omitted here, and a more complete proof can be found in that work. In addition, our problem requires further adjustments, which will be explained next.

Technical details. Let $W \in L^2[0, L]$ be a real function. For $\beta \in \mathbb{C}$, consider the operators

$$T_\beta^+ w = -w'' + \beta W(s) w \quad \text{and} \quad T_\beta^- w = -w'' + \beta W(s) w,$$

with domains given by

$$\operatorname{dom} T_\beta^+ = \{w \in H^2(0,L) : w(0) = w(L), w'(0) = w'(L)\}, \tag{4.5}$$

$$\operatorname{dom} T_\beta^- = \{w \in H^2(0,L) : w(0) = -w(L), w'(0) = -w'(L)\}, \tag{4.6}$$

respectively. Denote by $\{l_j^+(\beta)\}_{j\in\mathbb{N}}$ and $\{l_j^-(\beta)\}_{j\in\mathbb{N}}$ the eigenvalues of T_β^+ and T_β^-, respectively. For $\beta \in \mathbb{R}$ and $j \in \mathbb{N}$, define

$$\delta_j^+(\beta) := l_{2j+1}^+(\beta) - l_{2j}^+(\beta) \quad \text{and} \quad \delta_j^-(\beta) := l_{2j}^-(\beta) - l_{2j-1}^-(\beta).$$

Now,

$$\delta_{2j-1}(\beta) := \delta_j^-(\beta) \quad \text{and} \quad \delta_{2j}(\beta) := \delta_j^+(\beta).$$

Let $\{w^j\}_{j=-\infty}^{+\infty}$ be the Fourier coefficients of W:

$$W(s) = \sum_{j=-\infty}^{+\infty} \frac{1}{\sqrt{L}} w^j e^{2\pi j i s/L} \quad \text{in } L^2[0,L],$$

where $w^j = \overline{w^{-j}}$ for all $j \in \mathbb{Z}$.

The next theorem gives asymptotic behavior for $\delta_j(\beta)$, as $\beta \to 0$, in terms of the Fourier coefficients of W.

Theorem 4.8. *For each $j \in \mathbb{N}$,*

$$\delta_j(\beta) = \frac{2}{\sqrt{L}} |w^j|\, |\beta| + O(|\beta|^2), \quad \beta \to 0,\ \beta \in \mathbb{R}.$$

A detailed proof of Theorem 4.8 may be found in **[28]**.

Auxiliary problem. For each $\gamma > 0$ and $\theta \in \mathcal{C}$, consider the one-dimensional quadratic form

$$s_n^\gamma(\theta)(w) := \int_0^L (|w' + i\theta w|^2 + \gamma^2 W_n(s)|w|^2)\, \mathrm{d}s,$$

$\operatorname{dom} s_n^\gamma(\theta) := \{w \in H^1(0,L) : w(0) = w(L)\}$. The self-adjoint operator associated with $s_n^\gamma(\theta)$ is given by

$$S_n^\gamma(\theta)w := (-i\partial_s + \theta)^2 w + \gamma^2 W_n(s) w,$$

$\operatorname{dom} S_n^\gamma(\theta) := \{w \in H^2(0,L) : w(0) = w(L), w'(0) = w'(L)\}$. Denote by $\nu_{n,j}(\gamma,\theta)$ the jth eigenvalue of $S_n^\gamma(\theta)$ counted with multiplicity.

Now, consider

$$b_n^\gamma(\theta)(w) := b_n^{\varepsilon,\gamma}(\theta)(wu_n) = \int_0^L (|w' + i\theta w|^2 + V_n^\gamma(s)|w|^2)\,\mathrm{d}s,$$

$\operatorname{dom} b_n^\gamma(\theta) := \{w \in H^1(0,L) : w(0) = w(L)\}$, where $V_n^\gamma(s) := \gamma^2 W_n(s) - \gamma C_n^2(S)\alpha''(s)$. The self-adjoint operator associated with $b_n^\gamma(\theta)$ is

$$T_n^\gamma(\theta)w := (-i\partial_s + \theta)^2 w + V_n^\gamma(s)w,$$

$\operatorname{dom} T_n^\gamma(\theta) := \{w \in H^2(0,L) : w(0) = w(L), w'(0) = w'(L)\}$. Denote by $k_{n,j}(\gamma,\theta)$ the jth eigenvalue of $T_n^\gamma(\theta)$ counted with multiplicity. Take $c > \max\{\|V_n\|_\infty, \|W_n\|_\infty\}$. Straightforward calculations show that $K > 0$ exists so that

$$|(b_n^\gamma(\theta) + c)(w) - (s_n^\gamma(\theta) + c)(w)| \le K\,\gamma\,|(b_n^\gamma(\theta) + c)(w)|, \quad \text{for all } w \in \operatorname{dom} b_n^\gamma(\theta), \tag{4.7}$$

$\theta \in \mathcal{C}$ and $\gamma > 0$ small enough.

Inequality (4.7), [**1,** Theorem 2] and [**13,** Corollary 2.3] imply the following.

Corollary 4.9. *For each $j \in \mathbb{N}$, there exists a $\gamma_j > 0$ so that, for all $\gamma \in (0, \gamma_j)$,*

$$k_{n,j}(\gamma,\theta) = \nu_{n,j}(\gamma,\theta) + O(\gamma),$$

uniformly in $\mathcal{C}$.

Estimates I. We define

$$G_{n,j}(\gamma) := \begin{cases} (k_{n,j}(\gamma,\pi/L), k_{n,j+1}(\gamma,\pi/L)) & \text{for } j \text{ odd so that } k_{n,j}(\gamma,\pi/L) \neq k_{n,j+1}(\gamma,\pi/L), \\ (k_{n,j}(\gamma,0), k_{n,j+1}(\gamma,0)) & \text{for } j \text{ even so that } k_{n,j}(\gamma,0) \neq k_{n,j+1}(\gamma,0), \\ \emptyset & \text{otherwise,} \end{cases}$$

namely, if $G_{n,j}(\gamma) \neq \emptyset$, it is called the gap of the spectrum $\sigma(T_n^\gamma)$, where

$$T_n^\gamma w := -w'' + V_n^\gamma(s)w, \quad \operatorname{dom} T_n^\gamma = H^2(\mathbb{R}).$$

Similarly to the considerations of subsection 4.3 and Corollary 4.5, we have:

Corollary 4.10. *If $G_{n,j}(\gamma) \neq \emptyset$, there exist $\gamma_j > 0$ and $\varepsilon_j > 0$ so that, for all $\gamma \in (0, \gamma_j)$ and $\varepsilon \in (0, \varepsilon_j)$,*

$$\min_{\theta \in \mathcal{C}} E_{n,j+1}(\gamma, \varepsilon, \theta) - \max_{\theta \in \mathcal{C}} E_{n,j}(\gamma, \varepsilon, \theta) \geq \frac{1}{2} |G_{n,j}(\gamma)|.$$

Estimates II. Now, consider

$$\widetilde{G}_{n,j}(\gamma) := \begin{cases} (\nu_{n,j}(\gamma, \pi/L), \nu_{n,j+1}(\gamma, \pi/L)) & \text{for } j \text{ odd so that} \\ & \quad \nu_{n,j}(\gamma, \pi/L) \neq \nu_{n,j+1}(\gamma, \pi/L), \\ (\nu_{n,j}(\gamma, 0), \nu_{n,j+1}(\gamma, 0)) & \text{for } j \text{ even so that} \\ & \quad \nu_{n,j}(\gamma, 0) \neq \nu_{n,j+1}(\gamma, 0), \\ \emptyset & \text{otherwise;} \end{cases}$$

if $\widetilde{G}_{n,j}(\gamma) \neq \emptyset$, it is called the gap of $\sigma(S_n^\gamma)$, where

$$S_n^\gamma w := -w'' + \gamma^2 W_n(s) w, \quad \operatorname{dom} S_n^\gamma = H^2(\mathbb{R}).$$

Similarly to the proof of Theorem 1.3, consider the unitary operator $(u_\theta w)(s) = e^{-i\theta s} w(s)$. We define the self-adjoint operators $\widetilde{S}_n^\gamma(0) := u_0 S_n^\gamma(0) u_0^{-1}$ and $\widetilde{S}_n^\gamma(\pi/L) := u_{\pi/L} S_n^\gamma(\pi/L) u_{\pi/L}^{-1}$, whose eigenvalues are given by $\{\nu_{n,j}(\gamma, 0)\}_{j \in \mathbb{N}}$ and $\{\nu_{n,j}(\gamma, \pi/L)\}_{j \in \mathbb{N}}$, respectively. Furthermore, the domains of these operators are given by (4.5) and (4.6), respectively. Thus, we can see that $|\widetilde{G}_{n,j}(\gamma)| = \delta_j(\gamma^2)$ for all $j \in \mathbb{N}$, if we consider $\beta = \gamma^2$ and $W(s) = W_n(s)$ in Theorem 4.8.

With the previous information at hand, we now have the necessary conditions to prove the main theorem of this section.

Proof of Theorem 4.7. Recall that we have denoted the Fourier coefficients of W_n by $\{w_n^j\}_{j=-\infty}^{+\infty}$. Since W_n is not a constant function in $[0, L]$, there exists a $j \in \mathbb{N}$ so that $w_n^j \neq 0$. From Theorem 4.8,

$$|\widetilde{G}_{n,j}(\gamma)| = \frac{2}{\sqrt{L}} \gamma^2 |w_n^j| + O(\gamma^4), \quad \gamma \to 0.$$

This estimate and Corollary 4.9 imply that $|G_{n,j}(\gamma)| > 0$, for all $\gamma > 0$ small enough. By Corollary 4.10, Theorem 4.7 is proven by taking $C_{n,j}(\gamma) := |G_{n,j}(\gamma)|/2 > 0$. □

Remark 4.11. Since we suppose that $V_n(s)$ is a non null function in $[0, L]$, if $W_n(s) = 0$ for all $s \in \mathbb{R}$, we can consider $\widetilde{W}_n(s) := C_n^2(S)\alpha''(s)$ instead of $W_n(s)$ in this subsection. All of the previous results also hold in this case; the proofs are similar and will not be presented here.

APPENDIX

A.1. The self-adjoint operator associated with b_n. Recall the quadratic form

$$b_n(w) = \int_Q |w'u_n + \langle \nabla_y u_n, Ry\rangle \alpha'(s) w|^2 \mathrm{d}s,$$

$\operatorname{dom} b_n = H^1(I)$. The goal is to show that the operator T_n defined by (1.9), (1.10), (1.11) and (1.12) is the self-adjoint operator associated with b_n.

Consider the specific case where $I = (a, b)$ is a bounded interval. Some calculations show that

$$\begin{aligned} b_n(w) = &\int_a^b (|w'|^2 + V_n(s)|w|^2)\,\mathrm{d}s \\ &+ C_n^2(S)\alpha'(b)|w(b)|^2 - C_n^2(S)\alpha'(a)|w(a)|^2. \end{aligned}$$

Let $b_n(w, u)$ be the sesquilinear form associated with $b_n(w)$. We have

$$b_n(w, u) = \langle w, T_n u\rangle \quad \text{for all } w \in \operatorname{dom} b_n,\ v \in \operatorname{dom} T_n.$$

Then, T_n is self-adjoint operator associated with b_n. The case $I = \mathbb{R}$ can be proven in a similar way.

A.2. Γ-convergence. Let H be a (real or complex) Hilbert space and $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. The sequence of quadratic functionals $f_\varepsilon : H \to \overline{\mathbb{R}}$ strongly Γ-converges to $f : H \to \overline{\mathbb{R}}$, that is, $f_\varepsilon \stackrel{S\Gamma}{\to} f$, if and only if the following two conditions are satisfied:

(i) for every $v \in H$ and every $v_\varepsilon \to v$ in H, we have

$$\liminf_{\varepsilon \to 0} f_\varepsilon(v_\varepsilon) \geq f(v).$$

(ii) For every $v \in H$, there exists a sequence $v_\varepsilon \to v$ in H such that

$$\lim_{\varepsilon \to 0} f_\varepsilon(v_\varepsilon) = f(v).$$

If the strong convergence $v_\varepsilon \to v$ is replaced by the weak convergence $v_\varepsilon \rightharpoonup v$ in (i) and (ii), then we have a characterization of the weakly Γ-converge, i.e., $f_\varepsilon \overset{W\Gamma}{\to} f$.

The following result can be found in **[7]** wherein the version for real Hilbert spaces is proven; the generalization for complex Hilbert spaces is presented in **[8]**.

Proposition A.1. *Let d_ε and d be positive* (*or uniformly lower bounded*) *closed, sesquilinear forms in the Hilbert space H, and let D_ε and D be the corresponding positive self-adjoint operators. Then, the following statements are equivalent*:

(a) *$d_\varepsilon \overset{S\Gamma}{\to} d$ and, for each $\zeta \in H$, $d(\zeta) \le \liminf_{\varepsilon \to 0} d_\varepsilon(\zeta_\varepsilon)$ for all $\zeta_\varepsilon \to \zeta$ in H.*

(b) *$d_\varepsilon \overset{S\Gamma}{\to} d$ and $d_\varepsilon \overset{W\Gamma}{\to} d$.*

(c) *D_ε converges to D in the strong resolvent sense in $H_0 = \overline{\operatorname{dom} D} \subset H$, that is,*

$$\lim_{\varepsilon \to 0} R_{-\lambda}(D_\varepsilon)\zeta = R_{-\lambda}(D)P\zeta \quad \text{for all } \zeta \in H,\ \lambda > 0,$$

where P is the orthogonal projection onto H_0.

The next result is due to **[8]**.

Proposition A.2. *Let d_ε, $d \ge \beta > -\infty$, be closed, sesquilinear forms, let D_ε, $D \ge \beta\mathbf{1}$, be the corresponding self-adjoint operators, and let $\overline{\operatorname{dom} D} = H_0 \subset H$. Assume that the following three conditions hold*:

(a) *$d_\varepsilon \overset{S\Gamma}{\to} d$ and $d_\varepsilon \overset{W\Gamma}{\to} d$.*

(b) *The resolvent operator $R_{-\lambda}(D)$ is compact in H_0 for some real number $\lambda > |\beta|$.*

(c) *There exists a Hilbert space $\mathcal{K}$, compactly embedded in H so that, if the sequence (ψ_ε) is bounded in H and $(d_\varepsilon(\psi_\varepsilon))$ is also bounded, then (ψ_ε) is a bounded subset of $\mathcal{K}$.*

Then, D_ε converges in the norm resolvent sense to D in H_0 as $\varepsilon \to 0$.

Remark A.3. In both Propositions A.1 and A.2, the domain of D is not required to be dense in H, but it is required that rng$D \subset H_0$; we say that D is self-adjoint in H_0.

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