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CUNTZ-PIMSNER ALGEBRAS OF GROUP REPRESENTATIONS

VALENTIN DEACONU

ABSTRACT. Given a locally compact group G and a unitary representation $\rho : G \to U(\mathcal{H})$ on a Hilbert space $\mathcal{H}$, we construct a C^*-correspondence $\mathcal{E}(\rho) = \mathcal{H} \otimes_{\mathbb{C}} C^*(G)$ over $C^*(G)$ and study the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{E}(\rho)}$. We prove that, for G compact, $\mathcal{O}_{\mathcal{E}(\rho)}$ is strongly Morita equivalent to a graph C^*-algebra. If λ is the left regular representation of an infinite, discrete and amenable group G, we show that $\mathcal{O}_{\mathcal{E}(\lambda)}$ is simple and purely infinite, with the same K-theory as $C^*(G)$. If G is compact abelian, any representation decomposes into characters and determines a skew product graph. We illustrate with several examples, and we compare $\mathcal{E}(\rho)$ with the crossed product C^*-correspondence.

1. Introduction. In a seminal paper **[10]**, Kumjian studied the Cuntz-Pimsner algebra associated to a faithful representation π of a separable unital C^*-algebra A on a Hilbert space $\mathcal{H}$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$, where $\mathcal{K}(\mathcal{H})$ denotes the set of compact operators. He considered the C^*-correspondence $\mathcal{E} = \mathcal{H} \otimes_{\mathbb{C}} A$ over A with natural structure and proved that $\mathcal{O}_{\mathcal{E}}$ is simple and purely infinite. Moreover, its isomorphism class depends only upon the K-theory of A and the class of the unit $[1_A]$. In **[1]**, the authors used this type of construction to prove that any order-two automorphism of the K-theory of a unital Kirchberg algebra A satisfying UCT with $[1_A] = 0$ in $K_0(A)$ lifts to an order-two automorphism of A. For more about lifting automorphisms of K-groups to Kirchberg algebras, see **[8, 15]**.

In this paper, we study a similar C^*-correspondence $\mathcal{E}(\rho)$ over $A = C^*(G)$, the C^*-algebra of a locally compact group G, where the representation π of A is obtained by integrating a unitary representation $\rho : G \to U(\mathcal{H})$. In our setting, π is not necessarily faithful, and

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the intersection of $\pi(A)$ with the compact operators is not necessarily trivial. In particular, the left multiplication on $\mathcal{E}(\rho)$ may not be injective, and the resulting Cuntz-Pimsner algebra may not be simple or purely infinite. For compact groups, we prove that $\mathcal{O}_{\mathcal{E}}$ is strongly Morita equivalent to a graph C^*-algebra. Since any representation decomposes into a direct sum of irreducible representations, it suffices to study C^*-correspondences arising from these representations. We illustrate how basic operations on representations reflect on the associated graphs.

Graphs associated to representations of finite or compact groups were already considered in the work of McKay [**11**] and of Mann, Raeburn and Sutherland [**12**]. Given a representation ρ of a group G with set of irreducible representations $\widehat{G} = \{\pi_j\}$, they considered the graph with vertex set $\widehat{G}$ and incidence matrix $[m_{jk}]$, where

$$\rho \otimes \pi_j = \bigoplus_k m_{jk}\pi_k.$$

This kind of graph has connections with Doplicher-Roberts algebras, see [**2, 12**]. The graphs obtained from our $\mathcal{E}(\rho)$ are, in general, different from these graphs, since they may have sources.

For G infinite, discrete and amenable, it is known that the left regular representation λ extends to a faithful representation of $C^*(G)$. Since $C^*(G)$ is unital and the intersection with the compacts is trivial, it follows from [**10**] that the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{E}(\lambda)}$ is simple, purely infinite and KK-equivalent with $C^*(G)$. It is a challenge to understand the Cuntz-Pimsner algebra associated with any representation of an arbitrary group. Sometimes there are connections with C^*-algebras of topological graphs, as in the case $G = \mathbb{Z}$.

We compare the C^*-correspondence $\mathcal{E}(\rho)$ with the crossed product C^*-correspondence $\mathcal{D}(\rho)$ associated to a group action, see [**2**]. Recall that a representation of dimension $n \geq 2$ determines a quasi-free action on the Cuntz algebra $\mathcal{O}_n$ such that $\mathcal{O}_n \rtimes G \cong \mathcal{O}_{\mathcal{D}(\rho)}$. In the case where G is abelian, we review some examples studied by Kumjian and Kishimoto [**9**] and by Katsura, see [**7**], from a new viewpoint.

For more C^*-correspondences over group C^*-algebras, also see the recent preprint [**5**].

2. The C^*-correspondence of a group representations. Let G be a (second countable) locally compact group. A unitary group representation is a homomorphism $\rho : G \to U(\mathcal{H})$, where $\mathcal{H}$ is a (separable) Hilbert space such that, for any fixed $\xi \in \mathcal{H}$, the map $g \to \rho(g)\xi$ is continuous. The left regular representation is

$$\lambda : G \longrightarrow U(L^2(G)), \quad \lambda(g)\xi(h) = \xi(g^{-1}h).$$

A representation $\rho : G \to U(\mathcal{H})$ extends by integration to

$$\pi = \pi_\rho : C^*(G) \longrightarrow \mathcal{L}(\mathcal{H})$$

such that

$$\pi(f)\xi = \int_G f(t)\rho(t)\xi\, dt \quad \text{for } f \in L^1(G),\ \xi \in \mathcal{H}.$$

Definition 2.1. The *C^*-correspondence over $C^*(G)$ of a representation ρ* is $\mathcal{E} = \mathcal{E}(\rho) = \mathcal{H} \otimes_{\mathbb{C}} C^*(G)$ where, for $\xi, \eta \in \mathcal{H}$ and $a, b \in C^*(G)$, the inner product is given by

$$\langle \xi \otimes a, \eta \otimes b\rangle = \langle \xi, \eta\rangle a^*b,$$

and the right and left multiplications are

$$(\xi \otimes a) \cdot b = \xi \otimes ab, \qquad a \cdot (\xi \otimes b) = \pi_\rho(a)\xi \otimes b.$$

Here $\langle \xi, \eta\rangle$ denotes the *right-linear inner product in $\mathcal{H}$.*

It is easy to see that equivalent representations determine isomorphic C^*-correspondences.

Theorem 2.2. *If G is a compact group and $\rho : G \to U(\mathcal{H})$ is any unitary representation, then $\mathcal{O}_{\mathcal{E}(\rho)}$ is strongly Morita equivalent (SME) to a graph C^*-algebra. If $\pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{H})$ is injective, then the graph has no sources. If $\rho \cong \rho_1 \oplus \rho_2$, then the incidence matrix for the graph of ρ is the sum of incidence matrices for ρ_1 and ρ_2.*

Proof. By the Peter-Weyl theorem, the group C^*-algebra $C^*(G)$ decomposes as a direct sum of matrix algebras $A_i = M_{n(i)}$ with units p_i, indexed by the discrete set $\widehat{G}$ of equivalence classes of irreducible representations.

Let E be the graph with vertex space $E^0 = \widehat{G}$ and with edges joining the vertices v_i and v_j determined by the number of minimal components of the (non-zero) A_j-A_i C^*-correspondences $p_j\mathcal{E}(\rho)p_i$. If $p_j\mathcal{E}(\rho)p_i = 0$, there is no edge from v_i to v_j.

It follows that $\mathcal{O}_{\mathcal{E}(\rho)}$ is isomorphic to the C^*-algebra of the graph of C^*-correspondences in which we assign the algebra A_i at the vertex v_i and the minimal components of $p_j\mathcal{E}(\rho)p_i \neq 0$ for each edge joining v_i with v_j. By construction, this C^*-algebra is SME to $C^*(E)$ (see **[6, 13]**). For more on graphs of C^*-correspondences, see **[3]**.

When $\pi_\rho : C^*(G) \to \mathcal{L}(\mathcal{H})$ is injective, it follows that $p_j\mathcal{E}(\rho) \neq 0$, and therefore, v_j is not a source for all j. For the last part, note that $\mathcal{E}(\rho) \cong \mathcal{E}(\rho_1) \oplus \mathcal{E}(\rho_2)$. $\square$

We recall the following result of Kumjian, see **[10,** Theorem 3.1]:

Theorem 2.3. *Let A be a separable unital C^*-algebra, and let $\mathcal{E} = \mathcal{H} \otimes_{\mathbb{C}} A$, with left multiplication given by a faithful representation $\pi : A \to \mathcal{L}(\mathcal{H})$ such that $\pi(A) \cap \mathcal{K}(\mathcal{H}) = \{0\}$. Then $\mathcal{O}_{\mathcal{E}}$ is simple, purely infinite and KK-equivalent to A.*

Corollary 2.4. *Let G be infinite, discrete and amenable, and let $\lambda : G \to U(\ell^2(G))$ be the left regular representation. Then, $\mathcal{O}_{\mathcal{E}(\lambda)}$ is simple, purely infinite and KK-equivalent to $C^*(G)$.*

Proof. Since G is discrete, $C^*(G)$ is unital. Since G is amenable, it is known that the representation $\pi_\lambda : C^*(G) \to \mathcal{L}(\ell^2(G))$ induced by λ is faithful (see **[17,** Theorem A.18]). Since G is infinite, we also have

$$\pi_\lambda(C^*(G)) \cap \mathcal{K}(\ell^2(G)) = \{0\},$$

and we can apply the above theorem. $\square$

Note that the same conclusion holds for any representation ρ of an infinite discrete group G such that π_ρ is faithful.

3. Examples.

Example 3.1. Let $S_3 = \{(1), (12), (13), (23), (123), (132)\}$ be the symmetric group. Then, $\widehat{S_3} = \{\iota, \varepsilon, \sigma\}$, where

$$\iota : S_3 \longrightarrow U(\mathbb{C}), \quad \iota(g) = 1,$$

is the trivial representation,

$$\varepsilon : S_3 \to U(\mathbb{C}), \qquad \varepsilon(12) = -1, \qquad \varepsilon(123) = 1$$

is the signature representation, and

$$\sigma : S_3 \longrightarrow U(\mathbb{C}^2),$$

$$\sigma(12) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix},$$
$$\sigma(123) = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

is the (standard) irreducible two-dimensional representation. These representations have characters

$$\chi_\iota(g) = 1, \qquad \chi_\varepsilon(1) = 1, \qquad \chi_\varepsilon(12) = -1, \qquad \chi_\varepsilon(123) = 1$$

and

$$\chi_\sigma(1) = 2, \qquad \chi_\sigma(12) = 0, \qquad \chi_\sigma(123) = -1.$$

The group algebra $C^*(S_3)$ is isomorphic to $\mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ with unit $p_1 \oplus p_2 \oplus p_3 = (1/6)\chi_\iota \oplus (1/6)\chi_\varepsilon \oplus (1/3)\chi_\sigma$.

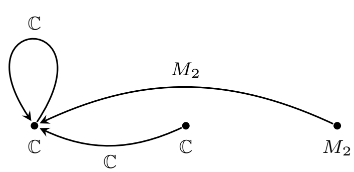
Any representation $\rho : S_3 \to U(\mathcal{H})$ extends to a representation

$$\pi_\rho : C^*(S_3) \longrightarrow \mathcal{L}(\mathcal{H}), \qquad \pi_\rho\Big(\sum a_g \delta_g\Big) = \sum a_g \rho(g),$$

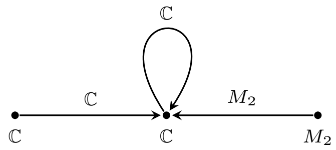
where, for $g \in S_3$, $a_g \in \mathbb{C}$, $\delta_g(h) = 1$ for $h = g$ and $\delta_g(h) = 0$ otherwise.

Since ρ decomposes as a direct sum of irreducibles (with multiplicities), we first list the graphs associated with the representations ι, ε and σ.

Since $\pi_\iota(p_1) = 1$, $\pi_\iota(p_2) = \pi_\iota(p_3) = 0$, we have $\mathcal{E}(\iota) = \mathbb{C} \otimes C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$, which gives the graph of C^*-correspondences:



Similarly, $\pi_\varepsilon(p_2) = 1$, $\pi_\varepsilon(p_1) = \pi_\varepsilon(p_3) = 0$, and $\mathcal{E}(\varepsilon) = \mathbb{C} \otimes C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus M_2(\mathbb{C})$ gives



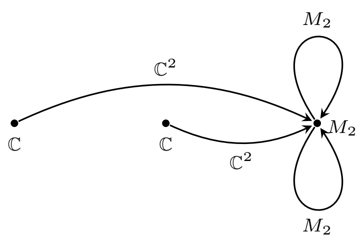
Since

$$\pi_\sigma(p_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\pi_\sigma(p_1) = \pi_\sigma(p_2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

it follows that $\mathcal{E}(\sigma) = \mathbb{C}^2 \otimes C^*(S_3) \cong \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \otimes M_2(\mathbb{C})$ determines



Example 3.2. For the permutation representation $\rho : S_3 \to U(\mathbb{C}^3)$, we have $\pi_\rho \cong \pi_\iota \oplus \pi_\sigma$ and

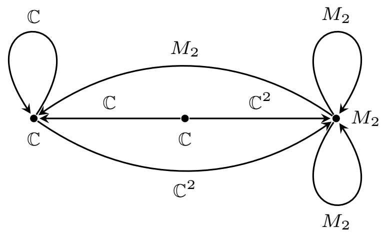
$$\begin{aligned} \pi_\rho(p_1) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \pi_\rho(p_2) &= 0, \\ \pi_\rho(p_3) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The C^*-correspondence $\mathcal{E}(\rho) = \mathbb{C}^3 \otimes C^*(S_3) \cong \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \otimes M_2$

decomposes as

$$\begin{aligned}\mathcal{E}(\rho) &= p_1\mathcal{E}(\rho)p_1 \oplus p_3\mathcal{E}(\rho)p_1 \oplus p_1\mathcal{E}(\rho)p_2 \oplus p_3\mathcal{E}(\rho)p_2 \oplus p_1\mathcal{E}(\rho)p_3 \oplus p_3\mathcal{E}(\rho)p_3 \\ &\cong \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus M_2 \oplus \mathbb{C}^2 \otimes M_2.\end{aligned}$$

Counting dimensions, we obtain the following graph of C^*-correspondences:



The subjacent graph E has incidence matrix

$$B_\rho = B_\iota + B_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix},$$

where the entry $B_\rho(v, w)$ counts the edges from w to v. It follows that $\mathcal{O}_{\mathcal{E}(\rho)}$ is SME to the graph algebra $C^*(E)$.

Using [**14,** Theorem 3.2], we obtain

$$K_0(C^*(E)) \cong \operatorname{coker} D \cong \mathbb{Z}, \qquad K_1(C^*(E)) \cong \ker D \cong 0,$$

where

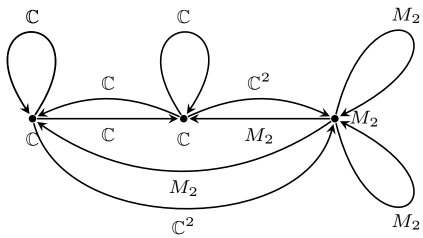
$$D = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ -1 & -2 \end{bmatrix}.$$

Compared with [**14**], note that we reverse the direction of the edges; thus, sinks become sources.

Remark 3.3. In all of the above cases the graphs have sources since the extension of the given representation to $C^*(S_3)$ is not one-to-one.

If we consider a representation ρ of S_3 which contains each of ι, ε and σ, for example, $\sigma \otimes \sigma = \iota \oplus \varepsilon \oplus \sigma$ or the left regular representation $\lambda = \iota \oplus \varepsilon \oplus 2\sigma$, then π_ρ will be injective and the graph associated to $\mathcal{E}(\rho)$ will have no sources. Its incidence matrix is obtained by adding the incidence matrices corresponding to ι, ε and σ, counting multiplicities.

For $\rho = \sigma \otimes \sigma$, we obtain the following graph of C^*-correspondences:



with incidence matrix

$$B_{\sigma\otimes\sigma} = B_\iota + B_\varepsilon + B_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Example 3.4. Any representation ρ of a cyclic group G is determined by a unitary $\rho(1) \in U(\mathcal{H})$ and decomposes as a direct sum or a direct integral of characters. Recall that this is merely a restatement of the spectral theorem for unitary operators.

If $G = \mathbb{Z}/n\mathbb{Z}$, then $\widehat{G} = \{\chi_1, \ldots, \chi_n\}$, and $\mathcal{E}(\rho)$ determines a graph with n vertices where the incidence matrix $[a_{ji}]$ is such that $a_{ji} = \dim \chi_j \mathcal{H} \chi_i$.

For $G = \mathbb{Z}$, we assume that $\mathcal{H} = L^2(X, \mu)$ for a measure space (X, μ) and that $\rho(1) = M_\varphi$, the multiplication operator with a function $\varphi : X \to \mathbb{T}$.

Then, $\mathcal{E}(\rho) = L^2(X, \mu) \otimes C^*(\mathbb{Z}) \cong C(\mathbb{T}, L^2(X, \mu))$ becomes a C^*-correspondence over $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, with operations:

$$\langle \xi, \eta \rangle(k) = \langle \xi(k), \eta(k) \rangle_{L^2(X,\mu)}, \quad \text{for } \xi,\ \eta \in C_c(\mathbb{Z}, L^2(X, \mu)),$$

and

$$(\xi \cdot f)(k) = \sum_k \xi(k) f(k), \qquad (f \cdot \xi)(k) = \sum_k f(k) \varphi^k \xi(k),$$
$$\text{for } f \in C_c(\mathbb{Z}), \quad \xi \in C_c(\mathbb{Z}, L^2(X, \mu)).$$

If $\dim \mathcal{H} = n$ is finite, then the function φ is given by $(z_1, \ldots, z_n) \in \mathbb{T}^n$, and $\mathcal{E}(\rho)$ is isomorphic to the C^*-correspondence of the topological graph with vertex space $E^0 = \mathbb{T}$, edge space $E^1 = \mathbb{T} \times \{1, 2, \ldots, n\}$, source map $s : E^1 \to E^0, s(z,k) = z$ and range map $r : E^1 \to E^0$, $r(z,k) = z_k z$.

If λ is the left regular representation of $G = \mathbb{Z}$ on $\ell^2(\mathbb{Z})$, it follows from Theorem 2.3 that $\mathcal{O}_{\mathcal{E}(\lambda)}$ is simple and purely infinite with the same K-theory as $C(\mathbb{T})$.

Example 3.5. Let $G = \mathbb{R}$, and let μ be the (normalized) Lebesgue measure μ on $\mathbb{R}$. Consider the representation

$$\rho : \mathbb{R} \longrightarrow U(L^2(\mathbb{R}, \mu)), \ (\rho(t)\xi)(s) = e^{its}\xi(s),$$

which extends to the Fourier transform π_ρ on $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$, where

$$(\pi_\rho(f)\xi)(s) = \int_{\mathbb{R}} f(t)e^{its}\xi(s)\, d\mu(t),$$
$$\text{for } f \in L^1(\mathbb{R}, \mu), \ \xi \in L^2(\mathbb{R}, \mu).$$

It is known that ρ is equivalent to the right regular representation of $\mathbb{R}$ (see [**16,** page 117, Theorem 2.2]) and that ρ is a direct integral of characters χ_t, where $\chi_t(s) = e^{its}$.

Then, $\mathcal{E}(\rho) = L^2(\mathbb{R}, \mu) \otimes C^*(\mathbb{R}) \cong C_0(\mathbb{R}, L^2(\mathbb{R}, \mu))$ becomes a C^*-correspondence over $C^*(\mathbb{R}) \cong C_0(\mathbb{R})$ such that the left multiplication is injective and $\pi_\rho(C_0(\mathbb{R})) \cap \mathcal{K}(L^2(\mathbb{R}, \mu)) = \{0\}$. It follows that $\mathcal{O}_{\mathcal{E}(\rho)}$ has the K-theory of $C_0(\mathbb{R})$; however, since $C_0(\mathbb{R})$ is not unital, we cannot apply Theorem 2.3 to conclude that this algebra is simple or purely infinite.

4. The crossed product C^*-correspondence. We want to compare our C^*-correspondence $\mathcal{E}(\rho)$ associated to a group representation with another C^*-correspondence from the literature. Let G be a locally compact group, and let $\rho : G \to U(\mathcal{H})$ be a representation with $\dim \mathcal{H} = n$ for $n \in \mathbb{N} \cup \{\infty\}$. In [**2**], we studied the crossed-product $\mathcal{O}_n \rtimes_\rho G$, using the crossed product C^*-correspondence $\mathcal{D}(\rho)$ over $C^*(G)$, constructed also from $\mathcal{H} \otimes_{\mathbb{C}} C^*(G)$ with the same inner product and right multiplication as $\mathcal{E}(\rho)$

$$\langle \xi, \eta \rangle(t) = \int_G \langle \xi(s), \eta(st) \rangle\, ds,$$

$$(\xi \cdot f)(t) = \int_G \xi(s) f(s^{-1}t)\, ds,$$

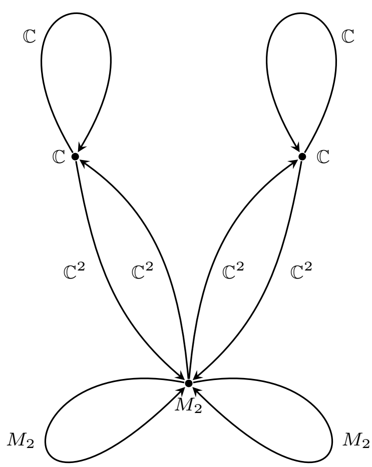
but with a different left multiplication

$$(h \cdot \xi)(t) = \int_G h(s)\rho(s)\xi(s^{-1}t)\, ds$$

for $\xi, \eta \in C_c(G, \mathcal{H})$ and $f, h \in C_c(G)$. This left multiplication is always one-to-one, which changes the structure of the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{D}(\rho)}$. We recall the following result (see [**4**]):

Theorem 4.1. *The representation ρ determines a quasi-free action of G on the Cuntz algebra $\mathcal{O}_n$ and $\mathcal{O}_{\mathcal{D}(\rho)} \cong \mathcal{O}_n \rtimes_\rho G$.*

Example 4.2. If $G = S_3$ and $\rho : S_3 \to U(\mathbb{C}^3)$ is the permutation representation, then $\mathcal{D}(\rho) = \mathbb{C}^3 \otimes C^*(S_3)$ with the above operations becomes a C^*-correspondence over $C^*(S_3)$, different from $\mathcal{E}(\rho)$ discussed in Example 3.2. It decomposes as $\mathbb{C} \oplus \mathbb{C} \oplus M_2 \oplus M_2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$, see [**2,** Example 6.4], and it determines the following graph of C^*-correspondences for $\mathcal{O}_3 \rtimes_\rho S_3$:



The subjacent graph has no sources, and the incidence matrix is

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

5. Quasi-free actions of abelian groups. If G is compact and abelian, then any representation ρ of dimension $n \geq 2$ (including $n = \infty$) decomposes into characters and determines a cocycle $c : E_n^1 \to \widehat{G}$, where E_n is the graph with one vertex and n edges. Recall that $C^*(E_n) = \mathcal{O}_n$.

It turns out that $\mathcal{O}_n \rtimes_\rho G$ is isomorphic to $C^*(E_n(c))$, where $E_n(c)$ is the skew product graph $(\widehat{G}, \widehat{G} \times E_n^1, r, s)$ with

$$r(\chi, e) = \chi c(e), \qquad s(\chi, e) = \chi,$$

for $\chi \in \widehat{G}$ and $e \in E_n^1$.

Example 5.1. If $G = \mathbb{T}$, and $\lambda : \mathbb{T} \to U(L^2(\mathbb{T}))$ is the left regular representation, then the algebra $\mathcal{O}_\infty \rtimes_\lambda \mathbb{T}$ is isomorphic to the graph algebra, where the vertices are labeled by $\mathbb{Z}$, and the incidence matrix has each entry equal to 1.

Example 5.2. If $G = \mathbb{R}$ and $\rho : \mathbb{R} \to U(\mathbb{C}^n)$ is a representation for $n \geq 2$, Kishimoto and Kumjian (see [**9**]) showed that $\mathcal{O}_n \rtimes_\rho \mathbb{R}$ is simple and purely infinite if the characters in the decomposition of ρ generate $\mathbb{R}$ as a closed semigroup.

In [**7**], Katsura determined the ideal structure of the crossed products $\mathcal{O}_n \rtimes_\rho G$, where G is a locally compact second countable abelian group. The action of G is determined by an n-dimensional representation ρ with characters $\{\omega_1, \ldots, \omega_n\}$ such that

$$\rho_t(S_i) = \omega_i(t) S_i,$$

for $t \in G$, where S_i are the generators of $\mathcal{O}_n$ for $i = 1, \ldots, n$. It is shown in [**7**] that $\mathcal{O}_n \rtimes_\rho G \cong \mathcal{O}_{\mathcal{D}}$, where $\mathcal{D}$ is the C^*-correspondence obtained from $\mathbb{C}^n \otimes C^*(G)$ with the obvious right multiplication and inner product, and with left multiplication given by

$$f \cdot (f_1, f_2, \ldots, f_n) = (\sigma_{\omega_1}(f) f_1, \sigma_{\omega_2}(f) f_2, \ldots, \sigma_{\omega_n}(f) f_n),$$

for $f, f_1, \ldots, f_n \in C_0(\widehat{G})$, where $(\sigma_\omega f)(\gamma) = f(\gamma + \omega)$.

Note that, in this case, our C^*-correspondence $\mathcal{E}(\rho)$ determines a different Cuntz-Pimsner algebra, since the left multiplication is not injective. The ideal structure of the algebras $\mathcal{O}_{\mathcal{E}(\rho)}$ will be considered in future work.

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University of Nevada, Department of Mathematics (084), Reno, NV 89557
Email address: **vdeaconu@unr.edu**