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FUNDAMENTAL GROUP OF SPACES OF SIMPLE POLYGONS

AHTZIRI GONZÁLEZ

ABSTRACT. The space of shapes of n-gons with marked vertices can be identified with $\mathbb{CP}^{n-2}$. The space of shapes of n-gons without marked vertices is the quotient of $\mathbb{CP}^{n-2}$ by a cyclic group of order n generated by the function which re-enumerates the vertices. In this paper, we prove that the subset corresponding to simple polygons, i.e., without self-intersections, in each case is open and has two homeomorphic, simply connected components.

Let $n \geq 3$ be an integer. Identifying the point $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ with the n-gon whose consecutive vertices are $z_1, z_2, \dots, z_n$, we obtain that $\mathbb{C}^n$ is the set of n-gons with marked vertices contained in $\mathbb{C}$. Consider the equivalence relation given by $(z_1, z_2, \dots, z_n) \sim (w_1, w_2, \dots, w_n)$, if and only if there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $w_i = az_i + b$ for $i = 1, \dots, n$. Note that any two equivalent n-gons in the $(n-1)$-dimensional subspace $V_n = \{(z_1, \dots, z_n) \colon z_1 = 0\}$ differ by a factor of $a \neq 0$. Then, there are canonical biholomorphisms between the complex projectivization of V_n, the complex projective space $\mathbb{CP}^{n-2}$ and the quotient $\mathsf{P}(n) := (\mathbb{C}^n \setminus \{(z, z, \dots, z)\})/{\sim}$, where the complex line $\{(z, z, \dots, z)\}$ corresponds to the zero of V_n. The space $\mathsf{P}(n)$ is called the *space of shapes of n-gons with marked vertices.*

In order to eliminate the marks on the vertices, we consider the action of the shift function $(z_1, z_2, \dots, z_n) \mapsto (z_2, \dots, z_n, z_1)$ in $\mathbb{C}^n$. This linear automorphism defines a biholomorphism $\delta \colon \mathsf{P}(n) \to \mathsf{P}(n)$ (Lemma 4.1). The quotient $\mathcal{P}(n) = \mathsf{P}(n)/\langle \delta \rangle$ is called the *space of shapes of n-gons without marked vertices.*

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Topology and geometry of certain subsets of the space of shapes of n-gons $\mathsf{P}(n)$ have been investigated by different authors. Some of this research includes: the topology of subsets corresponding to n-gons with fixed side lengths is studied in **[6]** (achieving a complete description for the cases $n = 4$, 5 and 6). The structure of the subsets corresponding to n-gons whose sides are parallel to fixed directions has been discussed in **[2]**. In **[7]**, the subset of $\mathsf{P}(5)$ determined by star-shaped pentagons is studied. Subsets corresponding to n-gons obtained as unfoldings of polyhedra with fixed conic angles are studied in **[10]**.

In this paper, we use classical techniques in topology to prove that the subsets of $\mathsf{P}(n)$ and $\mathcal{P}(n)$ corresponding to simple n-gons are open and have two simply connected homeomorphic components. We also provide a description of the local neighborhoods around the regular polygon in $\mathcal{P}(n)$. The topics treated here are divided into four sections: In Section 1, we mention the necessary definitions and remarks as well as the example of triangles. In Section 2, the first topological properties of the space of simple n-gons with marked vertices (Definition 1.1) are proved. In Section 3, it is proven that each component of such a space is simply connected. Finally, in Section 4, results regarding the space of simple n-gons without marked vertices are provided.

1. Preliminaries. Let $Z = (z_1, \ldots, z_n)$ be a point in $\mathbb{C}^n$. We denote by $\mathfrak{c}(Z)$ the set

$$\{\overline{z_1 z_2} \cup \overline{z_2 z_3} \cup \cdots \cup \overline{z_{n-1} z_n} \cup \overline{z_n z_1}\} \subset \mathbb{C},$$

where $\overline{ab}$ is the line segment between the complex numbers a and b.

Definition 1.1 (Simple polygons)**.** Let $Z \in \mathbb{C}^n$.

(1) We say that Z is *simple* if its vertices are pairwise distinct, and $\mathfrak{c}(Z)$ determines a Jordan curve in $\mathbb{C}$. We denote by $S(n) \subset \mathbb{C}^n$ the set of simple n-gons.

(2) If Z is simple, then the interior of Z is the bounded component of $\mathbb{C} \setminus \mathfrak{c}(Z)$ and is denoted by $\mathfrak{int}(Z)$. The closure of Z is the set $\mathfrak{cl}(Z) = \mathfrak{c}(Z) \cup \mathfrak{int}(Z)$.

(3) If Z is simple and $x \in \mathfrak{int}(Z)$, then we say that Z is *positively* (*negatively*) oriented if the winding number around x of $\mathfrak{c}(Z)$ with the orientation determined by the increasing order in the vertices is 1 (-1).

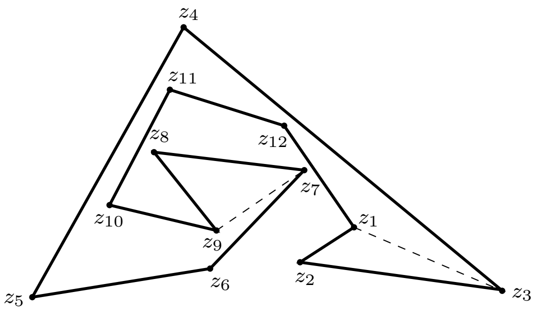

FIGURE 1. A 12-gon where z_2 and z_8 are the only vertices such that their diagonals are internal.

Definition 1.2 (Interior and exterior diagonals)**.** Let $Z \in \mathbb{C}^n$ be a simple n-gon with $n \geq 4$. A *diagonal* of Z is a segment $\overline{z_i z_j}$ with $j \neq i-1,\ i,\ i+1$. The diagonal $\overline{z_{i-1} z_{i+1}}$ is called the *diagonal of the vertex* z_i.

(1) The diagonal $\overline{z_i z_j}$ is interior to Z if $\overline{z_i z_j} \setminus \{z_i, z_j\} \subset \mathfrak{int}(Z)$.

(2) The diagonal $\overline{z_i z_j}$ is exterior to Z if $\overline{z_i z_j} \setminus \{z_i, z_j\} \subset (\mathbb{C} \smallsetminus \mathfrak{cl}(Z))$.

It is well known that, if $n \geq 4$, then every simple n-gon has an interior diagonal. Using this fact and induction over n, it can be proven that the interior of every simple n-gon can be divided, with internal diagonals, into $n-2$ triangles.

Remark 1.3. Let Z be a simple n-gon with $n \geq 4$. If we divide Z with diagonals into $n-2$ triangles, then, by a counting argument, there must be at least two triangles having two edges which also are edges of Z. Therefore, every simple n-gon has at least two non adjacent vertices such that their diagonals are internal (Figure 1).

We denote by $\eta\colon \mathbb{C}^n \setminus \{(z, z, \ldots, z)\} \to \mathsf{P}(n)$ the quotient projection, and by $[z_1, z_2, \ldots, z_n] \in \mathsf{P}(n)$ the shape of the n-gon $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \setminus \{(z, z, \ldots, z)\}$. In addition, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathcal{A}_{\mathbb{C}}$ denotes the complex affine group

$$\{f\colon \mathbb{C} \longrightarrow \mathbb{C}\colon f(z) = az + b,\ a \in \mathbb{C}^*,\ b \in \mathbb{C}\}.$$

Remark 1.4 (Fibration)**.** Note that $\eta^{-1}([z_1, \ldots, z_n]) = \{(az_1 + b, \ldots, az_n + b) \in \mathbb{C}^n\colon a \in \mathbb{C}^*,\ b \in \mathbb{C}\}$. In fact, there is a principal $\mathcal{A}_{\mathbb{C}}$-bundle

$$\mathcal{A}_{\mathbb{C}} \longrightarrow \mathbb{C}^n \setminus \{(z, z, \ldots, z)\} \xrightarrow{\eta} \mathsf{P}(n).$$

This fibration is not trivial since $\pi_1(\mathsf{P}(n)\times\mathcal{A}_{\mathbb{C}}) = \pi_1(\mathsf{P}(n)\times\mathbb{C}\times\mathbb{C}^*) = \mathbb{Z}$ ($\mathcal{A}_{\mathbb{C}}$ is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$); however, $\pi_1(\mathbb{C}^n \setminus \{(z, \ldots, z\}) = 0$ for $n > 2$.

Remark 1.5 (Local chart)**.** The chart $\{[z_1, z_2, \ldots, z_n] \in \mathsf{P}(n)\colon z_1 \neq z_2\} \to \mathbb{C}^{n-2}$, given by

$$[z_1, z_2, z_3, \ldots, z_n] \longmapsto \left(\frac{z_3 - z_1}{z_2 - z_1}, \ldots, \frac{z_n - z_1}{z_2 - z_1}\right),$$

defines an embedding of $\eta(S(n))$ into $\mathbb{C}^{n-2}$, i.e., $\eta(S(n))$ is entirely contained in the domain of this single chart of $\mathsf{P}(n)$.

Example 1.6 (Space of triangles)**.** The space of shapes of triangles with marked vertices $\mathsf{P}(3)$ is homeomorphic to the sphere $\mathbb{S}^2$. There are three options for the shapes $[0, 1, x + iy] \in \mathsf{P}(3)$: a) $y > 0$, b) $y < 0$, c) $y = 0$. Note that a) and b), respectively, correspond to positively and negatively oriented simple triangles, and c) corresponds to triangles having collinear vertices. The only shape which is not in the local chart of Remark 1.5 is $[0, 0, 1]$.

We conclude that the space of shapes of simple triangles $\eta(S(3))$ is the union of two open discs with common boundary. From Remark 1.4, it follows that $S(3)$ has two components which are homeomorphic to the product of an open disc of dimension two with $\mathbb{C} \times \mathbb{C}^*$ (any fiber bundle over the disc is trivial).

2. First results.

Proposition 2.1. *$S(n)$ is an open subset of $\mathbb{C}^n$.*

Proof. Let $Z = (z_1, z_2, \ldots, z_n) \in S(n)$. For $k \in \{1, 2, \ldots, n\}$, we define

$$\mathcal{L}_k = \{\overline{z_{k+1}z_{k+2}} \cup \overline{z_{k+2}z_{k+3}} \cup \cdots \cup \overline{z_{k-2}z_{k-1}}\}.$$

$\mathcal{L}_k$ is compact and does not contain the vertex z_k; therefore, $r_k = d(z_k, \mathcal{L}_k)$ is positive, where $d(A, B)$ is the Euclidean distance between the sets $A, B \subset \mathbb{C}$. The number $r_Z = \min\{r_1, r_2, \ldots, r_n\}$ is also positive. Let $B_{r_Z/3}(z_k)$ be the disc of radio $r_Z/3$, centered at z_k. Then, the polydisc,

$$\mathcal{U}_Z = \prod_{k=1}^{n} B_{r_Z/3}(z_k),$$

is an open neighborhood of Z contained in $S(n)$. $\square$

Lemma 2.2. *Let $n \geq 4$. If $J \subset \{1, 2, \ldots, n\}$ is such that $|J| \leq n - 3$, then the set*

$$S_J(n - |J|) := \left\{ Z \in S(n) \colon z_k = \frac{z_{k-1} + z_{k+1}}{2},\ k \in J \right\}$$

is an embedded copy of $S(n - |J|)$ in $S(n)$.

Proof. $S_J(n - |J|)$ is contained in the linear subspace

$$W(n - |J|) = \{Z \in \mathbb{C}^n \colon z_k = (z_{k-1} + z_{k+1})/2,\ k \in J\}.$$

If $L \colon W(n - |J|) \to \mathbb{C}^{n-|J|}$ is the restriction of the projection to the coordinates $\{1, 2, \ldots, n\} \setminus J$, then L is a linear isomorphism and $L(S_J(n - |J|)) = S(n - |J|)$. The restriction of L^{-1} to $S(n - |J|)$ provides an embedding into $S(n)$. $\square$

From now on, we denote $R_n = (1, e^{2\pi i/n}, (e^{2\pi i/n})^2, \ldots, (e^{2\pi i/n})^{n-1}) \in \mathbb{C}^n$, i.e., the regular n-gon with vertices at the nth-roots of the unity.

Theorem 2.3. *$S(n)$ has two homeomorphic path-connected components.*

Proof. The components correspond to the subsets of positively and negatively oriented n-gons, denoted by $S^+(n)$ and $S^-(n)$, respectively. It is clear that $S^+(n) \cap S^-(n) = \emptyset$, and the function

$$(z_1, z_2, \ldots, z_{n-1}, z_n) \longmapsto (z_1, z_n, z_{n-1}, \ldots, z_2)$$

is a homeomorphism between $S^+(n)$ and $S^-(n)$.

We will use induction to prove that, for every positively oriented simple n-gon Z, there is a curve $\gamma_Z\colon [0,1] \to S^+(n)$ which joins Z with R_n. For $n = 3$, the result follows from Example 1.6.

If $Z \in S^+(n)$, and $z_k \in Z$ is a vertex whose diagonal is interior to Z (Remark 1.3), then the curve

$$\gamma_1\colon [0,1] \longrightarrow \mathbb{C}^n,$$

$$\gamma_1(t) = (z_1, \ldots, (1-t)z_k + t\,(z_{k-1} + z_{k+1})\,/2, \ldots, z_n),$$

is contained in $S^+(n)$, $\gamma_1(0) = Z$ and $\gamma_1(1) \in S_{\{k\}}(n-1)$. From the induction hypothesis and Lemma 2.2, there is a curve

$$\gamma_2\colon [0,1] \longrightarrow S_{\{k\}}(n-1)$$

such that $\gamma_2(0) = \gamma_1(1)$, and $\gamma_2(1)$ is the n-gon

$$\left(1, \ldots, (e^{2\pi i/n})^{k-2}, \frac{(e^{2\pi i/n})^{k-2} + (e^{2\pi i/n})^{k}}{2}, (e^{2\pi i/n})^{k}, \ldots, (e^{2\pi i/n})^{n-1}\right),$$

i.e., $\gamma_2(1)$ is R_n with its kth vertex deformed to the middle point of its diagonal. The curve $\gamma_3\colon [0,1] \to \mathbb{C}^n$, which linearly joins $\gamma_2(1)$ with R_n, is also contained in $S^+(n)$. The desired curve is $\gamma_Z = \gamma_1 * \gamma_2 * \gamma_3$. $\square$

The next result easily follows from Proposition 2.1 and Theorem 2.3.

Corollary 2.4. *The set* $\eta(S(n)) = \eta(S^+(n)) \cup \eta(S^-(n)) \subset \mathsf{P}(n)$ *is open, and* $\eta(S^+(n))$ *and* $\eta(S^-(n))$ *are disjoint and path-connected, and the function* $[z_1, z_2, \ldots, z_n] \mapsto [z_1, z_n, \ldots, z_2]$ *is a homeomorphism between them.*

2.1. Deformable vertices.

Definition 2.5 (Deformable vertex)**.** Let $Z \in \mathbb{C}^n$ be a simple n-gon. We say that the vertex $z_k \in Z$ is *deformable* if $\mathfrak{cl}(z_{k-1}, z_k, z_{k+1}) \cap \mathfrak{c}(Z) = \overline{z_{k-1}z_k} \cup \overline{z_k z_{k+1}}$. If $z_k \in Z$ is deformable, then we denote $\widetilde{z}_k = (z_{k-1} + z_{k+1})/2$. We denote by $D_k(n) \subset \mathbb{C}^n$ the *set of simple n-gons whose kth vertex is deformable.*

Remark 2.6. The vertex z_k in the simple n-gon $Z = (z_1, z_2, \ldots, z_n)$ is deformable if and only if one of the following conditions holds:

(a) $\overline{z_{k-1}z_{k+1}}$ is interior to Z;

(b) z_k belongs to $\overline{z_{k-1}z_{k+1}}$;

(c) $\overline{z_{k-1}z_{k+1}}$ is exterior to Z, and the polygonal

$$\mathcal{L}_k = \{\overline{z_{k+1}z_{k+2}} \cup \overline{z_{k+2}z_{k+3}} \cup \cdots \cup \overline{z_{k-2}z_{k-1}}\}$$

does not intersect $\mathfrak{int}(z_{k-1}, z_k, z_{k+1})$.

Note that the triangles do not have deformable vertices. In the 12-gon of Figure 1, z_2, z_8 and z_9 are the only deformable vertices. In this case, $\overline{z_3z_5}$ is an exterior diagonal, but z_4 is not deformable.

Proposition 2.7. *If Z is a simple n-gon with $n \geq 4$, then Z has at least three deformable vertices.*

Proof. If Z is convex, then each diagonal $\overline{z_{k-1}z_{k+1}}$ satisfies condition (a) or (b) from Remark 2.6. If Z is not convex, then, by Remark 1.3, it is sufficient to prove that there exists a vertex whose diagonal satisfies condition (c) from Remark 2.6.

Let $E(Z)$ be the polygon which is the convex hull of Z. Suppose that $\overline{z_iz_j}$, with $i < j-1$, is a diagonal of Z which is an edge of $E(Z)$. Let $\widehat{Z}$ be the polygon such that z_i and z_j are vertices of $\widehat{Z}$ and $\mathfrak{int}(\widehat{Z}) \subset (\mathfrak{cl}(E(Z)) \setminus \mathfrak{cl}(Z))$ (the possibilities are $\widehat{Z} = (z_i, z_{i+1}, \ldots, z_{j-1}, z_j)$ or $\widehat{Z} = (z_j, z_{j+1}, \ldots, z_n, z_1, \ldots, z_{i-1}, z_i)$). Then, there are the cases:

Case I. *$\widehat{Z}$ is a triangle.* In this case, $j = i+2$ or $j = i-2$, and in both cases, $\overline{z_iz_j}$ is the desired diagonal.

Case II. *$\widehat{Z}$ is a simple k-gon with $4 \leq k < n$.* By Remark 1.3, $\widehat{Z}$ has at least one vertex z_l different from z_i and z_j and such that $\overline{z_{l-1}z_{l+1}}$ is interior to $\widehat{Z}$. The desired diagonal is $\overline{z_{l-1}z_{l+1}}$. □

Lemma 2.8 (Properties of the $D_k(n)$)**.** *Let $n \geq 4$ and $1 \leq k \leq n$.*

(i) *$S_{\{k\}}(n-1)$ is a strong deformation retract of $D_k(n)$.*
(ii) *$D_1(n)$ is homeomorphic to $D_k(n)$.*

(iii) $S(n) = \bigcup_{k=3}^{n} D_k(n)$.
(iv) $D_k(n) \subset \mathbb{C}^n$ *is open.*

Proof. The function $(t, (z_1, \dots, z_k, \dots, z_n)) \mapsto (z_1, \dots, (1-t)z_k + t\widetilde{z}_k, \dots, z_n)$ defines a strong deformation retraction from $D_k(n)$ to $S_{\{k\}}(n)$. A homeomorphism between $D_1(n)$ and $D_k(n)$ is given by $(z_1, \dots, z_k, \dots, z_n) \mapsto (z_k, \dots, z_n, z_1, \dots)$ (function which re-enumerates the vertices). Property (iii) follows easily from Proposition 2.7.

In order to prove property (iv), using property (ii), it is sufficient to show that $D_2(n)$ is open. Let $Z = (z_1, \dots, z_n) \in D_2(n)$. Since $\mathfrak{cl}(z_1, z_2, z_3)$ is compact and disjoint from $\{z_l\}$ for $l \in \{4, 5, \dots, n\}$, then $r_l = \min\{r_Z, d(z_l, \mathfrak{cl}(z_1, z_2, z_3))\}$ is a positive number (r_Z was defined in the Proof of Proposition 2.1). If $\mathfrak{r} = \min\{r_4, \dots, r_n\}$, then the polydisc $\prod_{k=1}^{n} B_{\mathfrak{r}/3}(z_k)$ is an open neighborhood of Z which is contained in $D_2(n)$. □

3. $\mathsf{S}(n)$ **is simply connected.** From now on, we shall work in the component $\mathsf{S}(n) = \eta(S^+(n))$ of $\eta(S(n))$. By Corollary 2.4, the results obtained will also be valid in the component $\eta(S^-(n))$.

In the proof of the main theorem we shall use the next easy corollary of Van Kampen's theorem [**5,** p.43].

Lemma 3.1. *Let* $U_1, U_2, \dots, U_m$ *be open and simply connected subsets of* X *such that* $\bigcap_{k=1}^{m} U_k \neq \emptyset$, $\bigcup_{k=1}^{m} U_k = X$ *and, for all* $1 \le j < k \le m$, $U_j \cap U_k$ *is path-connected. Then,* X *is simply connected.*

During the proof of the next result, we will use the notation: $\mathsf{S}_J(n - |J|) = \eta(S_J(n - |J|) \cap S^+(n))$ and $\mathsf{D}_k(n) = \eta(D_k(n) \cap S^+(n))$.

Theorem 3.2. $\mathsf{S}(n) \subset \mathsf{P}(n)$ *is simply connected.*

Proof. We proceed by induction over n. Example 1.6 shows the case $n = 3$.

We will prove that the sets $\mathsf{D}_k(n)$ with $k = 3, 4, \dots, n$, satisfy the hypothesss of Lemma 3.1. From Lemma 2.8 (iii) and (iv), we know that the sets $\mathsf{D}_k(n)$ are open and $\mathsf{S}(n) = \bigcup_{k=3}^{n} \mathsf{D}_k(n)$. The intersection $\bigcap_{k=3}^{n} \mathsf{D}_k(n)$ is not empty since it contains the convex n-gons.

The strong deformation retraction in the Proof of Lemma 2.8 respects the action of the complex affine group, and hence, descends to a well-defined strong deformation retraction from $\mathsf{D}_k(n)$ to $\mathsf{S}_{\{k\}}(n)$. Using the induction hypothesis and Lemma 2.2, we conclude that $\mathsf{D}_k(n)$ is simply connected for all k.

It only remains to show that $\mathsf{D}_j(n) \cap \mathsf{D}_k(n)$ is path-connected for $3 \leq j < k \leq n$. There are two cases.

Case $j < k-1$. Let $Z, W \in \mathsf{D}_j(n) \cap \mathsf{D}_k(n)$. Deforming their jth and kth vertices to the points $\widetilde{z}_j$, $\widetilde{z}_k$ and $\widetilde{w}_k$, $\widetilde{w}_j$, we obtain two simple n-gons Z' and W' which belong to $\mathsf{S}_{\{j,k\}}(n-2)$. Using Lemma 2.2 and Theorem 2.3, the result is proven.

Case $j = k-1$. We will suppose that $j = 3$ and $k = 4$; the other cases are similar. During the proof, we will work with representatives of simple n-gons with their first and second vertices fixed at 0 and 1, respectively.

Let $Z = [0, 1, z_3, \dots, z_n] \in \mathsf{D}_3(n) \cap \mathsf{D}_4(n)$. If $n > 4$, then, for each vertex z_k, with $k = 6, 7, \dots, n, 1$, we denote by $\mathfrak{R}_k \subset \mathbb{C}$ the infinite ray which begins at z_5 and contains the segment $\overline{z_5 z_k}$. Since z_3 is deformable and Z is simple, the ray $\mathfrak{R}_k \setminus \overline{z_5 z_k}$ intersects $\overline{z_4 1}$ at a single point or does not intersect $\overline{z_4 1}$. We denote $\mu_k(Z) \in \mathbb{C}$ as the point $(\mathfrak{R}_k \setminus \overline{z_5 z_k}) \cap \overline{z_4 1}$, if it exists; if such an intersection is empty, set $\mu_k(Z) = 1$ (note that always $\mu_k(Z) \neq z_4$). Let $\mu(Z) \in \overline{z_4 1}$ be the closest $\mu_k(Z)$ to z_4 (Figure 2). For the case $n = 4$, $\mu(Z) = 1$ for all $Z \in \mathsf{S}(4)$. The functions

$$\mu_k \colon \mathsf{D}_3(n) \cap \mathsf{D}_4(n) \to \mathbb{C},$$

$$Z \longmapsto \mu_k(Z),$$

with $k = 6, 7, \dots, n, 1$, vary continuously with the vertices of Z, and therefore, the function $\mu \colon \mathsf{D}_3(n) \cap \mathsf{D}_4(n) \to \mathbb{C}$, $Z \mapsto \mu(Z)$, is continuous.

We define $z_3' := (\mu(Z) + z_4)/2$ for all $Z \in \mathsf{D}_3(n) \cap \mathsf{D}_4(n)$, and

$$G \colon [0,1] \times (\mathsf{D}_3(n) \cap \mathsf{D}_4(n)) \to \mathsf{P}(n),$$

the function given by

$$(t, Z) = (t, [0, 1, z_3, z_4, \dots, z_n]) \longmapsto [0, 1, (1-t)z_3 + t z_3', z_4, \dots, z_n].$$

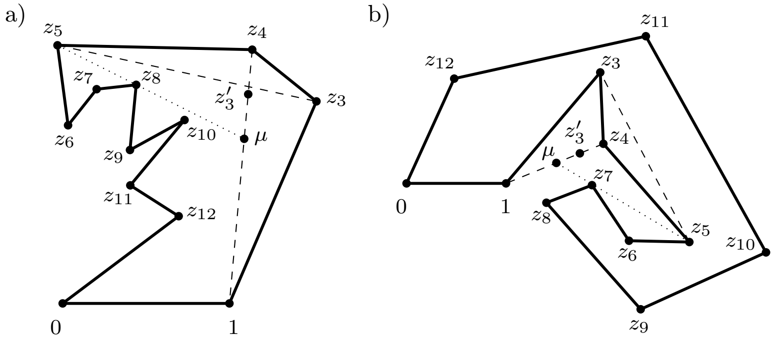

FIGURE 2. The points $\mu = \mu(Z)$ and z_3' for two 12-gons. a) The diagonals $\overline{1z_4}$ and $\overline{z_3z_5}$ are interior. b) $\overline{1z_4}$ is exterior and $\overline{z_3z_5}$ is interior.

We proceed to prove that G is a strong deformation retraction from $\mathsf{D}_3(n) \cap \mathsf{D}_4(n)$ to $G(1, \mathsf{D}_3(n) \cap \mathsf{D}_4(n))$. Continuity of G follows from continuity of μ. $G(0, -)$ is the identity map, and, for all $t \in [0, 1]$ and $Z \in G(1, \mathsf{D}_3(n) \cap \mathsf{D}_4(n))$, $G(t, Z) = Z$ is satisfied. Let $Z \in \mathsf{D}_3(n) \cap \mathsf{D}_4(n)$. It only remains to prove that $G(t, Z) \in \mathsf{D}_3(n) \cap \mathsf{D}_4(n)$ for all $t \in [0, 1]$. Note that $G(-, Z)$ does not move the diagonal $\overline{1z_4}$, and therefore, the third vertex $z_3(t) = (1 - t)z_3 + tz_3'$ of $G(t, Z)$ is deformable for all $t \in [0, 1]$. We claim that z_4 is deformable in $G(t, Z)$ for all $t \in [0, 1]$. The proof is broken into three cases:

Case I. $\overline{1z_4} \cup \overline{z_3z_5} \subset \mathfrak{cl}(Z)$ *or* $\overline{1z_4} \cup \overline{z_3z_5} \subset \mathbb{C} \setminus \mathfrak{int}(Z)$. In these cases, the quadrilateral $(z_3, \mu(Z), z_5, z_4)$ is convex and $\mathfrak{int}(z_3, \mu(Z), z_5, z_4)$ contains no vertices of Z, Figure 2 a). Since z_3' is the middle point of $\overline{\mu(Z)z_4}$, the segment $\overline{z_3z_3'}$ is contained in $\mathfrak{cl}(z_3, \mu(Z), z_5, z_4)$. By definition of $\mu(Z)$ and the assumption that z_3 is deformable in Z, it is implied that, for all t, the open segment $\overline{z_5z_3(t)} \setminus \{z_5, z_3(t)\}$ does not intersect $\mathfrak{c}(Z)$. We conclude that $\overline{z_5z_3(t)}$ is an interior diagonal of $G(t, Z)$ if $\overline{z_5z_3}$ is interior to Z, and $\overline{z_5z_3(t)}$ is an exterior diagonal of $G(t, Z)$ if $\overline{z_5z_3}$ is exterior to Z. Then, z_4 is a deformable vertex of $G(t, Z)$ for all $t \in [0, 1]$.

Case II. $\overline{1z_4} \subset \mathfrak{cl}(Z)$ *and* $\overline{z_3z_5} \subset \mathbb{C} \setminus \mathfrak{int}(Z)$. There are three sub-cases:

(i) If the quadrilateral $(1, z_3, z_5, z_4)$ is convex, then $\mathfrak{int}(1, z_3, z_5, z_4)$ does not contain any vertex of Z, and hence, $\mu(Z) = 1$. We conclude that $z_3' = \widetilde{z}_3$ and $\overline{z_3z_3'} \subset \mathfrak{cl}(1, z_3, z_5, z_4)$, and therefore, $z_4 \in G(t, Z)$ is deformable.

(ii) If $z_3 \in \mathfrak{int}(1, z_5, z_4)$, then z_3 is the only vertex of Z in the convex region $\mathfrak{int}(\mu(Z), z_5, z_4)$ (it may occur that $\mu(Z) = 1$). Since $\overline{z_3z_3'} \subset \mathfrak{cl}(\mu(Z), z_5, z_4)$, $z_4 \in G(t, Z)$ is deformable.

(iii) If $z_4 \in \mathfrak{int}(1, z_3, z_5)$, then z_4 is the only vertex of Z in the convex region $\mathfrak{int}(\mu(Z), z_3, z_5)$. By construction, $\overline{z_3z_3'} \subset \mathfrak{cl}(\mu(Z), z_3, z_5)$ and $\overline{z_3z_3'} \cap \overline{z_5z_4} = \emptyset$; thus, $z_4 \in G(t, Z)$ is deformable.

Case III. $\overline{1z_4} \subset \mathbb{C} \setminus \mathfrak{int}(Z)$ *and* $\overline{z_3z_5} \subset \mathfrak{cl}(Z)$. There are three analogous sub-cases as in Case II (Figure 3 b) shows an example analogous to Case II (iii)). The proof of each is also similar.

The fact that Cases I–III cover all possibilities can be deduced from the fact that, since z_3 and z_4 are deformable vertices, the diagonals $\overline{z_3z_5}$ and $\overline{1z_4}$ must each lie either in $\mathfrak{cl}(Z)$ or in $\mathbb{C} \setminus \mathfrak{int}(Z)$ (see Remark 2.6). We conclude that $G(1, \mathsf{D}_3(n) \cap \mathsf{D}_4(n))$ is a strong deformation retract of $\mathsf{D}_3(n) \cap \mathsf{D}_4(n)$.

Consider the function $f\colon \mathsf{S}_{\{3\}}(n-1) \to G(1, \mathsf{D}_3(n) \cap \mathsf{D}_4(n))$, defined by $[0, 1, \widetilde{z}_3, z_4, \ldots, z_n] \mapsto [0, 1, z_3', z_4, \ldots, z_n]$. Note that f is well defined, and, in fact, is a bijection since, for all $Z \in \mathsf{D}_3(n) \cap \mathsf{D}_4(n)$, the value of z_3' is determined by the vertices $0, 1, z_4, \ldots, z_n$. Continuity of f is immediate from continuity of μ, and continuity of f^{-1} follows from continuity of $(1, z_3', z_4) \mapsto (1, (1 + z_4)/2, z_4)$. Then, f is a homeomorphism between $\mathsf{S}_{\{3\}}(n - 1)$ and $G_1(\mathsf{D}_3(n) \cap \mathsf{D}_4(n))$. From Theorem 2.3, it follows that $\mathsf{D}_3(n) \cap \mathsf{D}_4(n)$ is path-connected. $\square$

Theorem 3.3. *$\pi_1(S^+(n))$ is a cyclic group, and, if $k \geq 3$, then $\pi_k(S^+(n))$ is isomorphic to $\pi_k(\mathsf{S}(n))$.*

Proof. The long exact sequence in homotopy for the fibration from Remark 1.4 [**5,** page 376] looks like:

$$\cdots \longrightarrow \pi_k(\mathbb{C}\times\mathbb{C}^*) \longrightarrow \pi_k(S^+(n)) \longrightarrow \pi_k(\mathsf{S}(n)) \longrightarrow \pi_{k-1}(\mathbb{C}\times\mathbb{C}^*) \longrightarrow \cdots.$$

If $k \geq 3$, then $\pi_k(\mathbb{C} \times \mathbb{C}^*) = \pi_{k-1}(\mathbb{C} \times \mathbb{C}^*) = 0$, and therefore, $\pi_k(S^+(n)) \cong \pi_k(\mathsf{S}(n))$. For $k = 1$, consider the next part of the sequence:

$$\cdots \longrightarrow \pi_2(S^+(n)) \longrightarrow \pi_2(\mathsf{S}(n)) \longrightarrow \mathbb{Z} \xrightarrow{f} \pi_1(S^+(n)) \longrightarrow 0 \longrightarrow \cdots,$$

here, using that $\pi_1(\mathbb{C} \times \mathbb{C}^*) = \mathbb{Z}$ and Theorem 3.2. By exactness, f is onto, and therefore, $\pi_1(S^+(n))$ is cyclic. □

4. Unlabelled polygons. We denote by $\widehat{\delta}\colon \mathbb{C}^n \to \mathbb{C}^n$ the linear isomorphism $(z_1, z_2, \ldots, z_{n-1}, z_n) \mapsto (z_2, \ldots, z_{n-1}, z_n, z_1)$. Clearly, $\widehat{\delta}^n$ is the identity map in $\mathbb{C}^n$.

Lemma 4.1. *$\widehat{\delta}$ defines a biholomorphism $\delta\colon \mathsf{P}(n) \to \mathsf{P}(n)$.*

Proof. Note that $Z, W \in \mathbb{C}^n$ have the same shape if and only if $\widehat{\delta}(Z)$ and $\widehat{\delta}(W)$ have the same shape. Then, there exists a $\delta\colon \mathsf{P}(n) \to \mathsf{P}(n)$ such that $\delta \circ \eta = \eta \circ \widehat{\delta}$. The expression of δ in the chart of Remark 1.5 is:

$$(z_3, z_4, \ldots, z_n) \mapsto \left(\frac{z_4 - 1}{z_3 - 1}, \ldots, \frac{z_n - 1}{z_3 - 1}, \frac{1}{1 - z_3}\right).$$

By taking different charts, it can be proven that this expression defines a biholomorphism in $\mathsf{P}(n)$. □

It is clear that δ^n is the identity map in $\mathsf{P}(n)$. We will denote by δ the restriction of δ to the open set $\mathsf{S}(n)$, and by R_n the projection of the regular n-gon $\eta(R_n)$.

Lemma 4.2. *If $n = mk$, then there exists a $Z \in \mathsf{S}(n)$ such that $\delta^l(Z) \neq Z$ when $1 \leq l < m$ and $\delta^m(Z) = Z$. In addition, for $Z \in \mathsf{S}(n)$, $\delta(Z) = Z$ if and only if $Z = \mathsf{R}_n$.*

Proof. If $n = mk$, then the shape of the n-gon obtained by marking $m - 1$ equidistant points on the edges of R_k (Figure 3) satisfies the desired conditions. If $\delta(Z) = Z$, then Z must have equal angles and edge lengths, and therefore $Z = \mathsf{R}_n$. □

Theorem 4.3. *Let $D_{\mathsf{R}_n}\delta$ be the differential of δ at R_n. Then, it is conjugate to the diagonal matrix with entries $e^{4i\pi/n}, e^{6i\pi/n}, \ldots, e^{2(n-1)i\pi/n}$.*

Proof. Let $\langle Z, W\rangle = \sum_i z_i\overline{w}_i$ be the standard Hermitian product. The set

$$\beta = \{P_k = (1, e^{2\pi ik/n}, (e^{2\pi ik/n})^2, \dots, (e^{2\pi ik/n})^{n-1}\}$$

with $k = 0, 1, \dots, n-1$, is a basis of $\mathbb{C}^n$ [**9,** Proposition 3]. The products between these vectors are

$$\langle P_k, P_l\rangle = \sum_{m=0}^{n-1} (e^{2\pi i(k-l)/n})^m = \begin{cases} n & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

and therefore, β is orthogonal. In addition, β is a basis of eigenvectors for $\widehat{\delta}$ since $\widehat{\delta}(P_k) = e^{2ik\pi/n}P_k$ for all k.

The space of shapes of n-gons can be obtained as the complex projectivization of the subspace

$$V = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n\colon z_1 + z_2 + \cdots + z_n = 0\}.$$

It is clear that $\beta \setminus \{P_0\}$ is basis of V, and therefore, $\widehat{\delta}(V) = V$. A presentation of the tangent space $T_{\mathsf{R}_n}\mathsf{P}(n)$ is given by the linear subspace $\langle P_2, P_3, \dots, P_{n-1}\rangle$ since the direction determined by $P_1 = R_n$ is nullified by the projectivization. The action of $\widehat{\delta}$ in this subspace corresponds to the action of $D_{\mathsf{R}_n}\delta$ in $T_{\mathsf{R}_n}\mathsf{P}(n)$. We conclude that $D_{\mathsf{R}_n}\delta$ is conjugated to the matrix:

$$\Delta_n = \begin{pmatrix} e^{4\pi i/n} & 0 & \cdots & 0 \\ 0 & e^{6\pi i/n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2(n-1)\pi i/n} \end{pmatrix}. \qquad \square$$

Definition 4.4 (Polygons without marked vertices)**.** We call the quotients $\mathcal{P}(n) = \mathsf{P}(n)/\langle\delta\rangle$ and $\mathcal{S}(n) = \mathsf{S}(n)/\langle\delta\rangle$ the *spaces of shapes of n-gons and simple n-gons without marked vertices*, respectively. We denote by σ the quotient projection $\mathsf{S}(n) \to \mathcal{S}(n)$.

Remark 4.5. Let g^{FS} be the Fubini-Study metric on $\mathsf{P}(n)$ (remember that $\mathsf{P}(n)$ is biholomorphic to $\mathbb{CP}^{n-2}$). Since $\widehat{\delta}$ is an isometry of $(\mathbb{C}^n, \langle\ ,\ \rangle)$, then g^{FS} defines a metric in the orbifold $\mathcal{P}(n)$.

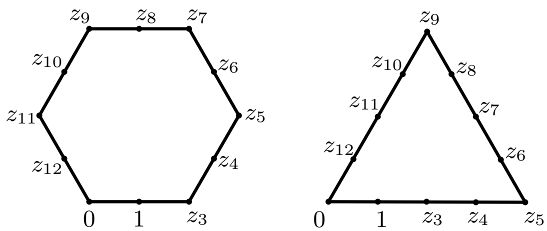

FIGURE 3. Two 12-gons where a power of δ is the identity map.

Example 4.6 (Triangles without marked vertices)**.** The action of δ in $\mathsf{P}(3)$ is topologically conjugate (through the chart in Remark 1.5) with the action of the Möbius map $1/(1-z)$ in the Riemann sphere $\widehat{\mathbb{C}}$. Therefore, $\mathcal{P}(3)$ is a sphere with spherical metric (g^{FS} in $\mathbb{CP}^1$ is the spherical metric), and two conic singularities of conic angle equal to $2\pi/3$. In this case, $\mathcal{S}(3)$ is topologically an open disc corresponding to the superior hemisphere in $\mathcal{P}(3)$.

4.1. Topological properties of $\mathcal{S}(n)$.

Theorem 4.7. *$\mathcal{S}(n)$ is simply connected.*

Proof. The group $\langle\delta\rangle$ acts discontinuously in $\mathsf{S}(n)$ and fixes R_n. From Theorem 3.2 and Armstrong's theorem [**1**], it follows that $\pi_1(\mathcal{S}(n)) \cong \pi_1(\mathsf{S}(n)) = 1$. □

Theorem 4.8. *The local neighborhoods of $\sigma(\mathsf{R}_n)$ in $\mathcal{P}(n)$ are homeomorphic to the cone $((\mathbb{S}^{2n-5}/\Delta_n) \times [0,1))/\{(x,0)\}$.*

Proof. Since $g^{FS}_{\mathsf{R}_n}(v,v) = g^{FS}_{\mathsf{R}_n}(D_{\mathsf{R}_n}\delta v, D_{\mathsf{R}_n}\delta v)$ for all $v \in T_{\mathsf{R}_n}\mathsf{S}(n)$, then $D_{\mathsf{R}_n}\delta$ preserves the level spheres

$$E_s = \{v \in T_{\mathsf{R}_n}\mathsf{S}(n) \colon g^{FS}_{\mathsf{R}_n}(v,v) = s\}.$$

From Theorem 4.3, it follows that $T_{\mathsf{R}_n}\mathsf{S}(n)/\langle D_{\mathsf{R}_n}\delta\rangle$ is homeomorphic to $((\mathbb{S}^{2n-5}/\langle\Delta_n\rangle) \times [0,1))/\{(x,0)\}$. Note that, if n is a prime number, then this quotient is the cone over a generalized lens space [**5,** page 144].

Let $d_{FS}\colon \mathsf{S}(n)\times\mathsf{S}(n)\to\mathbb{R}$ be the distance in $\mathsf{P}(n)$ associated to g^{FS}. It can be proven that δ is an isometry of $(\mathsf{P}(n),d_{FS})$. Then, for all $Z\in\mathsf{S}(n)$ and $1\le k\le n$, we have

$$d_{FS}(\mathsf{R}_n,Z)=d_{FS}(\delta^k(\mathsf{R}_n),\delta^k(Z))=d_{FS}(\mathsf{R}_n,\delta^k(Z)).$$

Let $\exp_{\mathsf{R}_n}\colon T_{\mathsf{R}_n}\mathsf{S}(n)\to\mathsf{S}(n)$ be the exponential map of g^{FS}. Then, there exists a number $r>0$ such that $\exp_{\mathsf{R}_n}$ is an isometry in the neighborhood

$$\mathcal{U}=\{v\in T_{\mathsf{R}_n}\mathsf{S}(n)\colon g^{FS}_{\mathsf{R}_n}(v,v)<r\}$$

[**3,** page 65]. We conclude that, for $s<r$, the images $\mathcal{E}_s=\exp_{\mathsf{R}_n}(E_s)$ are the level spheres $\{Z\in\mathsf{S}(n)\colon d_{FS}(\mathsf{R}_n,Z)=s\}$ of d_{FS} around R_n. In addition, the spheres $\mathcal{E}_s$ remain invariant under the action of the cyclic group $\langle\delta\rangle$.

Since δ takes the geodesic with initial condition $v_0\in T_{\mathsf{R}_n}\mathsf{S}(n)$ to the geodesic with initial condition $D_{\mathsf{R}_n}\delta(v_0)$, then it follows that $\delta\circ\exp_{\mathsf{R}_n}=\exp_{\mathsf{R}_n}\circ D_{\mathsf{R}_n}\delta$ (i.e., $\exp_{\mathsf{R}_n}$ conjugates the actions of δ and $D_{\mathsf{R}_n}\delta$). Therefore, for all $s<r$, the following holds:

$$\mathcal{E}_s/\langle\delta\rangle\cong E_s/\langle D_{\mathsf{R}_n}\delta\rangle\cong E_s/\langle\Delta_n\rangle\cong\mathbb{S}^{2n-5}/\langle\Delta_n\rangle.$$

We conclude that the neighborhood $\sigma(\exp(\mathcal{U}))\subset\mathcal{P}(n)$ is homeomorphic to the cone mentioned. $\square$

In Example 4.6, the level spheres $\mathcal{E}_s$ and the quotients $\mathcal{E}_s/\langle\delta\rangle$ are circles, and therefore, $\mathcal{P}(3)$ is a manifold. The next result shows what occurs when n is the power of a prime number.

Corollary 4.9. *If $n>3$ and $n=p^k$ with p a prime number, then the space $\mathcal{P}(n)$ is not a manifold.*

Proof. If U is a local neighborhood around $\sigma(\mathsf{R_n})$, as in Theorem 4.8, then, by Kwun's theorem [**8**], it is sufficient to prove that $H_1(\mathbb{S}^{2n-5}/\langle\Delta_n\rangle)$ is not trivial. By Theorems 4.3 and 4.8, the subgroup of $\langle\delta\rangle$, which fixes some points in the level spheres $\mathcal{E}_s$, is exactly $\langle\delta^p\rangle$. Using Armstrong's theorem [**1**], we conclude that $\pi_1(\mathbb{S}^{2n-5}/\langle\Delta_n\rangle)$;

hence, also $H_1(\mathbb{S}^{2n-5}/\langle\Delta_n\rangle)$, is isomorphic to $\langle\delta\rangle/\langle\delta^p\rangle$, which is cyclic of order p. □

The same argument shows that $\pi_1(\mathcal{E}_s/\langle\delta\rangle)$ is trivial if n is not a power of a prime number. For these cases, we do not have any properties of the quotients $\mathcal{E}_s/\langle\delta\rangle$.

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Universidad Nacional Autónoma de México (UNAM), Campus Morelia, Centro de Ciencias Matemáticas, C.P. 58190, Morelia, Michoacán, México
Email address: **ahtziri@matmor.unam.mx**