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LOCAL HOMOLOGY, KOSZUL HOMOLOGY AND SERRE CLASSES

KAMRAN DIVAANI-AAZAR, HOSSEIN FARIDIAN AND MASSOUD TOUSI

ABSTRACT. Given a Serre class $\mathcal{S}$ of modules, we compare the containment of Koszul homology, Ext modules, Tor modules, local homology and local cohomology in $\mathcal{S}$ up to a given bound $s \geq 0$. As applications, we give a full characterization of Noetherian local homology modules. Furthermore, we establish a comprehensive vanishing result which readily leads to formerly known descriptions of the numerical invariants' width and depth in terms of Koszul homology, local homology and local cohomology. In addition, we recover a few renowned vanishing criteria scattered throughout the literature.

1. Introduction. Throughout this paper, R denotes a commutative Noetherian ring with identity, and $\mathcal{M}(R)$ designates the category of R-modules.

The theory of local cohomology was introduced by Grothendieck approximately six decades ago and has since been tremendously developed by various algebraic geometers and commutative algebraists.

There exists a dual theory, the theory of local homology, which was initiated by Matlis [**18, 19**] in 1974, and its investigation was continued by Simon [**26, 27**]. After the ground-breaking achievements of Greenlees and May [**13**], as well as Tarrío, et al. [**29**], a new era in the study of this theory was undoubtedly begun, see e.g., [**8, 9, 11, 12, 14, 17, 22, 25**]. However, the theory of local homology has not been addressed as it merits, and numerous counterparts of the results in local cohomology are yet to be explored and unearthed in local homology.

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One interesting problem in local cohomology is the investigation of the Artinian property. This is very well accomplished in [**21,** Theorem 5.5]. The dual problem, i.e., the Noetherian property of local homology, will be, among other things, carefully probed in this paper.

A survey among the results [**1,** Theorem 2.10], [**2,** Theorem 2.9], [**5,** Corollary 3.1], [**6,** Lemma 2.1], [**10,** Proposition 1, Corollary 1], [**15,** Lemma 4.2], [**21,** Theorem 2.1], [**24,** Propositions 7.1, 7.2, 7.4] and [**32,** Lemma 1.2] reveals an in-depth connection between local cohomology, Ext modules, Tor modules and Koszul homology in terms of their containment in a Serre class of modules. The purpose of this paper is to bring the local homology into play and uncover the true connection among all these homology and cohomology theories, and consequently, to illuminate and enhance the aforementioned results.

In order to shed some light on this revelation, we observe that, given elements $\underline{a} = a_1, \ldots, a_n \in R$ and an R-module M, the Koszul homology $H_i(\underline{a}; M) = 0$ for every $i < 0$ or $i > n$, and

$$H_n(\underline{a}; M) \cong (0 :_M (\underline{a})) \cong \operatorname{Ext}^0_R(R/(\underline{a}), M)$$

and

$$H_0(\underline{a}; M) \cong M/(\underline{a})M \cong \operatorname{Tor}^R_0(R/(\underline{a}), M).$$

These isomorphisms suggest the existence of an intimate connection between the Koszul homology on one hand, and Ext and Tor modules on the other hand. The connection seems to manifest as in the following casual diagram for any given integer $s \geq 0$:

$$\underbrace{\operatorname{Ext}^0_R(R/(\underline{a}), M), \ldots, \operatorname{Ext}^s_R(R/(\underline{a}), M)}$$

$$\updownarrow$$

$$\overbrace{H_n(\underline{a}; M), \ldots, H_{n-s}(\underline{a}; M)}, \ldots, \underbrace{H_s(\underline{a}; M), \ldots, H_0(\underline{a}; M)}$$

$$\updownarrow$$

$$\overbrace{\operatorname{Tor}^R_s(R/(\underline{a}), M), \ldots, \operatorname{Tor}^R_0(R/(\underline{a}), M)}$$

As the above diagram depicts, the Koszul homology acts as a connecting device in attaching Ext modules to Tor modules in an elegant manner. One such connection could be the containment in some Serre class of modules that we shall study in the following sections. More specifically, we prove the following as our main results, see Definitions 2.1, 2.3, 2.5 and 2.7.

Theorem 1.1. *Let $\mathfrak{a} = (a_1, \dots, a_n)$ be an ideal of R, $\underline{a} = a_1, \dots, a_n$, and M an R-module. Let $\mathcal{P}$ be a Serre property. Then, the following three conditions are equivalent for any given $s \geq 0$:*

(i) $H_i(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every* $0 \leq i \leq s$;

(ii) $\operatorname{Tor}_i^R(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every finitely generated R-module N with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$;

(iii) $\operatorname{Tor}_i^R(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for some finitely generated R-module N with* $\operatorname{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

If, in addition, $\mathcal{P}$ satisfies the condition $\mathfrak{D}_{\mathfrak{a}}$, then the above three conditions are equivalent to the following condition:

(iv) $H_i^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$ *for every* $0 \leq i \leq s$.

Dually:

Theorem 1.2. *Let $\mathfrak{a} = (a_1, \dots, a_n)$ be an ideal of R, $\underline{a} = a_1, \dots, a_n$, and M an R-module. Let $\mathcal{P}$ be a Serre property. Then, the following three conditions are equivalent for any given $s \geq 0$:*

(i) $H_{n-i}(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every* $0 \leq i \leq s$;

(ii) $\operatorname{Ext}_R^i(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every finitely generated R-module N with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$;

(iii) $\operatorname{Ext}_R^i(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for some finitely generated R-module N with* $\operatorname{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

If, in addition, $\mathcal{P}$ satisfies the condition $\mathfrak{C}_{\mathfrak{a}}$, then the above three conditions are equivalent to the following condition:

(iv) $H_{\mathfrak{a}}^i(M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every* $0 \leq i \leq s$.

With these results at our disposal, by specializing when $s = n$ and the Serre property is Noetherian, Artinian, or zero, we draw a handful of fruitful conclusions on Noetherianness, Artinianness and vanishing of the five foregoing homology and cohomology theories.

2. Prerequisites. First and foremost, we recall the notion of a Serre class of modules.

Definition 2.1. A class $\mathcal{S}$ of R-modules is said to be a *Serre class* if, given any short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, we have $M \in \mathcal{S}$ if and only if $M', M'' \in \mathcal{S}$.

The following example showcases the stereotypical instances of Serre classes.

Example 2.2. Let $\mathfrak{a}$ be an ideal of R. Then, the following classes of modules are Serre classes:

(i) the zero class;
(ii) the class of all Noetherian R-modules;
(iii) the class of all Artinian R-modules;
(iv) the class of all minimax R-modules;
(v) the class of all minimax and $\mathfrak{a}$-cofinite R-modules, see [**20,** Corollary 4.4];
(vi) the class of all weakly Laskerian R-modules;
(vii) the class of all Matlis reflexive R-modules.

We are concerned with a change of rings when dealing with Serre classes. Therefore, we adopt the following notion of Serre property to exclude any possible ambiguity in the statement of the results.

Definition 2.3. A property $\mathcal{P}$ concerning modules is said to be a *Serre property* if

$$\mathcal{S}_{\mathcal{P}}(R) := \{M \in \mathcal{M}(R) \mid M \text{ satisfies the property } \mathcal{P}\}$$

is a Serre class for every ring R.

Given a Serre property, the corresponding notions of depth and width naturally arise.

Definition 2.4. Let $\mathfrak{a}$ be an ideal of R and M an R-module. Let $\mathcal{P}$ be a Serre property. Then:

(i) Define the $\mathcal{P}$-depth of M *with respect to* $\mathfrak{a}$ to be

$$\mathcal{P}\text{-}\operatorname{depth}_R(\mathfrak{a}, M) := \inf\{i \geq 0 \mid \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \notin \mathcal{S}_{\mathcal{P}}(R)\}.$$

(ii) Define the $\mathcal{P}$-width of M *with respect to* $\mathfrak{a}$ to be

$$\mathcal{P}\text{-}\operatorname{width}_R(\mathfrak{a}, M) := \inf\{i \geq 0 \mid \operatorname{Tor}^R_i(R/\mathfrak{a}, M) \notin \mathcal{S}_{\mathcal{P}}(R)\}.$$

It is clear that, upon letting $\mathcal{P}$ be the Serre property of being zero, we recover the classical notions of depth and width.

The definitions of local cohomology and local homology functors are recalled next. Let $\mathfrak{a}$ be an ideal of R. We let

$$\Gamma_{\mathfrak{a}}(M) := \{x \in M \mid \mathfrak{a}^t x = 0 \text{ for some } t \geq 1\}$$

for an R-module M and $\Gamma_{\mathfrak{a}}(f) := f|_{\Gamma_{\mathfrak{a}}(M)}$ for an R-homomorphism $f : M \to N$. This provides the so-called $\mathfrak{a}$-torsion functor $\Gamma_{\mathfrak{a}}(-)$ on the category of R-modules. The ith local cohomology functor with respect to $\mathfrak{a}$ is defined to be

$$H^i_{\mathfrak{a}}(-) := R^i\Gamma_{\mathfrak{a}}(-)$$

for every $i \geq 0$. In addition, the cohomological dimension of $\mathfrak{a}$ with respect to M is

$$\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \geq 0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Recall that an R-module M is said to be $\mathfrak{a}$-torsion if $M = \Gamma_{\mathfrak{a}}(M)$. It is well known that any $\mathfrak{a}$-torsion R-module M enjoys a natural $\widehat{R}^{\mathfrak{a}}$-module structure in such a way that the lattices of its R-submodules and $\widehat{R}^{\mathfrak{a}}$-submodules coincide. In addition, we have $\widehat{R}^{\mathfrak{a}} \otimes_R M \cong M$ both as R-modules and $\widehat{R}^{\mathfrak{a}}$-modules.

Similarly, we let

$$\Lambda^{\mathfrak{a}}(M) := \widehat{M}^{\mathfrak{a}} = \varprojlim_n (M/\mathfrak{a}^n M)$$

for an R-module M and $\Lambda^{\mathfrak{a}}(f) := \widehat{f}$ for an R-homomorphism $f : M \to N$. This provides the so-called $\mathfrak{a}$-adic completion functor $\Lambda^{\mathfrak{a}}(-)$ on the category of R-modules. The ith local homology functor with respect to $\mathfrak{a}$ is defined to be

$$H_i^{\mathfrak{a}}(-) := L_i\Lambda^{\mathfrak{a}}(-)$$

for every $i \geq 0$. In addition, the homological dimension of $\mathfrak{a}$ with respect to M is

$$\operatorname{hd}(\mathfrak{a}, M) := \sup\{i \geq 0 \mid H_i^{\mathfrak{a}}(M) \neq 0\}.$$

The next conditions are also required to be imposed on Serre classes when considering local homology and local cohomology.

Definition 2.5. Let $\mathcal{P}$ be a Serre property and $\mathfrak{a}$ an ideal of R. We say that $\mathcal{P}$ satisfies the *condition* $\mathfrak{D}_{\mathfrak{a}}$ if the following statements hold:

(i) if R is $\mathfrak{a}$-adically complete and $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(R)$ for some R-module M, then $H_0^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(R)$.
(ii) For any $\mathfrak{a}$-torsion R-module M, we have $M \in \mathcal{S}_{\mathcal{P}}(R)$ if and only if $M \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$.

Example 2.6. Let $\mathfrak{a}$ be an ideal of R. Then, we have:

(i) The Serre property of being zero satisfies the condition $\mathfrak{D}_{\mathfrak{a}}$. Use [**26,** Lemma 5.1 (iii)].
(ii) The Serre property of being Noetherian satisfies the condition $\mathfrak{D}_{\mathfrak{a}}$. See [**11,** Lemma 2.5].

Aghapournahr and Melkersson [**3,** Definition 2.1] defined the condition $\mathfrak{C}_{\mathfrak{a}}$ for Serre classes. Adopting their definition, we have:

Definition 2.7. Let $\mathcal{P}$ be a Serre property and $\mathfrak{a}$ an ideal of R. We say that $\mathcal{P}$ satisfies the *condition* $\mathfrak{C}_{\mathfrak{a}}$ if the containment $(0 :_M \mathfrak{a}) \in \mathcal{S}_{\mathcal{P}}(R)$ for some R-module M implies that $\Gamma_{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(R)$.

Example 2.8. Let $\mathfrak{a}$ be an ideal of R. Then, we have:

(i) The Serre property of being zero satisfies the condition $\mathfrak{C}_{\mathfrak{a}}$. Indeed, if M is an R-module such that $(0 :_M \mathfrak{a}) = 0$, then it can easily be seen that $\Gamma_{\mathfrak{a}}(M) = 0$.
(ii) The Serre property of being Artinian satisfies the condition $\mathfrak{C}_{\mathfrak{a}}$. This follows from the Melkersson's Criterion [**21,** Theorem 1.3].

(iii) A Serre class which is closed under taking direct limits is called a torsion theory.

Many torsion theories exist, e.g., given any R-module L, the class

$$\mathfrak{T}_L := \{M \in \mathcal{M}(R) \mid \operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(L)\}$$

is a torsion theory. It can easily be seen that any torsion theory satisfies the condition $\mathfrak{C}_{\mathfrak{a}}$.

Next, we briefly describe the Koszul homology. The Koszul complex $K^R(a)$ on an element $a \in R$ is the R-complex

$$K^R(a) := \operatorname{Cone}(R \xrightarrow{a} R),$$

and the Koszul complex $K^R(\underline{a})$ on a sequence of elements $\underline{a} = a_1, \ldots, a_n \in R$ is the R-complex

$$K^R(\underline{a}) := K^R(a_1) \otimes_R \cdots \otimes_R K^R(a_n).$$

It is easy to see that $K^R(\underline{a})$ is a complex of finitely generated free R-modules concentrated in degrees $n, \ldots, 0$. Given any R-module M, there is an isomorphism of R-complexes

$$K^R(\underline{a}) \otimes_R M \cong \Sigma^n \operatorname{Hom}_R(K^R(\underline{a}), M),$$

sometimes referred to as the self-duality property of the Koszul complex. Accordingly, we define the Koszul homology of the sequence $\underline{a}$ with coefficients in M by setting

$$H_i(\underline{a}; M) := H_i(K^R(\underline{a}) \otimes_R M) \cong H_{i-n}(\operatorname{Hom}_R(K^R(\underline{a}), M))$$

for every $i \geq 0$.

3. Proofs of the main results. In this section, we study the containment of Koszul homology, Ext modules, Tor modules, local homology and local cohomology in Serre classes.

In the proof of Theorem 1.1 below, we use the straightforward observation that, given elements $\underline{a} = a_1, \ldots, a_n \in R$, a finitely generated R-module N, and an R-module M, if M belongs to a Serre class $\mathcal{S}$, then $H_i(\underline{a}; M) \in \mathcal{S}$, $\operatorname{Ext}^i_R(N, M) \in \mathcal{S}$ and $\operatorname{Tor}^R_i(N, M) \in \mathcal{S}$ for every $i \geq 0$. For a proof, refer to [**4,** Lemma 2.1].

Proof of Theorem 1.1.

(i) $\Rightarrow$ (iii). Set $N = R/\mathfrak{a}$. Let F be a free resolution of $R/\mathfrak{a}$ consisting of finitely generated R-modules. Then, the R-complex $F \otimes_R M$ is isomorphic to an R-complex of the form

$$\cdots \longrightarrow M^{s_2} \xrightarrow{\partial_2} M^{s_1} \xrightarrow{\partial_1} M^{s_0} \longrightarrow 0.$$

We note that

$$\operatorname{Tor}_0^R(R/\mathfrak{a}, M) \cong \operatorname{coker} \partial_1 \cong H_0(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R), \tag{3.1}$$

by assumption. If $s = 0$, then we are done. Suppose that $s \geq 1$. The short exact sequence

$$0 \longrightarrow \operatorname{im} \partial_1 \longrightarrow M^{s_0} \longrightarrow \operatorname{coker} \partial_1 \longrightarrow 0$$

induces the exact sequence

$$H_{i+1}(\underline{a}; \operatorname{coker} \partial_1) \longrightarrow H_i(\underline{a}; \operatorname{im} \partial_1) \longrightarrow H_i(\underline{a}; M^{s_0}).$$

This assumption, together with the display (3.1) imply that the two lateral terms of the above exact sequence belong to $\mathcal{S}_{\mathcal{P}}(R)$; thus, $H_i\left(\underline{a}; \operatorname{im} \partial_1\right) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$. The short exact sequence

$$0 \longrightarrow \ker \partial_1 \longrightarrow M^{s_1} \longrightarrow \operatorname{im} \partial_1 \longrightarrow 0,$$

yields the exact sequence

$$H_{i+1}(\underline{a}; \operatorname{im} \partial_1) \longrightarrow H_i(\underline{a}; \ker \partial_1) \longrightarrow H_i(\underline{a}; M^{s_1}).$$

Therefore, $H_i(\underline{a}; \ker \partial_1) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s-1$. Since $s \geq 1$, we see that $H_0(\underline{a}; \ker \partial_1) \in \mathcal{S}_{\mathcal{P}}(R)$. The short exact sequence

$$0 \longrightarrow \operatorname{im} \partial_2 \longrightarrow \ker \partial_1 \longrightarrow \operatorname{Tor}_1^R(R/\mathfrak{a}, M) \longrightarrow 0, \tag{3.2}$$

implies the exact sequence

$$H_0(\underline{a}; \ker \partial_1) \longrightarrow H_0(\underline{a}; \operatorname{Tor}_1^R(R/\mathfrak{a}, M)) \longrightarrow 0.$$

Therefore, $H_0(\underline{a}; \operatorname{Tor}_1^R(R/\mathfrak{a}, M)) \in \mathcal{S}_{\mathcal{P}}(R)$. However, $\mathfrak{a} \operatorname{Tor}_1^R(R/\mathfrak{a}, M) = 0$; thus,

$$\operatorname{Tor}_1^R(R/\mathfrak{a}, M) \cong H_0(\underline{a}; \operatorname{Tor}_1^R(R/\mathfrak{a}, M)) \in \mathcal{S}_{\mathcal{P}}(R).$$

If $s = 1$, then we are done. Suppose that $s \geq 2$. The short exact sequence (3.2) induces the exact sequence

$$H_{i+1}(\underline{a}; \operatorname{Tor}_1^R(R/\mathfrak{a}, M)) \longrightarrow H_i(\underline{a}; \operatorname{im} \partial_2) \longrightarrow H_i(\underline{a}; \ker \partial_1).$$

It follows that $H_i(\underline{a}; \operatorname{im} \partial_2) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s-1$. The short exact sequence

$$0 \longrightarrow \ker \partial_2 \longrightarrow M^{s_2} \longrightarrow \operatorname{im} \partial_2 \longrightarrow 0,$$

yields the exact sequence

$$H_{i+1}(\underline{a}; \operatorname{im} \partial_2) \longrightarrow H_i(\underline{a}; \ker \partial_2) \longrightarrow H_i(\underline{a}; M^{s_2}).$$

Hence, $H_i(\underline{a}; \ker \partial_2) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s-2$. Since $s \geq 2$, we see that $H_0(\underline{a}; \ker \partial_2) \in \mathcal{S}_{\mathcal{P}}(R)$. The short exact sequence

$$0 \longrightarrow \operatorname{im} \partial_3 \longrightarrow \ker \partial_2 \longrightarrow \operatorname{Tor}_2^R(R/\mathfrak{a}, M) \longrightarrow 0,$$

yields the exact sequence

$$H_0(\underline{a}; \ker \partial_2) \longrightarrow H_0(\underline{a}; \operatorname{Tor}_2^R(R/\mathfrak{a}, M)) \longrightarrow 0.$$

As before, we conclude that

$$\operatorname{Tor}_2^R(R/\mathfrak{a}, M) \cong H_0(\underline{a}; \operatorname{Tor}_2^R(R/\mathfrak{a}, M)) \in \mathcal{S}_{\mathcal{P}}(R).$$

If $s = 2$, then we are done. Proceeding in this manner, we see that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$.

(iii) $\Rightarrow$ (ii). Let L be a finitely generated R-module with $\operatorname{Supp}_R(L) \subseteq V(\mathfrak{a})$. By induction on s, we show that $\operatorname{Tor}_i^R(L, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$. Let $s = 0$. Then, $N \otimes_R M \in \mathcal{S}(R)$. Using Gruson's filtration lemma [**30,** Theorem 4.1], there is a finite filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{t-1} \subseteq L_t = L,$$

such that L_j/L_{j-1} is isomorphic to a quotient of a finite direct sum of copies of N for every $1 \leq j \leq t$. Given any $1 \leq j \leq t$, the exact sequence

$$0 \longrightarrow K_j \longrightarrow N^{r_j} \longrightarrow L_j/L_{j-1} \longrightarrow 0, \tag{3.3}$$

induces the exact sequence

$$N^{r_j} \otimes_R M \longrightarrow (L_j/L_{j-1}) \otimes_R M \longrightarrow 0.$$

It follows that $(L_j/L_{j-1}) \otimes_R M \in \mathcal{S}_{\mathcal{P}}(R)$. Now, the short exact sequence

$$0 \longrightarrow L_{j-1} \longrightarrow L_j \longrightarrow L_j/L_{j-1} \longrightarrow 0, \tag{3.4}$$

yields the exact sequence

$$L_{j-1} \otimes_R M \longrightarrow L_j \otimes_R M \longrightarrow (L_j/L_{j-1}) \otimes_R M.$$

Successively using the above exact sequence, letting $j = 1, \ldots, t$, implies that $L \otimes_R M \in \mathcal{S}_{\mathcal{P}}(R)$.

Now, let $s \geq 1$, and suppose that the results holds for $s - 1$. The induction hypothesis implies that $\mathrm{Tor}_i^R(L, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s - 1$. Hence, it suffices to show that $\mathrm{Tor}_s^R(L, M) \in \mathcal{S}_{\mathcal{P}}(R)$. Given any $1 \leq j \leq t$, the short exact sequence (3.3) induces the exact sequence

$$\mathrm{Tor}_s^R(N^{r_j}, M) \longrightarrow \mathrm{Tor}_s^R(L_j/L_{j-1}, M) \longrightarrow \mathrm{Tor}_{s-1}^R(K_j, M).$$

The induction hypothesis shows that $\mathrm{Tor}_{s-1}^R(K_j, M) \in \mathcal{S}_{\mathcal{P}}(R)$; thus, from the above exact sequence we obtain that $\mathrm{Tor}_s^R(L_j/L_{j-1}, M) \in \mathcal{S}_{\mathcal{P}}(R)$. Now, the short exact sequence (3.4) yields the exact sequence

$$\mathrm{Tor}_s^R(L_{j-1}, M) \longrightarrow \mathrm{Tor}_s^R(L_j, M) \longrightarrow \mathrm{Tor}_s^R(L_j/L_{j-1}, M).$$

A successive use of the above exact sequence, letting $j = 1, \ldots, t$, implies that $\mathrm{Tor}_s^R(L, M) \in \mathcal{S}_{\mathcal{P}}(R)$.

(ii) $\Rightarrow$ (i). The hypothesis implies that $\mathrm{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$. The R-complex $K^R(\underline{a}) \otimes_R M$ is isomorphic to an R-complex of the form

$$0 \longrightarrow M \xrightarrow{\partial_n} M^{t_{n-1}} \longrightarrow \cdots \longrightarrow M^{t_2} \xrightarrow{\partial_2} M^{t_1} \xrightarrow{\partial_1} M \longrightarrow 0.$$

We have

$$H_0(\underline{a}; M) \cong \operatorname{coker} \partial_1 \cong \mathrm{Tor}_0^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R), \tag{3.5}$$

by assumption. If $s = 0$, then we are done. Suppose that $s \geq 1$. The short exact sequence

$$0 \longrightarrow \operatorname{im} \partial_1 \longrightarrow M \longrightarrow \operatorname{coker} \partial_1 \longrightarrow 0,$$

induces the exact sequence

$$\mathrm{Tor}_{i+1}^R(R/\mathfrak{a}, \operatorname{coker} \partial_1) \longrightarrow \mathrm{Tor}_i^R(R/\mathfrak{a}, \operatorname{im} \partial_1) \longrightarrow \mathrm{Tor}_i^R(R/\mathfrak{a}, M).$$

This assumption, along with the display (3.5), implies that the two lateral terms of the above exact sequence belong to $\mathcal{S}_{\mathcal{P}}(R)$; thus, $\operatorname{Tor}_i^R(R/\mathfrak{a}, \operatorname{im}\partial_1) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \le i \le s$. The short exact sequence

$$0 \longrightarrow \ker\partial_1 \longrightarrow M^{t_1} \longrightarrow \operatorname{im}\partial_1 \longrightarrow 0$$

yields the exact sequence

$$\operatorname{Tor}_{i+1}^R(R/\mathfrak{a}, \operatorname{im}\partial_1) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, \ker\partial_1) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, M^{t_1})$$

Therefore, $\operatorname{Tor}_i^R(R/\mathfrak{a}, \ker\partial_1) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \le i \le s-1$. Since $s \ge 1$, we see that $(R/\mathfrak{a}) \otimes_R \ker\partial_1 \in \mathcal{S}_{\mathcal{P}}(R)$. The short exact sequence

$$0 \longrightarrow \operatorname{im}\partial_2 \longrightarrow \ker\partial_1 \longrightarrow H_1(\underline{a}; M) \longrightarrow 0 \tag{3.6}$$

implies the exact sequence

$$(R/\mathfrak{a}) \otimes_R \ker\partial_1 \longrightarrow (R/\mathfrak{a}) \otimes_R H_1(\underline{a}; M) \longrightarrow 0.$$

Therefore, $(R/\mathfrak{a}) \otimes_R H_1(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R)$. However, $\mathfrak{a}H_1(\underline{a}; M) = 0$; thus,

$$H_1(\underline{a}; M) \cong (R/\mathfrak{a}) \otimes_R H_1(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R).$$

If $s = 1$, then we are done. Suppose that $s \ge 2$. The short exact sequence (3.6) induces the exact sequence

$$\operatorname{Tor}_{i+1}^R(R/\mathfrak{a}, H_1(\underline{a}; M)) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, \operatorname{im}\partial_2) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, \ker\partial_1).$$

It follows that $\operatorname{Tor}_i^R(R/\mathfrak{a}, \operatorname{im}\partial_2) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \le i \le s-1$. The short exact sequence

$$0 \longrightarrow \ker\partial_2 \longrightarrow M^{t_2} \longrightarrow \operatorname{im}\partial_2 \longrightarrow 0$$

yields the exact sequence

$$\operatorname{Tor}_{i+1}^R(R/\mathfrak{a}, \operatorname{im}\partial_2) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, \ker\partial_2) \longrightarrow \operatorname{Tor}_i^R(R/\mathfrak{a}, M^{t_2}).$$

Thus, $\operatorname{Tor}_i^R(R/\mathfrak{a}, \ker\partial_2) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \le i \le s-2$. Since $s \ge 2$, we see that $(R/\mathfrak{a}) \otimes_R \ker\partial_2 \in \mathcal{S}_{\mathcal{P}}(R)$. The short exact sequence

$$0 \longrightarrow \operatorname{im}\partial_3 \longrightarrow \ker\partial_2 \longrightarrow H_2(\underline{a}; M) \longrightarrow 0$$

implies the exact sequence

$$(R/\mathfrak{a}) \otimes_R \ker\partial_2 \longrightarrow (R/\mathfrak{a}) \otimes_R H_2(\underline{a}; M) \longrightarrow 0.$$

As before, we conclude that

$$H_2(\underline{a};M) \cong (R/\mathfrak{a}) \otimes_R H_2(\underline{a};M) \in \mathcal{S}_{\mathcal{P}}(R).$$

If $s = 2$, then we are done. Proceeding in this manner, we infer that $H_i(\underline{a};M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$.

Now, assume that the Serre property $\mathcal{P}$ satisfies the condition $\mathfrak{D}_{\mathfrak{a}}$. We first note that, since $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is an $\mathfrak{a}$-torsion R-module, it has an $\widehat{R}^{\mathfrak{a}}$-module structure such that

$$\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \cong \widehat{R}^{\mathfrak{a}} \otimes_R \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \cong \operatorname{Tor}_i^{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}} \otimes_R M),$$

for every $i \geq 0$ both as R-modules and $\widehat{R}^{\mathfrak{a}}$-modules. Moreover, by **[27,** Lemma 2.3], we have

$$H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes_R M),$$

for every $i \geq 0$ both as R-modules and $\widehat{R}^{\mathfrak{a}}$-modules. With this preparation we prove:

(iv) $\Rightarrow$ (iii). We let $N = R/\mathfrak{a}$, and show that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$. First, suppose that R is $\mathfrak{a}$-adically complete. From **[11,** Lemma 2.6], there is a first quadrant spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^R(R/\mathfrak{a}, H_q^{\mathfrak{a}}(M)) \underset{p}{\Longrightarrow} \operatorname{Tor}_{p+q}^R(R/\mathfrak{a}, M). \tag{3.7}$$

The hypothesis implies that $E^2_{p,q} \in \mathcal{S}_{\mathcal{P}}(R)$ for every $p \geq 0$ and $0 \leq q \leq s$. Let $0 \leq i \leq s$. There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^i = \operatorname{Tor}_i^R(R/\mathfrak{a}, M)$$

such that $U^p/U^{p-1} \cong E^{\infty}_{p,i-p}$ for every $0 \leq p \leq i$. Since $E^{\infty}_{p,i-p}$ is a subquotient of $E^2_{p,i-p}$ and $0 \leq i-p \leq s$, we infer that

$$U^p/U^{p-1} \cong E^{\infty}_{p,i-p} \in \mathcal{S}_{\mathcal{P}}(R)$$

for every $0 \leq p \leq i$. A successive use of the short exact sequence

$$0 \longrightarrow U^{p-1} \longrightarrow U^p \longrightarrow U^p/U^{p-1} \longrightarrow 0,$$

letting $p = 0, \ldots, i$, implies that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$.

Now, consider the general case. Since

$$H_i^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes_R M) \cong H_i^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}}),$$

for every $0 \leq i \leq s$, the special case implies that

$$\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \cong \operatorname{Tor}_i^{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}} \otimes_R M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}}),$$

for every $0 \leq i \leq s$. However, $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is $\mathfrak{a}$-torsion; thus, the condition $\mathfrak{D}_{\mathfrak{a}}$ implies that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$.

(ii) $\Rightarrow$ (iv). It follows from the hypothesis that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$. First, suppose that R is $\mathfrak{a}$-adically complete. We argue by induction on s. Let $s = 0$. Then, $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(R)$, whence $H_0^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(R)$ by the condition $\mathfrak{D}_{\mathfrak{a}}$.

Next, let $s \geq 1$, and suppose that the result holds for $s-1$. The induction hypothesis implies that $H_i^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s-1$. Hence, it suffices to show that $H_s^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(R)$. Consider the spectral sequence (3.7). By the hypothesis, $E_{p,q}^2 \in \mathcal{S}_{\mathcal{P}}(R)$ for every $p \in \mathbb{Z}$ and $0 \leq q \leq s-1$. There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^s = \operatorname{Tor}_s^R(R/\mathfrak{a}, M)$$

such that $U^p/U^{p-1} \cong E_{p,s-p}^{\infty}$ for every $0 \leq p \leq s$. Since $\operatorname{Tor}_s^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$, we conclude that

$$E_{0,s}^{\infty} \cong U^0/U^{-1} \cong U^0 \in \mathcal{S}_{\mathcal{P}}(R).$$

Let $r \geq 2$, and consider the differentials

$$E_{r,s-r+1}^r \xrightarrow{d_{r,s-r+1}^r} E_{0,s}^r \xrightarrow{d_{0,s}^r} E_{-r,s+r-1}^r = 0.$$

Since $s-r+1 \leq s-1$, and $E_{r,s-r+1}^r$ is a subquotient of $E_{r,s-r+1}^2$, the hypothesis implies that $E_{r,s-r+1}^r \in \mathcal{S}_{\mathcal{P}}(R)$, and consequently, $\operatorname{im} d_{r,s-r+1}^r \in \mathcal{S}_{\mathcal{P}}(R)$ for every $r \geq 2$. We thus obtain

$$E_{0,s}^{r+1} \cong \ker d_{0,s}^r / \operatorname{im} d_{r,s-r+1}^r = E_{0,s}^r / \operatorname{im} d_{r,s-r+1}^r,$$

and consequently, a short exact sequence

$$0 \longrightarrow \operatorname{im} d_{r,s-r+1}^r \longrightarrow E_{0,s}^r \longrightarrow E_{0,s}^{r+1} \longrightarrow 0. \tag{3.8}$$

There is an integer $r_0 \geq 2$ such that $E_{0,s}^{\infty} \cong E_{0,s}^{r+1}$ for every $r \geq r_0$. It follows that $E_{0,s}^{r_0+1} \in \mathcal{S}_{\mathcal{P}}(R)$. Now, the short exact sequence (3.8) implies that $E_{0,s}^{r_0} \in \mathcal{S}_{\mathcal{P}}(R)$. Using (3.8) inductively, we conclude that

$$H_s^{\mathfrak{a}}(M)/\mathfrak{a}H_s^{\mathfrak{a}}(M) \cong E_{0,s}^2 \in \mathcal{S}_{\mathcal{P}}(R).$$

Therefore, by [**11,** Lemma 2.4] and the condition $\mathfrak{D}_{\mathfrak{a}}$, we get

$$H_s^{\mathfrak{a}}(M) \cong H_0^{\mathfrak{a}}(H_s^{\mathfrak{a}}(M)) \in \mathcal{S}_{\mathcal{P}}(R).$$

Now, consider the general case. Since $\operatorname{Tor}_i^R(R/\mathfrak{a}, M)$ is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(R)$ for every $0 \leq i \leq s$, we deduce that

$$\operatorname{Tor}_i^{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}}/\mathfrak{a}\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}} \otimes_R M) \cong \operatorname{Tor}_i^R(R/\mathfrak{a}, M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}}),$$

for every $0 \leq i \leq s$. However, the special case yields that

$$H_i^{\mathfrak{a}}(M) \cong H_i^{\mathfrak{a}\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}} \otimes_R M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}}),$$

for every $0 \leq i \leq s$. □

The following, special case may be of independent interest.

Corollary 3.1. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Let $\mathcal{P}$ be a Serre property satisfying the condition $\mathfrak{D}_{\mathfrak{a}}$. Then, the following conditions are equivalent*:

(i) $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(R)$.
(ii) $H_0^{\mathfrak{a}}(M) \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$.
(iii) $\widehat{M}^{\mathfrak{a}} \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$.

Proof.

(i) ⇔ (ii). Follows from Theorem 1.1 upon letting $s = 0$.

(ii) ⇒ (iii). By [**26,** Lemma 5.1(i)], the natural homomorphism $H_0^{\mathfrak{a}}(M) \to \widehat{M}^{\mathfrak{a}}$ is surjective. Thus, the result follows.

(iii) ⇒ (i). Since $\widehat{M}^{\mathfrak{a}} \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$, we see that $\widehat{M}^{\mathfrak{a}}/\mathfrak{a}\widehat{M}^{\mathfrak{a}} \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$. However, by [**26,** Theorem 1.1], we have $\widehat{M}^{\mathfrak{a}}/\mathfrak{a}\widehat{M}^{\mathfrak{a}} \cong M/\mathfrak{a}M$. It follows that $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})$. But, $M/\mathfrak{a}M$ is $\mathfrak{a}$-torsion; thus, by the condition $\mathfrak{D}_{\mathfrak{a}}$, we have $M/\mathfrak{a}M \in \mathcal{S}_{\mathcal{P}}(R)$. □

Using similar arguments, we can prove the dual result to Theorem 1.1. It is worth noting that Theorem 1.2 is proven in [**3,** Theorem 2.9]. However, the condition $\mathfrak{C}_{\mathfrak{a}}$ is assumed to be satisfied in all four statements. Here, we only require that the condition $\mathfrak{C}_{\mathfrak{a}}$ be satisfied for the last statement. Moreover, the techniques used there are quite different than those used here.

Proof of Theorem 1.2. The proof is similar to that of Theorem 1.1. However, instead of the spectral sequence (3.7), we use the third quadrant spectral sequence

$$E^2_{p,q} = \operatorname{Ext}^{-p}_R(R/\mathfrak{a}, H^{-q}_{\mathfrak{a}}(M)) \underset{p}{\Longrightarrow} \operatorname{Ext}^{-p-q}_R(R/\mathfrak{a}, M). \qquad \square$$

If we let the integer s exhaust the entire nonzero range of Koszul homology, i.e., $s = n$, then we can effectively combine Theorems 1.1 and 1.2 to obtain the following result, which in turn generalizes [**5,** Corollary 3.1].

Corollary 3.2. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* R, $\underline{a} = a_1, \ldots, a_n$, *and* M *an* R*-module. Let* $\mathcal{P}$ *be a Serre property. Then, the following conditions are equivalent*:

(i) $H_i(\underline{a}; M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every* $0 \leq i \leq n$.

(ii) $\operatorname{Tor}^R_i(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every finitely generated* R*-module* N *with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(iii) $\operatorname{Tor}^R_i(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for some finitely generated* R*-module* N *with* $\operatorname{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq n$.

(iv) $\operatorname{Ext}^i_R(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for every finitely generated* R*-module* N *with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(v) $\operatorname{Ext}^i_R(N, M) \in \mathcal{S}_{\mathcal{P}}(R)$ *for some finitely generated* R*-module* N *with* $\operatorname{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq n$.

Proof.

(i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iv). Since $H_i(\underline{a}; M) = 0$ for every $i > n$, these equivalences follow from Theorems 1.1 and 1.2, respectively.

(i) $\Leftrightarrow$ (iii). Follows from Theorem 1.1.

(i) $\Leftrightarrow$ (v). Follows from Theorem 1.2. $\square$

The following corollaries describe the numerical invariants $\mathcal{P}$-depth and $\mathcal{P}$-width in terms of Koszul homology, local homology and local cohomology.

Corollary 3.3. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* R, $\underline{a} = a_1, \ldots, a_n$, *and* M *an* R*-module. Let* $\mathcal{P}$ *be a Serre property. Then, the following assertions hold*:

(i) $\mathcal{P}\text{-depth}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H_{n-i}(\underline{a}; M) \notin \mathcal{S}_{\mathcal{P}}(R)\}$.

(ii) $\mathcal{P}\text{-width}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H_i(\underline{a}; M) \notin \mathcal{S}_{\mathcal{P}}(R)\}$.

(iii) *We have* $\mathcal{P}\text{-depth}_R(\mathfrak{a}, M) < \infty$ *if and only if* $\mathcal{P}\text{-width}_R(\mathfrak{a}, M) < \infty$. *Moreover, in this case, we have*

$$\sup\{i \geq 0 \mid H_i(\underline{a}; M) \notin \mathcal{S}_{\mathcal{P}}(R)\} + \mathcal{P}\text{-depth}_R(\mathfrak{a}, M) = n.$$

Proof.

(i) and (ii). Follow from Theorems 1.1 and 1.2.

(iii) The first assertion follows from Corollary 3.2. For the second assertion, note that

$$\inf\{i \geq 0 \mid H_{n-i}(\underline{a}; M) \notin \mathcal{S}_{\mathcal{P}}(R)\} = n - \sup\{i \geq 0 \mid H_i(\underline{a}; M) \notin \mathcal{S}_{\mathcal{P}}(R)\}. \quad \square$$

Corollary 3.4. *Let* $\mathfrak{a}$ *be an ideal of* R *and* M *an* R*-module. Let* $\mathcal{P}$ *be a Serre property. Then, the following assertions hold*:

(i) *if* $\mathcal{P}$ *satisfies the condition* $\mathfrak{C}_{\mathfrak{a}}$, *then*

$$\mathcal{P}\text{-depth}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H^i_{\mathfrak{a}}(M) \notin \mathcal{S}_{\mathcal{P}}(R)\}.$$

(ii) *If* $\mathcal{P}$ *satisfies the condition* $\mathfrak{D}_{\mathfrak{a}}$, *then*

$$\mathcal{P}\text{-width}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H^{\mathfrak{a}}_i(M) \notin \mathcal{S}_{\mathcal{P}}(\widehat{R}^{\mathfrak{a}})\}.$$

(iii) *If* $\mathcal{P}$ *satisfies both conditions* $\mathfrak{C}_{\mathfrak{a}}$ *and* $\mathfrak{D}_{\mathfrak{a}}$, *and* $\mathcal{P}\text{-depth}_R(\mathfrak{a}, M) < \infty$, *then*

$$\mathcal{P}\text{-depth}_R(\mathfrak{a}, M) + \mathcal{P}\text{-width}_R(\mathfrak{a}, M) \leq \operatorname{ara}(\mathfrak{a}).$$

Proof.

(i) and (ii). Follow from Theorems 1.1 and 1.2.

(iii) Clear by (i), (ii) and Corollary 3.3 (iii). $\square$

4. Noetherianness and Artinianness. In this section, we apply the results of Section 3 to obtain some characterizations of Noetherian local homology modules and Artinian local cohomology modules. The following result, together with Corollary 4.4, generalizes [**24,** Propositions 7.1, 7.2, 7.4] when applied to modules.

Proposition 4.1. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* R, $\underline{a} = a_1, \ldots, a_n$ *and* M *an* R*-module. Then, the following assertions are equivalent for any given* $s \geq 0$:

(i) $H_i(\underline{a}; M)$ *is a finitely generated* R*-module for every* $0 \leq i \leq s$.

(ii) $\mathrm{Tor}_i^R(N, M)$ *is a finitely generated* R*-module for every finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iii) $\mathrm{Tor}_i^R(N, M)$ *is a finitely generated* R*-module for some finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iv) $H_i^{\mathfrak{a}}(M)$ *is a finitely generated* $\widehat{R}^{\mathfrak{a}}$*-module for every* $0 \leq i \leq s$.

Proof. Obvious in view of Example 2.6 (ii) and Theorem 1.1. □

The next corollary provides a characterization of Noetherian local homology modules in its full generality.

Corollary 4.2. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* R, $\underline{a} = a_1, \ldots, a_n$, *and* M *an* R*-module. Then, the following assertions are equivalent*:

(i) $H_i^{\mathfrak{a}}(M)$ *is a finitely generated* $\widehat{R}^{\mathfrak{a}}$*-module for every* $i \geq 0$.

(ii) $H_i(\underline{a}; M)$ *is a finitely generated* R*-module for every* $0 \leq i \leq n$.

(iii) $H_i(\underline{a}; M)$ *is a finitely generated* R*-module for every* $0 \leq i \leq \mathrm{hd}(\mathfrak{a}, M)$.

(iv) $\mathrm{Tor}_i^R(N, M)$ *is a finitely generated* R*-module for every finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(v) $\mathrm{Tor}_i^R(N, M)$ *is a finitely generated* R*-module for some finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq \mathrm{hd}(\mathfrak{a}, M)$.

(vi) $\mathrm{Ext}_R^i(N, M)$ *is a finitely generated* R*-module for every finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(vii) $\mathrm{Ext}_R^i(N, M)$ *is a finitely generated* R*-module for some finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq n$.

Proof.

(ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). Follows from Corollary 3.2.

(iii) $\Leftrightarrow$ (v). Follows from Proposition 4.1 upon setting $s = \mathrm{hd}(\mathfrak{a}, M)$.

(i) $\Leftrightarrow$ (iii). Since $H_i^{\mathfrak{a}}(M) = 0$ for every $i > \mathrm{hd}(\mathfrak{a}, M)$, the result follows from Proposition 4.1.

(i) $\Leftrightarrow$ (iv). Follows from Proposition 4.1. $\square$

It should be noted that a slightly weaker version of the following result has been proved in [**21,** Theorem 5.5] by using a different method.

Proposition 4.3. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Then, the following assertions are equivalent for any given $s \geq 0$:*

- (i) *$H_{n-i}(\underline{a}; M)$ is an Artinian R-module for every $0 \leq i \leq s$.*
- (ii) *$\mathrm{Ext}_R^i(N, M)$ is an Artinian R-module for every finitely generated R-module N with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, and for every $0 \leq i \leq s$.*
- (iii) *$\mathrm{Ext}_R^i(N, M)$ is an Artinian R-module for some finitely generated R-module N with $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, and for every $0 \leq i \leq s$.*
- (iv) *$H_{\mathfrak{a}}^i(M)$ is an Artinian R-module for every $0 \leq i \leq s$.*

Proof. Obvious in view of Example 2.8 (ii) and Theorem 1.2. $\square$

The next corollary provides a characterization of Artinian local cohomology modules in its full generality.

Corollary 4.4. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Then, the following assertions are equivalent:*

- (i) *$H_{\mathfrak{a}}^i(M)$ is an Artinian R-module for every $i \geq 0$.*
- (ii) *$H_i(\underline{a}; M)$ is an Artinian R-module for every $0 \leq i \leq n$.*
- (iii) *$H_{n-i}(\underline{a}; M)$ is an Artinian R-module for every $0 \leq i \leq \mathrm{cd}(\mathfrak{a}, M)$.*
- (iv) *$\mathrm{Ext}_R^i(N, M)$ is an Artinian R-module for every finitely generated R-module N with $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, and for every $i \geq 0$.*

(v) $\mathrm{Ext}^i_R(N, M)$ *is an Artinian R-module for some finitely generated R-module N with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq \mathrm{cd}(\mathfrak{a}, M)$.

(vi) $\mathrm{Tor}^R_i(N, M)$ *is an Artinian R-module for every finitely generated R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(vii) $\mathrm{Tor}^R_i(N, M)$ *is an Artinian R-module for some finitely generated R-module N with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq n$.

Proof.

(ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii). Follows from Corollary 3.2.

(iii) $\Leftrightarrow$ (v). Follows from Proposition 4.3 upon setting $s = \mathrm{cd}(\mathfrak{a}, M)$.

(i) $\Leftrightarrow$ (iii). Since $H^i_{\mathfrak{a}}(M) = 0$ for every $i > \mathrm{cd}(\mathfrak{a}, M)$, the result follows from Proposition 4.3.

(i) $\Leftrightarrow$ (iv). Follows from Proposition 4.3. □

5. Vanishing results. In this section, we treat the vanishing results. Since the property of being zero is a Serre property that satisfies the condition $\mathfrak{D}_{\mathfrak{a}}$, we obtain the next result.

Proposition 5.1. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of R,* $\underline{a} = a_1, \ldots, a_n$, *and M an R-module. Then, the following assertions are equivalent for any given* $s \geq 0$:

(i) $H_i(\underline{a}; M) = 0$ *for every* $0 \leq i \leq s$.

(ii) $\mathrm{Tor}^R_i(N, M) = 0$ *for every R-module N with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iii) $\mathrm{Tor}^R_i(N, M) = 0$ *for some finitely generated R-module N with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iv) $H^{\mathfrak{a}}_i(M) = 0$ *for every* $0 \leq i \leq s$.

Proof. Immediate from Theorem 1.1. For part (ii), note that every module is a direct limit of its finitely generated submodules, and the Tor functor commutes with direct limits. □

The following result is proven in **[28,** Theorem 4.4] using a different method; here, it is an immediate by-product of Proposition 5.1.

Corollary 5.2. *Let M be an R-module and N a finitely generated R-module. Then, the following conditions are equivalent for any given* $s \geq 0$:

(i) $\operatorname{Tor}_i^R(N, M) = 0$ *for every* $0 \leq i \leq s$.
(ii) $\operatorname{Tor}_i^R(R/\operatorname{ann}_R(N), M) = 0$ *for every* $0 \leq i \leq s$.

Proof. Immediate from Proposition 5.1. □

We observe that Corollary 5.3 below generalizes [**30,** Corollary 4.3], which states that, if N is a faithful finitely generated R-module, then $N \otimes_R M = 0$ if and only if $M = 0$.

Corollary 5.3. *Let M be an R-module and N a finitely generated R-module. Then, the following conditions are equivalent*:

(i) $M \otimes_R N = 0$.
(ii) $M = \operatorname{ann}_R(N)M$.

In particular, we have

$$\operatorname{Supp}_R(M \otimes_R N) = \operatorname{Supp}_R(M/\operatorname{ann}_R(N)M).$$

Proof. For the equivalence of (i) and (ii), let $s = 0$ in Corollary 5.2. For the second part, note that, given any $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $\operatorname{ann}_R(N)_\mathfrak{p} = \operatorname{ann}_{R_\mathfrak{p}}(N_\mathfrak{p})$, and thus, $(M \otimes_R N)_\mathfrak{p} = 0$ if and only if $(M/\operatorname{ann}_R(N)M)_\mathfrak{p} = 0$. □

The support formula in Corollary 5.3 generalizes the well-known formula

$$\operatorname{Supp}_R(M \otimes_R N) = \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N),$$

which holds whenever M and N are both assumed to be finitely generated.

Using Corollary 3.4, we intend to obtain two somewhat known descriptions of the numerical invariant $\operatorname{width}_R(\mathfrak{a}, M)$ in terms of the Koszul homology and local homology. However, we need some generalizations of the five-term exact sequences. To the best of our knowledge, the only place where such generalizations may be found is [**23,** Corollaries 10.32, 10.34]. However, the statements there are incorrect and no proof is presented. Hence, due to the lack of a suitable reference, we deem it appropriate to include the correct statements with proofs for the convenience of the reader.

Lemma 5.4. *Let $E^2_{p,q} \underset{p}{\Rightarrow} H_{p+q}$ be a spectral sequence. Then, the following assertions hold*:

(i) *if $E^2_{p,q} \underset{p}{\Rightarrow} H_{p+q}$ is first quadrant and there is an integer $n \geq 1$ such that $E^2_{p,q} = 0$ for every $q \leq n-2$, then there is a five-term exact sequence*

$$H_{n+1} \longrightarrow E^2_{2,n-1} \longrightarrow E^2_{0,n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

(ii) *If $E^2_{p,q} \underset{p}{\Rightarrow} H_{p+q}$ is third quadrant and there is an integer $n \geq 1$ such that $E^2_{p,q} = 0$ for every $q \geq 2-n$, then there is a five-term exact sequence*

$$0 \longrightarrow E^2_{-1,1-n} \longrightarrow H_{-n} \longrightarrow E^2_{0,-n} \longrightarrow E^2_{-2,1-n} \longrightarrow H_{-n-1}.$$

Proof.

(i) Consider the following homomorphisms

$$0 = E^2_{4,n-2} \xrightarrow{d^2_{4,n-2}} E^2_{2,n-1} \xrightarrow{d^2_{2,n-1}} E^2_{0,n} \xrightarrow{d^2_{0,n}} E^2_{-2,n+1} = 0.$$

We thus have

$$E^3_{2,n-1} \cong \ker d^2_{2,n-1} / \operatorname{im} d^2_{4,n-2} \cong \ker d^2_{2,n-1}$$

and

$$E^3_{0,n} \cong \ker d^2_{0,n} / \operatorname{im} d^2_{2,n-1} = \operatorname{coker} d^2_{2,n-1}.$$

Let $r \geq 3$. Consider the following homomorphisms

$$0 = E^r_{r+2,n-r} \xrightarrow{d^r_{r+2,n-r}} E^r_{2,n-1} \xrightarrow{d^r_{2,n-1}} E^r_{2-r,n+r-2} = 0.$$

We thus have

$$E^{r+1}_{2,n-1} \cong \ker d^r_{2,n-1} / \operatorname{im} d^r_{r+2,n-r} \cong E^r_{2,n-1}.$$

Therefore,

$$\ker d^2_{2,n-1} \cong E^3_{2,n-1} \cong E^4_{2,n-1} \cong \cdots \cong E^\infty_{2,n-1}.$$

Furthermore, consider the following homomorphisms

$$0 = E^r_{r,n-r+1} \xrightarrow{d^r_{r,n-r+1}} E^r_{0,n} \xrightarrow{d^r_{0,n}} E^r_{-r,n+r-1} = 0.$$

We thus have

$$E^{r+1}_{0,n} \cong \ker d^r_{0,n} / \operatorname{im} d^r_{r,n-r+1} \cong E^r_{0,n}.$$

Therefore,

$$\operatorname{coker} d^2_{2,n-1} \cong E^3_{0,n} \cong E^4_{0,n} \cong \cdots \cong E^\infty_{0,n}.$$

Hence, we obtain the following exact sequence

$$(5.1) \qquad 0 \longrightarrow E^\infty_{2,n-1} \longrightarrow E^2_{2,n-1} \xrightarrow{d^2_{2,n-1}} E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow 0.$$

There is a finite filtration

$$0 = U^{-1} \subseteq U^0 \subseteq \cdots \subseteq U^{n+1} = H_{n+1}$$

such that $E^\infty_{p,n+1-p} \cong U^p/U^{p-1}$ for every $0 \le p \le n+1$. If $3 \le p \le n+1$, then $n+1-p \le n-2$, so $E^\infty_{p,n+1-p} = 0$. It follows that

$$U^2 = U^3 = \cdots = U^{n+1}.$$

Now, since $E^\infty_{2,n-1} \cong U^2/U^1$, we get an exact sequence

$$H_{n+1} \longrightarrow E^\infty_{2,n-1} \longrightarrow 0.$$

Splicing this exact sequence to the exact sequence (5.1), we obtain the following exact sequence

$$(5.2) \qquad H_{n+1} \longrightarrow E^2_{2,n-1} \xrightarrow{d^2_{2,n-1}} E^2_{0,n} \longrightarrow E^\infty_{0,n} \longrightarrow 0.$$

On the other hand, there is a finite filtration

$$0 = V^{-1} \subseteq V^0 \subseteq \cdots \subseteq V^n = H_n$$

such that $E^\infty_{p,n-p} \cong V^p/V^{p-1}$ for every $0 \le p \le n$. If $2 \le p \le n$, then $n-p \le n-2$; thus, $E^\infty_{p,n-p} = 0$. It follows that

$$V^1 = V^2 = \cdots = V^n.$$

Since $E^\infty_{0,n} \cong V^0/V^{-1} = V^0$ and $E^\infty_{1,n-1} \cong V^1/V^0$, we get the short exact sequence

$$0 \longrightarrow E^\infty_{0,n} \longrightarrow H_n \longrightarrow E^\infty_{1,n-1} \longrightarrow 0.$$

Splicing this short exact sequence to the exact sequence (5.2) yields the exact sequence

$$(5.3) \qquad H_{n+1} \longrightarrow E^2_{2,n-1} \xrightarrow{d^2_{2,n-1}} E^2_{0,n} \longrightarrow H_n \longrightarrow E^{\infty}_{1,n-1} \longrightarrow 0.$$

Let $r \geq 2$, and consider the homomorphisms

$$0 = E^r_{r+1,n-r} \xrightarrow{d^r_{r+1,n-r}} E^r_{1,n-1} \xrightarrow{d^r_{1,n-1}} E^r_{1-r,n+r-2} = 0.$$

We thus have

$$E^{r+1}_{1,n-1} \cong \ker d^r_{1,n-1} / \operatorname{im} d^r_{r+1,n-r} \cong E^r_{1,n-1}.$$

Therefore,

$$E^2_{1,n-1} \cong E^3_{1,n-1} \cong \cdots \cong E^{\infty}_{1,n-1}.$$

Hence, from the exact sequence (5.3), we obtain the desired exact sequence

$$H_{n+1} \longrightarrow E^2_{2,n-1} \xrightarrow{d^2_{2,n-1}} E^2_{0,n} \longrightarrow H_n \longrightarrow E^2_{1,n-1} \longrightarrow 0.$$

(ii) Similar to (i). □

For the next result, we need to recall the notion of coassociated prime ideals. Given an R-module M, a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ is said to be a coassociated prime ideal of M if $\mathfrak{p} = \operatorname{ann}_R(M/N)$ for some submodule N of M such that M/N is an Artinian R-module. The set of coassociated prime ideals of M is denoted by $\operatorname{Coass}_R(M)$.

Corollary 5.5. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Then, the following assertions hold*:

(i)

$$\operatorname{width}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H_i(\underline{a}; M) \neq 0\} = \inf\{i \geq 0 \mid H^{\mathfrak{a}}_i(M) \neq 0\}.$$

(ii) $\operatorname{Tor}^R_{\operatorname{width}_R(\mathfrak{a},M)}(R/\mathfrak{a}, M) \cong (R/\mathfrak{a}) \otimes_R H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)$.

(iii) $\Lambda^{\mathfrak{a}}(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) \cong \varprojlim_n \operatorname{Tor}^R_{\operatorname{width}_R(\mathfrak{a},M)}(R/\mathfrak{a}^n, M)$.

(iv)

$$\operatorname{Coass}_R(\operatorname{Tor}^R_{\operatorname{width}_R(\mathfrak{a},M)}(R/\mathfrak{a}, M)) = \operatorname{Coass}_R(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) \cap V(\mathfrak{a}).$$

Proof.

(i) Follows immediately from Corollaries 3.3 and 3.4.

(ii) Consider the first quadrant spectral sequence

$$E^2_{p,q} = \operatorname{Tor}^R_p(R/\mathfrak{a}, H^{\mathfrak{a}}_q(M)) \underset{p}{\Longrightarrow} \operatorname{Tor}^R_{p+q}(R/\mathfrak{a}, M).$$

Let $n = \operatorname{width}_R(\mathfrak{a}, M)$. Then, by (i), $E^2_{p,q} = 0$ for every $q \leq n-1$. Now, Lemma 5.4 (i) gives the exact sequence

$$\begin{aligned}\operatorname{Tor}^R_{n+1}(R/\mathfrak{a}, M) \longrightarrow \operatorname{Tor}^R_2(R/\mathfrak{a}, H^{\mathfrak{a}}_{n-1}(M)) \longrightarrow (R/\mathfrak{a}) \otimes_R H^{\mathfrak{a}}_n(M) \\ \longrightarrow \operatorname{Tor}^R_n(R/\mathfrak{a}, M) \longrightarrow \operatorname{Tor}^R_1(R/\mathfrak{a}, H^{\mathfrak{a}}_{n-1}(M)) \longrightarrow 0.\end{aligned}$$

However,

$$\operatorname{Tor}^R_2(R/\mathfrak{a}, H^{\mathfrak{a}}_{n-1}(M)) = 0 = \operatorname{Tor}^R_1(R/\mathfrak{a}, H^{\mathfrak{a}}_{n-1}(M));$$

thus,

$$(R/\mathfrak{a}) \otimes_R H^{\mathfrak{a}}_n(M) \cong \operatorname{Tor}^R_n(R/\mathfrak{a}, M).$$

(iii) Using (ii) and the facts that $H^{\mathfrak{a}^n}_i(M) \cong H^{\mathfrak{a}}_i(M)$ and $\operatorname{width}_R(\mathfrak{a}^n, M) = \operatorname{width}_R(\mathfrak{a}, M)$, for every n, $i \geq 0$, we obtain:

$$\begin{aligned}\Lambda^{\mathfrak{a}}(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) &= \varprojlim_n (H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)/\mathfrak{a}^n H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) \\ &\cong \varprojlim_n \operatorname{Tor}^R_{\operatorname{width}_R(\mathfrak{a},M)}(R/\mathfrak{a}^n, M).\end{aligned}$$

(iv) Using (ii) and [**31,** Theorem 1.21], we have

$$\begin{aligned}&\operatorname{Coass}_R(\operatorname{Tor}^R_{\operatorname{width}_R(\mathfrak{a},M)}(R/\mathfrak{a},M)) = \operatorname{Coass}_R((R/\mathfrak{a}) \otimes_R H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) \\ &\quad = \operatorname{Supp}_R(R/\mathfrak{a}) \cap \operatorname{Coass}_R(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)) \\ &\quad = V(\mathfrak{a}) \cap \operatorname{Coass}_R(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M)).\end{aligned}$$

□

Note that Corollary 5.5 (iii) is proven in [**27,** Proposition 2.5] by deploying a different method. On the other hand, in parallel with Corollary 5.10 (iii) below, one may wonder if intersection with $V(\mathfrak{a})$ in Corollary 5.5 (iv) is redundant, in other words, $\operatorname{Coass}_R(H^{\mathfrak{a}}_{\operatorname{width}_R(\mathfrak{a},M)}(M))$ may be contained in $V(\mathfrak{a})$. However, the following example shows that this is not the case in general.

Example 5.6. Let $R := \mathbb{Q}[X,Y]_{(X,Y)}$ and $\mathfrak{m} := (X,Y)R$. Then, $\operatorname{width}_R(\mathfrak{m}, R) = 0$ and

$$H_0^{\mathfrak{m}}(R) \cong \widehat{R}^{\mathfrak{m}} \cong \mathbb{Q}[[X,Y]].$$

For each $n \in \mathbb{Z}$, let $\mathfrak{p}_n := (X - nY)R$. Then, it is easy to see that $R/\mathfrak{p}_n \cong \mathbb{Q}[Y]_{(Y)}$, and thus, it is not a complete local ring. By **[33,** Theorem 2.4],

$$\begin{aligned}\operatorname{Coass}_R(H_0^{\mathfrak{m}}(R)) &= \operatorname{Coass}_R(\widehat{R}^{\mathfrak{m}}) \\ &= \{\mathfrak{m}\} \cup \{\mathfrak{p} \in \operatorname{Spec} R \mid R/\mathfrak{p} \text{ is not a complete local ring}\}.\end{aligned}$$

Hence, $\operatorname{Coass}_R(H_0^{\mathfrak{m}}(R))$ is not a finite set, while $\operatorname{Coass}_R(H_0^{\mathfrak{m}}(R)) \cap V(\mathfrak{m}) = \{\mathfrak{m}\}$, in particular, $\operatorname{Coass}_R(H_0^{\mathfrak{m}}(R)) \not\subseteq V(\mathfrak{m})$.

Since the property of being zero is a Serre property that satisfies the condition $\mathfrak{C}_{\mathfrak{a}}$, we obtain the following result.

Proposition 5.7. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Then, the following assertions are equivalent for any given $s \geq 0$:*

(i) $H_{n-i}(\underline{a}; M) = 0$ *for every* $0 \leq i \leq s$.

(ii) $\operatorname{Ext}^i_R(N, M) = 0$ *for every finitely generated R-module N with* $\operatorname{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iii) $\operatorname{Ext}^i_R(N, M) = 0$ *for some finitely generated R-module N with* $\operatorname{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq s$.

(iv) $H^i_{\mathfrak{a}}(M) = 0$ *for every* $0 \leq i \leq s$.

Proof. Immediate from Theorem 1.2. □

The following result is proven in **[28,** Theorem 3.2] using a different method. Here, it is an immediate by-product of Proposition 5.7.

Corollary 5.8. *Let M be an R-module and N a finitely generated R-module. Then, the following conditions are equivalent for any given $s \geq 0$:*

(i) $\operatorname{Ext}^i_R(N, M) = 0$ *for every* $0 \leq i \leq s$.

(ii) $\operatorname{Ext}^i_R(R/\operatorname{ann}_R(N), M) = 0$ *for every* $0 \leq i \leq s$.

Proof. Immediate from Proposition 5.7. □

The following, special case may be of independent interest.

Corollary 5.9. *Let M be an R-module and N a finitely generated R-module. Then, the following conditions are equivalent*:

(i) $\mathrm{Hom}_R(N, M) = 0$.
(ii) $(0 :_M \mathrm{ann}_R(N)) = 0$.

Proof. Let $s = 0$ in Corollary 5.8. □

It is easy to deduce from Corollary 5.9 that, given a finitely generated R-module N, we have $\mathrm{Hom}_R(N, M) \neq 0$ if and only if there are elements $x \in N$ and $0 \neq y \in M$ with $\mathrm{ann}_R(x) \subseteq \mathrm{ann}_R(y)$, known as the Hom vanishing lemma [**7,** page 11].

We state the dual result to Corollary 5.5 for the sake of integrity and completeness. Corollary 5.10 (ii) and (iii) below are stated in [**16,** Proposition 1.1], and it is only specified that part (ii) can be deduced from a spectral sequence. In addition, a proof of this result is offered in [**20,** Corollary 2.3] by using different techniques.

Corollary 5.10. *Let $\mathfrak{a} = (a_1, \ldots, a_n)$ be an ideal of R, $\underline{a} = a_1, \ldots, a_n$, and M an R-module. Then, the following assertions hold*:

(i) $\mathrm{depth}_R(\mathfrak{a}, M) = \inf\{i \geq 0 \mid H_{n-i}(\underline{a}; M) \neq 0\} = \inf\{i \geq 0 \mid H^i_{\mathfrak{a}}(M) \neq 0\}$.
(ii) $\mathrm{Ext}_R^{\mathrm{depth}_R(\mathfrak{a},M)}(R/\mathfrak{a}, M) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H_{\mathfrak{a}}^{\mathrm{depth}_R(\mathfrak{a},M)}(M))$.
(iii) $\mathrm{Ass}_R(\mathrm{Ext}_R^{\mathrm{depth}_R(\mathfrak{a},M)}(R/\mathfrak{a}, M)) = \mathrm{Ass}_R(H_{\mathfrak{a}}^{\mathrm{depth}_R(\mathfrak{a},M)}(M))$.

Proof.

(i) Follows immediately from Corollaries 3.3 and 3.4.

(ii) Consider the third quadrant spectral sequence

$$E^2_{p,q} = \mathrm{Ext}_R^{-p}(R/\mathfrak{a}, H_{\mathfrak{a}}^{-q}(M)) \underset{p}{\Longrightarrow} \mathrm{Ext}_R^{-p-q}(R/\mathfrak{a}, M),$$

and use Lemma 5.4 (ii).

(iii) Follows from (ii). □

Finally, we present the following, comprehensive vanishing result.

Corollary 5.11. *Let* $\mathfrak{a} = (a_1, \ldots, a_n)$ *be an ideal of* R, $\underline{a} = a_1, \ldots, a_n$ *and* M *an* R*-module. Then, the following assertions are equivalent*:

(i) $H_i(\underline{a}; M) = 0$ *for every* $0 \leq i \leq n$.

(ii) $H_i(\underline{a}; M) = 0$ *for every* $0 \leq i \leq \mathrm{hd}(\mathfrak{a}, M)$.

(iii) $H_i(\underline{a}; M) = 0$ *for every* $n - \mathrm{cd}(\mathfrak{a}, M) \leq i \leq n$.

(iv) $\mathrm{Tor}_i^R(N, M) = 0$ *for every* R*-module* N *with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(v) $\mathrm{Tor}_i^R(N, M) = 0$ *for some finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq \mathrm{hd}(\mathfrak{a}, M)$.

(vi) $\mathrm{Ext}^i_R(N, M) = 0$ *for every finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) \subseteq V(\mathfrak{a})$, *and for every* $i \geq 0$.

(vii) $\mathrm{Ext}^i_R(N, M) = 0$ *for some finitely generated* R*-module* N *with* $\mathrm{Supp}_R(N) = V(\mathfrak{a})$, *and for every* $0 \leq i \leq \mathrm{cd}(\mathfrak{a}, M)$.

(viii) $H_i^{\mathfrak{a}}(M) = 0$ *for every* $i \geq 0$.

(ix) $H^i_{\mathfrak{a}}(M) = 0$ *for every* $i \geq 0$.

Proof. Follows from Propositions 5.1 and 5.7. □

The next corollary is proven in [**27,** Corollary 1.7]. However, it is an immediate consequence of the results thus far obtained.

Corollary 5.12. *Let* $\mathfrak{a}$ *be an ideal of* R *and* M *an* R*-module. Then,* $\mathrm{depth}_R(\mathfrak{a}, M) < \infty$ *if and only if* $\mathrm{width}_R(\mathfrak{a}, M) < \infty$. *Moreover, in this case, we have*

$$\mathrm{depth}_R(\mathfrak{a}, M) + \mathrm{width}_R(\mathfrak{a}, M) \leq \mathrm{ara}(\mathfrak{a}).$$

Proof. Clear by Corollary 3.4 (iii). □

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Alzahra University, Department of Mathematics, Vanak 19834, Tehran, Iran and Institute for Research in Fundamental Sciences (IPM), School of Mathematics, P.O. Box 19395-5746, Tehran, Iran
Email address: **kdivaani@ipm.ir**

Shahid Beheshti University, Department of Mathematics, G.C., 1983963113 Evin, Tehran, Iran
Email address: **h.faridian@yahoo.com**

Shahid Beheshti University, Department of Mathematics, G.C., 1983963113 Evin, Tehran, Iran
Email address: **mtousi@ipm.ir**