ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 4, 2018

PERIODIC SOLUTIONS TO NONLINEAR WAVE EQUATIONS WITH x-DEPENDENT COEFFICIENTS AT RESONANCE

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ABSTRACT. In this paper, the unique existence of generalized solutions to periodic boundary value problems for a class of systems of nonlinear equations with x-dependent coefficients is discussed under a resonance condition. The argument presented makes use of the global inverse theorem and Galerkin's method.

1. Introduction. There is much research on solutions of nonlinear wave equations with ordinary coefficients, see **[2, 3, 9, 10, 11, 13]** and the references therein. The solvability of nonlinear equations with resonance conditions has been considered by many authors (see **[4, 5, 6, 7, 8, 12]**). By using the global inverse theorem and Galerkin's methods, Chen **[3]** studied the semilinear elastic beam equations at resonance. Iannacci and Nkashama in **[5]** treated the general operator equation at resonance and obtained some very general abstract results which unify and generalize most of the known existence theorems for the case where the resonance occurs at the eigenvalue r_N and the nonlinearity lies between consecutive eigenvalues. Compared with nonlinear equations with ordinary coefficients, there are fewer papers regarding nonlinear wave equations with variational coefficients **[1, 6, 12]**. In **[12]**, Wang and An investigated the following nonlinear

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2010 AMS *Mathematics subject classification.* Primary 35B10, 35L05, 35L70.

Keywords and phrases. Wave equations, x-dependent coefficients, resonance, unique existence, high-dimensional.

This research was supported in part by the Natural Science Foundation of China, grant Nos. 11501288 and 11701271, by the Specialized Research Foundation for the Doctoral Program of Higher Education, grant No. 20100091110033, by the 985 Project at Nanjing University, grant No. 9112020301, by A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions, by the Natural Science Foundation of Jiangsu Province, grant No. BK20150934 and by the Natural Science Foundation of the Jiangsu Higher Education Institutions, grant Nos. 16KJB110010 and 14KJB110009. The second author is the corresponding author.

Received by the editors on December 3, 2016, and in revised form on July 27, 2017.

DOI:10.1216/RMJ-2018-48-4-1291

one-dimensional wave equation with x-dependent coefficients:

$$\rho(x)u_{tt}-(\rho(x)u_x)_x-f(x,t,u(x,t))=h(x,t)\quad \text{for all } (x,t)\in\Omega, \tag{1.1}$$

with boundary conditions:

$$-\rho(0)u_x(0,t)=0,\qquad u(\pi,t)=0\quad \text{for all } t\in(0,T), \tag{1.2}$$

and T-periodic conditions:

$$u(x,0)=u(x,T),\qquad u_t(x,0)=u_t(x,T)\quad \text{for all } x\in(0,\pi), \tag{1.3}$$

where $\Omega=(0,\pi)\times(0,T)$. They considered the resonance conditions which are of Landesman-Lazer type by using the Leray-Schauder degree. In [**1**], by using the subdifferential method, Barbu and Pavel investigated equation (1.1) with the Dirichlet boundary condition $u(0,t) = u(\pi,t)=0$ and periodic condition (1.3). However, it is noted that they were mainly concerned with the existence of solutions for (1.1) and did not consider the uniqueness of the solution. Moreover, they only considered the scalar equation.

The main difficulty of the presence of resonance is that the problem of existence of periodic solutions may not have solutions for general nonlinearity f. Simply speaking, in the non-resonance case, the nonlinearity f must lie between an interval $[p,q]$ that does not contain any eigenvalues of the linear operator associated with the equations. However, the resonance case allows the nonlinearity to interact with the eigenvalues. More precisely, the nonlinearity lies between two consecutive eigenvalues and possibly "touches" the eigenvalues to some extent. Therefore, the resonance case could be trickier than the non-resonance case. In this paper, we are concerned with the existence and uniqueness of a generalized solution of system (1.1)–(1.3) at resonance of a particular type: the nonlinearity simultaneously interacts with the two consecutive eigenvalues. Our approach is based on a global inverse theorem and Galerkin's method.

Throughout the paper, the following assumptions are satisfied.

(H_1) $f(x,t,u):\Omega\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous and continuously differentiable with respect to u.

(H_2) $\rho(x)\in H^2(0,\pi), \rho(x)\geqslant s>0, x\in(0,\pi)$ and $\eta_0=\operatorname{ess\,inf}\eta_\rho(x)>0$, where

$$\eta_\rho(x) = \frac{1}{2}\frac{\rho''}{\rho'} - \frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2.$$

(H_3) $T = \pi(2k-1)/p$, $p, k \in \mathbf{N}$.

The outline of this paper is as follows. Some properties of the linear part of the equation with which we are concerned are presented in Section 2. Section 3 is devoted to the notation and two useful lemmas used in this paper. In Section 4, we give our main result and its proof. The main result is given in Theorem 4.3.

2. Properties of the linear part of the equation. We denote by $L^p(\Omega)$, $1 \leqslant p \leqslant +\infty$, the Lebesgue spaces, i.e., spaces of measurable real-valued functions whose pth power of the absolute value is Lebesgue integrable.

$$H^m(\Omega) = \{v \in L^2(\Omega) \mid D^\alpha v \in L^2(\Omega), |\alpha| \leqslant m\}, \quad m = 1, 2,$$

are the usual Sobolev spaces with the norm

$$\|v\|_m = \bigg(\sum_{|\alpha| \leqslant m} \|D^\alpha v\|^2_{L^2(\Omega)}\bigg)^{1/2}.$$

Consider the following scalar linear equation

$$\text{(2.1)} \quad \begin{cases} \rho(x)u_{tt} - (\rho(x)u_x)_x = \rho(x)h(x,t) & \text{for all } (x,t) \in \Omega, \\ -\rho(0)u_x(0,t) = 0, \quad u(\pi,t) = 0 & \text{for all } t \in (0,T), \\ u(x,0) = u(x,T), \quad u_t(x,0) = u_t(x,T) & \text{for all } x \in (0,\pi). \end{cases}$$

Let

$$C_2^\pi(\overline{\Omega}) = \{v \in C^2(\overline{\Omega}) : v_x(0,t) = 0, \\ v(\pi,t) = 0, v(x,0) = v(x,T), v_t(x,0) = v_t(x,T)\}.$$

It is easy to see that $C_2^\pi(\overline{\Omega})$ is dense in $L^2(\Omega)$. We say that $u \in L^2(\Omega)$ is a weak solution of (2.1) if, for all $v \in C_2^\pi(\overline{\Omega})$,

$$\text{(2.2)} \quad \int_\Omega u(x,t)(\rho(x)v_{tt} - (\rho(x)v_x)_x)\,dx\,dt = \int_\Omega \rho(x)h(x,t)v(x,t)\,dx\,dt.$$

Set $\operatorname{dom}(L) = \{u \in L^2(\Omega);$ there exists an $h \in L^2(\Omega)$ such that (2.2) holds$\}$, and define $L : \operatorname{dom}(L) \to L^2(\Omega)$ by $Lu = h$ for $u \in \operatorname{dom}(L)$, i.e., $Lu = h$ if and only if (2.2) holds for all $v \in C_2^{\pi}(\overline{\Omega})$. This operator is well defined and is called the *weak solution operator*. Define the inner product in $L^2(\Omega)$ by

$$\langle f, g\rangle = \int_\Omega \rho(x) f(x,t)\overline{g(x,t)}\,dx\,dt, \quad f, g \in L^2(\Omega).$$

The norm in $L^2(\Omega)$ is given by

$$\|u(x,t)\| = \left(\int_\Omega \rho(x)|u(x,t)|^2\,dx\,dt\right)^{1/2}.$$

Then, we have the following complete orthornormal system of eigenfunctions $\{\psi_m(t)\varphi_n(x) : m \in \mathbf{Z}, n \in \mathbf{N}^* = \{0\}\bigcup \mathbf{N}\}$ in $L^2(\Omega)$, see **[12]**, where

$$\psi_m(t) = \frac{1}{\sqrt{T}} e^{i\mu_m t}, \qquad \mu_m = \frac{2m\pi}{T}, \quad m \in \mathbf{Z}, \tag{2.3}$$

and $\lambda_n, \varphi_n(x)$ are determined by the Strum-Liouville problem:

$$(\rho(x)\varphi_n'(x))' = \rho(x)\lambda_n^2\varphi_n(x), \qquad \varphi_n'(0) = \varphi_n(\pi) = 0, \quad n \in \mathbf{N}^*, \tag{2.4}$$

where $\varphi_n' = (d/dx)\varphi_n$.

Denote by

$$u_{mn} = \int_\Omega \rho(x)u(x,t)\varphi_n(x)\overline{\psi_m(t)}\,dx\,dt$$

the Fourier coefficient of

$$u = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}^*}} u_{mn}\varphi_n(x)\psi_m(t)$$

in $L^2(\Omega)$ and

$$h_{mn} = \int_\Omega \rho(x)h(x,t)\varphi_n(x)\overline{\psi_m(t)}\,dx\,dt$$

the Fourier coefficient of

$$h = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}^*}} h_{mn}\varphi_n(x)\psi_m(t)$$

in $L^2(\Omega)$. According to Parseval's formula, we have

$$\|u\|^2 = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}^*}} |u_{mn}|^2, \qquad \|h\|^2 = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}^*}} |h_{mn}|^2.$$

Substituting $v = \varphi_n(x)\overline{\psi_m(t)}$ into (2.2) yields that $Lu = h$ if and only if

$$(\lambda_n^2 - \mu_m^2)u_{mn} = h_{mn}.$$

We can derive some of the main properties of L as follows.

Lemma 2.1 ([**1**]). *L is self-adjoint in $L^2(\Omega)$ with closed range* ImL *given by*

$$\mathrm{Im}L = \mathrm{Span}\{\varphi_n\psi_m; m \in \boldsymbol{Z}, n \in \boldsymbol{N},\ \textit{with}\ \lambda_n \neq |\mu_m|\} = (\ker L)^\perp$$

with null space

$$\ker L = \mathrm{Span}\{\varphi_n\psi_m; m \in \boldsymbol{Z}, n \in \boldsymbol{N},\ \textit{with}\ \lambda_n = |\mu_m|\}.$$

The right inverse of L (denoted by $K = L^{-1}_{\mathrm{dom}(L)\cap\mathrm{Im}L}$) is compact and continuous.

Moreover, we have:

$$\|Kh\|_{H^1(\Omega)} \leqslant C\|Kh\|_{L^2(\Omega)}, \qquad \|Kh\|_{L^\infty(\Omega)} \leqslant C\|Kh\|_{L^2(\Omega)}$$

and

$$\langle Kh, h\rangle \geqslant -\alpha^{-1}\|h\|^2, \qquad \langle Ku, u\rangle \geqslant -\alpha^{-1}\|Lu\|^2,$$
$$h \in \mathrm{Im}L, \quad u \in \mathrm{dom}(L),$$

where $\alpha^{-1} = \inf\{\mu_m^2 - \lambda_n^2; \mu_m^2 > \lambda_n^2\}$.

We denote by $\sigma(L) = \{\lambda_n^2 - \mu_m^2 \mid m \in \mathbf{Z},\ n \in \mathbf{N}^*\}$ the spectrum set of L. It is a consequence of Lemma 2.1 that each $r \neq 0$, $r \in \sigma(L)$ is an eigenvalue of finite multiplicity and that $\sigma(L)$ has no finite cluster point.

3. Preliminaries. Before discussion of the existence of solutions to system (1.1)–(1.3), in what follows, we give some notation and useful lemmas. Let $H = [L^2(\Omega)]^n$, with integer $n \geqslant 1$. Then, H is a real Hilbert space with inner product

$$\langle u, v\rangle = \int_0^\pi \int_0^T \rho(x)(u(x,t), v(x,t))\, dx\, dt, \quad u, v \in H. \tag{3.1}$$

Note that $(\cdot,\cdot)$ denotes the Euclidean inner product in $\mathbf{R}^n$, and the norms induced by $\langle\cdot,\cdot\rangle$ and $(\cdot,\cdot)$ are denoted by $\|\cdot\|$ and $|\cdot|$, respectively.

We call $u \in H$ the weak solution to system (1.1)–(1.3) if it satisfies

$$\begin{aligned}\int_\Omega (u(x,t), (\rho(x)v_{tt} - (\rho(x)v_x)_x))\, dx\, dt - \int_\Omega (f(x,t,u), v(x,t))\, dx\, dt \\ = \int_\Omega (h(x,t), v(x,t))\, dx\, dt \quad \text{for all } v \in [C_2^\pi(\overline{\Omega})]^n.\end{aligned} \tag{3.2}$$

System (1.1)–(1.3) is said to be at *resonance* if the following condition holds, i.e., there exist two constant symmetric $n \times n$ matrices A and B such that

$$A \leqslant \frac{f_u'(t,x,u)}{\rho(x)} \leqslant B, \qquad \bigcup_{j=1}^n [\nu_{1i}, \nu_{2i}] \bigcap \sigma(L) \neq \emptyset,$$

where $\nu_{11} \leqslant \nu_{12} \leqslant \cdots \leqslant \nu_{1n}$ and $\nu_{21} \leqslant \nu_{22} \leqslant \cdots \leqslant \nu_{2n}$ are the eigenvalues of A and B, respectively, with $\nu_{1i} \leqslant \nu_{2i}$, for $i = 1, \ldots, n$. For two $n \times n$ symmetric matrices A and B, by $B \geqslant A$, we mean that $((B-A)\xi, \xi) \geqslant 0$ for all $\xi \in \mathbf{R}^n$.

To be more precise, we will prove in this paper that, if f is continuously differentiable and $f_u'(t,x,u)$ is symmetric satisfying

$$A + \alpha(\|u\|)I \leqslant \frac{f_u'(t,x,u)}{\rho(x)} \leqslant B - \beta(\|u\|)I \tag{3.3}$$

and

$$\int_0^{+\infty} \min\{\alpha(s), \beta(s)\}\, ds = +\infty, \tag{3.4}$$

then there exists a unique weak solution to (1.1)–(1.3), where I is an $n \times n$ identity matrix, A and B are two real symmetric matrices (and we assume that $\nu_{1i}, \nu_{2i} \in \sigma(L)$ are consecutive, for $i = 1, 2, \ldots, n$, where ν_{1i} and ν_{2i} are the eigenvalues of A and B, respectively) and $\alpha(s)$ and $\beta(s)$ are two continuous and nonincreasing functions from $[0, \infty)$ to

$(0,\infty)$. We note that $\nu_{1i}, \nu_{2i} \in \sigma(L)$, $\nu_{1i} \neq \nu_{2i}$, are consecutive for $i = 1, \dots, n$, provided

$$\bigcup_{j=1}^{n} (\nu_{1i}, \nu_{2i}) \bigcap \sigma(L) = \emptyset$$

and $\nu_{11} \leqslant \nu_{12} \leqslant \cdots \leqslant \nu_{1n}$, $\nu_{21} \leqslant \nu_{22} \leqslant \cdots \leqslant \nu_{2n}$, $\nu_{1i} \leqslant \nu_{2i}$ for $i = 1, \dots, n$.

Next, we introduce the following two lemmas which play an important role in this paper.

Lemma 3.1 ([**7**]). *Let H be a real Hilbert space, $T \in C^1(H, H)$, and assume that $T'(u)$ is everywhere invertible for all $u \in H$. Then, T is a global diffeomorphism onto H if w satisfies $\|T'(u)^{-1}\| \leqslant w(\|u\|)$. Here, w is a nondecreasing mapping and satisfies the following conditions*:

$$w : \boldsymbol{R}_+ \longrightarrow \boldsymbol{R}_+, \qquad \int_0^\infty \frac{dt}{w(t)} = \infty, \qquad w(t) > 0, \quad t > 0. \tag{3.5}$$

Lemma 3.2 ([**3, 8**]). *Let H be a vector space such that, for subspaces Y and Z, $H = Z \bigoplus Y$. If Z is finite-dimensional and X is a subspace of H such that $X \cap Y = \{0\}$ and $\dim X = \dim Z$, then $H = X \bigoplus Y$.*

4. Main result and proof. We will follow the setup in Mawhin [**8**]. Set $v_{mn}(x,t) = \varphi_n(x)\psi_m(t)$, $m \in \mathbf{Z}$, $n \in \mathbf{N}^*$, and let $\{e_k \mid 1 \leqslant k \leqslant n\}$ denote an orthonormal basis in $\mathbf{R}^n$. Then, for every $u \in H$, we have the Fourier series

$$u = \sum_{k=1}^{n} \sum_{(m,n) \in \mathbf{Z} \times \mathbf{N}^*} u_{k,m,n} v_{mn} e_k, \tag{4.1}$$

where $u_{k,m,n}$ satisfies $u_{k,m,n} = \overline{u_{k,-m,n}}$ to make the series real. We define

$$\mathrm{D}(L) = \left\{ u \in H \;\middle|\; \sum_{k=1}^{n} \sum_{(m,n) \in \mathbf{Z} \times \mathbf{N}^*} (\lambda_n^2 - \mu_m^2)^2 |u_{k,m,n}|^2 < \infty \right\} \tag{4.2}$$

and

$$L : \mathrm{D}(L) \subset H \longmapsto H, \tag{4.3}$$

$$u \longmapsto \sum_{k=1}^{n} \sum_{(m,n)\in \mathbf{Z}\times\mathbf{N}^*} (\lambda_n^2 - \mu_m^2) u_{k,m,n} v_{mn} e_k.$$

From Lemma 2.1, it is easy to verify that L is a self-adjoint operator. We assume that there exists a constant $C \geqslant 0$ such that, for all $u \in \mathbf{R}^n$,

$$\|f'_u(t,x,u)\| \leqslant C. \tag{4.4}$$

It is well known that the mapping N defined on H by

$$Nu(x,t) = \frac{f(t,x,u(x,t))}{\rho(x)} \quad \text{almost everywhere on } \Omega$$

continuously maps H into itself. Then, the existence of the weak solution for (1.1)–(1.3) is equivalent to the existence of a solution $u \in \mathrm{D}(L)$ for the equation in H:

$$Lu - Nu = \overline{h}, \quad \overline{h} = h/\rho. \tag{4.5}$$

We shall now construct Galerkin's approximate equations for (4.5) in a manner similar to Mawhin [**8**]. Let $\{a_k \mid 1 \leqslant k \leqslant n\}$ and $\{b_k \mid 1 \leqslant k \leqslant n\}$ be two orthonormal bases in $\mathbf{R}^n$ such that

$$Aa_k = \nu_{1k} a_k, \qquad Bb_k = \nu_{2k} b_k, \quad k = 1, 2, \ldots, n.$$

For every $j \in \mathbf{N}$, define a subspace H_j of H by

$$H_j = \left\{ \sum_{k=1}^{n} \sum_{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j} u_{k,m,n} v_{mn} b_k, u_{k,m,n} = \overline{u_{k,-m,n}} \right\}, \tag{4.6}$$

where

$$(\mathbf{Z}\times\mathbf{N}^*)_j = \{(m,n) \in \mathbf{Z}\times\mathbf{N}^* : |\lambda_n^2 - \mu_m^2| \leqslant j,\ |\lambda_n| \leqslant j\}.$$

Note that, by this construction, the restriction of L to $\mathrm{D}(L) \bigcap H_j$ has, in contrast with L, a spectrum bounded below and above, and the spectrum set is made of eigenvalues having finite multiplicity. Moreover, $\bigcup_{j\in\mathbf{N}} H_j$ is dense in H. If we denote by $P_j : H \mapsto H$ the orthogonal projection onto H_j, $j \in \mathbf{N}$, Galerkin's approximate equations for (4.5) will be

$$(4.7) \qquad Lu_j - P_jNu_j = P_j\overline{h}, \qquad u_j \in \mathrm{D}(L)\bigcap H_j = H_j, \quad j \in \mathbf{N}.$$

We are now in a position to prove the existence of Galerkin's approximate solutions. Let $j \in \mathbf{N}$ be fixed. We introduce a direct sum decomposition of H_j:

$$X_j = \left\{ \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2\geqslant\nu_{2k}}} u_{k,m,n}v_{mn}b_k, u_{k,m,n} = \overline{u_{k,-m,n}} \right\},$$

$$Y_j = \left\{ \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2<\nu_{2k}}} u_{k,m,n}v_{mn}b_k, u_{k,m,n} = \overline{u_{k,-m,n}} \right\},$$

$$Z_j = \left\{ \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2<\nu_{1k}}} u_{k,m,n}v_{mn}a_k, u_{k,m,n} = \overline{u_{k,-m,n}} \right\}.$$

Clearly, $H_j = X_j \bigoplus Y_j$ (the orthogonal direct sum) and, since $\nu_{1i}, \nu_{2i} \in \sigma(L)$ are consecutive, $\dim Y_j = \dim Z_j < +\infty$.

Now, we establish a unique existence result. We note that some of the ideas in the proof are modeled after [**8,** Lemma 2].

Lemma 4.1. *If conditions* (3.3) *and* (3.4) *hold for all* $(x,t) \in \Omega$, *then equation* (4.7) *has, for each* $j \in \boldsymbol{N}$ *and each* $\overline{h} \in H$, *a unique solution* $u_j \in \mathrm{D}(L)\bigcap H_j$ *and there exists a constant* $C = C(\nu_{11}, \dots, \nu_{1n}, \nu_{21}, \dots, \nu_{2n}, \|\overline{h}\|)$ *such that* $\|u_j\| \leqslant C$ *for all* $j \in \boldsymbol{N}$.

Proof. We show that the mapping $F_j : H_j \mapsto H_j$, defined by

$$F_ju_j = Lu_j - P_jNu_j,$$

for every $u_j \in H_j$ satisfies all the conditions of Lemma 3.1. The continuous Fréchet differentiability of F_j is trivial. Now, if $u_j \in H_j$ and

$x_j \in \mathrm{D}(L) \bigcap X_j = X_j$ with

$$x_j = \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2 \geqslant \nu_{2k}}} x_{kmn} v_{mn} b_k,$$

we have

(4.8)
$$\begin{aligned}
&\langle Lx_j - P_j N'(u_j)x_j, x_j\rangle \\
&\quad = \langle Lx_j, x_j\rangle - \langle P_j N'(u_j)x_j, x_j\rangle \\
&\quad \geqslant \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2 \geqslant \nu_{2k}}} (\lambda_n^2 - \mu_m^2)|x_{kmn}|^2 - \langle (B - \beta(\|u_j\|)I)x_j, x_j\rangle \\
&\quad = \sum_{k=1}^{n} \sum_{\substack{(m,n)\in(\mathbf{Z}\times\mathbf{N}^*)_j \\ \lambda_n^2-\mu_m^2 \geqslant \nu_{2k}}} (\lambda_n^2 - \mu_m^2 - [\nu_{2k} - \beta(\|u_j\|)])|x_{kmn}|^2 \\
&\quad \geqslant \beta(\|u_j\|)\|x_j\|^2.
\end{aligned}$$

Similarly, if $z_j \in \mathrm{D}(L) \bigcap Z_j = Z_j$, we obtain

$$\langle Lz_j - P_j N'(u_j)z_j, z_j\rangle \leqslant -\alpha(\|u_j\|)\|z_j\|^2. \tag{4.9}$$

Inequalities (4.8) and (4.9) imply that $X_j \bigcap Z_j = \{0\}$, which, together with $\dim Y_j = \dim Z_j < \infty$, and Lemma 3.2, show that $H_j = X_j \bigoplus Z_j$ algebraically, and hence, topologically.

Consequently, if $u_j \in H_j, v_j \in H_j$ and $v_j = x_j + z_j$ with $x_j \in X_j$ and $z_j \in Z_j$, using (4.8), (4.9), and the symmetry of L and $P_j N'(u_j)$, we obtain

$$\begin{aligned}
\langle F_j'(u_j)v_j, x_j - z_j\rangle &= \langle F_j'(u_j)x_j, x_j\rangle - \langle F_j'(u_j)z_j, z_j\rangle \\
&\geqslant \beta(\|u_j\|)\|x_j\|^2 + \alpha(\|u_j\|)\|z_j\|^2 \\
&\geqslant \gamma(\|u_j\|)\|v_j\|^2,
\end{aligned}$$

where $\gamma(s) = \min\{\alpha(s), \beta(s)\}$. Furthermore, we have

$$\gamma(\|u_j\|)\|v_j\|^2 \leqslant \|F_j'(u_j)v_j\|(\|x_j\| + \|z_j\|)$$

and

$$\gamma(\|u_j\|)\|v_j\| \leqslant \gamma(\|u_j\|)(\|x_j\| + \|z_j\|) \leqslant 2\|F_j'(u_j)v_j\|, \tag{4.10}$$

since $\sqrt{a^2+b^2} \leqslant a + b$ and $(a+b)^2 \leqslant 2(a^2+b^2)$ for $a \geqslant 0$, $b \geqslant 0$.

We will prove that the conditions of Lemma 3.1 are satisfied. First, we show that $F_j'(u_j)$ is a one-to-one mapping. In fact, suppose that $w_1 \neq w_2$, $w_1, w_2 \in H_j$, such that $F_j'(u_j)w_1 = F_j'(u_j)w_2$. Then, from (4.10), we have

$$\begin{aligned} 0 &= \|F_j'(u_j)w_1 - F_j'(u_j)w_2\| = \|F_j'(u_j)(w_1 - w_2)\| \\ &\geqslant \frac{\gamma(\|u_j\|)}{2}\|w_1 - w_2\| > 0, \end{aligned}$$

which is a contradiction. Also, by (4.10), we can prove that $F_j'(u_j)H_j$ is a closed subspace of H_j. Next, we prove that $F_j'(u_j)H_j = H_j$. For this purpose, we assume that there exists a $v \in (F_j(u_j)H_j)^{\perp}$ with $v \neq 0$. Then, $\langle v, F_j'(u_j)w\rangle = 0$ for all $w \in H_j$. Let $v = v_X + v_Z, v_X \in X_j, v_Z \in Z_j$ and set $w = v_Z - v_X$. Then,

$$0 = \langle v, F_j'(u_j)w\rangle \geqslant \beta(\|u_j\|)\|v_X\|^2 + \alpha(\|u_j\|)\|v_Z\|^2 \geqslant \frac{\gamma(\|u_j\|)}{2}\|w\|^2.$$

This is a contradiction since $X_j \bigcap Z_j = \{0\}$; hence, $F_j'(u_j)H_j = H_j$. Note that

$$\|F_j'(u_j)^{-1}\| \leqslant \frac{2}{\gamma(\|u_j\|)}. \tag{4.11}$$

With (4.11) and Lemma 3.1, we can see that $F_j : H_j \mapsto H_j$ is a homeomorphism. In order to obtain the estimate for unique solution u_j to equation (4.7), by $F_j(u_j) = P_j\overline{h}$, we obtain that

$$u_j = F_j^{-1}(P_j\overline{h}) - F_j^{-1}(F_j(0));$$

hence, using the integral mean value theorem, we obtain

$$\begin{aligned} \|u_j\| &= \|F_j^{-1}(P_j\overline{h}) - F_j^{-1}(F_j(0))\| \\ &\leqslant \int_0^1 \|(F_j{}')^{-1}(F_j(0) + \theta(P_j\overline{h} - F_j(0)))\|\, d\theta \|P_j\overline{h} - F_j(0)\|. \end{aligned} \tag{4.12}$$

Noting that

$$\|F_j(0) + \theta(P_j\overline{h} - F_j(0))\| \leqslant \|P_j\overline{h}\| + \|F_j(0)\| \leqslant \|\overline{h}\| + \|N(0)\|,$$

it follows from (4.10) and (4.12) that

$$\|u_j\| \leqslant \frac{2}{\gamma(\|\overline{h}\| + \|N(0)\|)}\|\overline{h}\| + \|N(0)\| \tag{4.13}$$

with the right-hand side member independent of j. The proof is complete. □

The convergence result for Galerkin's method associated to nonlinear perturbations of L may be found from the following lemma.

Lemma 4.2 ([**8**])**.** *Assume that there exists a sequence $\{H_j\}$ of finite-dimensional vector subspaces of H such that*

$$\begin{aligned} &H_j \subset H_{j+1}, \qquad \widetilde{L}\Big(\mathrm{D}(\widetilde{L})\bigcap H_j\Big) \subset H_j, \quad j \in \boldsymbol{N}, \\ &\qquad\qquad H = \overline{\bigcup_{j\in \boldsymbol{N}} H_j}, \end{aligned} \tag{4.14}$$

and let $P_j : H \to H$ be the orthogonal projector onto H_j, $j \in \boldsymbol{N}$. Let $\widetilde{N} : H \to H$ be a continuous monotone mapping which takes bounded sets into bounded sets. Assume that, for some $\overline{h} \in H$ and some $r > 0$, the equation

$$\widetilde{L}u_j - P_j\widetilde{N}u_j = P_j\overline{h}, \tag{4.15}$$

has a unique solution $u_j \in \mathrm{D}(\widetilde{L})\bigcap H_j$ such that $\|u_j\| \leqslant r$, $j \in \boldsymbol{N}$. Then, $\widetilde{L}u - \widetilde{N}u = h$ has at least one solution $u \in \mathrm{D}(\widetilde{L})$ such that $\|u\| \leqslant r$.

Now, we present our main theorem.

Theorem 4.3. *If conditions* (3.3) *and* (3.4) *hold for all $(x,t) \in \Omega$ and $u \in H$, then the system* (1.1)–(1.3) *has a unique generalized solution.*

Proof. Since $\nu_{1k}, \nu_{2k} \in \sigma(L)$, for $k = 1, 2, \ldots, n$ are consecutive, then there exists an integer $0 \leqslant p \leqslant n$ such that

$$\begin{aligned} &\nu_{1k} \leqslant \nu_{2k}, \qquad \nu_{1k} \leqslant 0, \quad 1 \leqslant k \leqslant p, \\ &\nu_{2k} \geqslant \nu_{1k} \geqslant 0, \quad p+1 \leqslant k \leqslant n. \end{aligned}$$

Now, define the operators S_+ and S_- on $\mathbf{R}^n$ as follows:

$$S_+x = \sum_{k=p+1}^{n} \xi_k a_k, \qquad S_-x = \sum_{k=1}^{p} \xi_k b_k$$

for every $x = \sum_{k=1}^{n} \xi_k a_k$ in $\mathbf{R}^n$. Next, we show that $\mathrm{Im}S_+ \bigcap \mathrm{Im}S_- = \{0\}$. In fact, for $x \in \mathrm{Im}S_+$, we have

$$\begin{aligned}((B-\beta(\|u\|))x,x) &\geqslant ((A+\alpha(\|u\|))x,x)\\ &= \sum_{k=p+1}^{n} (\nu_{1k}+\alpha(\|u\|))\xi_k^2\\ &\geqslant \min_{p+1\leqslant k\leqslant n}(\nu_{1k}+\alpha(\|u\|))\bigg(\sum_{k=p+1}^{n}\xi_k^2\bigg),\end{aligned}$$

and, for $x \in \mathrm{Im}S_-$, we have

$$\begin{aligned}((A+\alpha(\|u\|))x,x) &\leqslant ((B-\beta(\|u\|))x,x)\\ &= \sum_{k=1}^{p}(\nu_{2k}-\beta(\|u\|))\xi_k^2\\ &\leqslant \max_{1\leqslant k\leqslant p}(\nu_{2k}-\beta(\|u\|))\bigg(\sum_{k=1}^{p}\xi_k^2\bigg).\end{aligned}$$

Since $\min_{p+1\leqslant k\leqslant n}(\nu_{1k}+\alpha(\|u\|)) > 0$ and $\max_{1\leqslant k\leqslant p}(\nu_{2k}-\beta(\|u\|)) < 0$, we have $\mathrm{Im}S_+ \bigcap \mathrm{Im}S_- = \{0\}$. From Lemma 3.2, we have $\mathbf{R}^n = \mathrm{Im}S_+ \bigoplus \mathrm{Im}S_-$.

If we now define the operators $\widetilde{S_+}$ and $\widetilde{S_-}$ on H by

$$\widetilde{S_\pm}u(x,t) = S_\pm(u(x,t)) \quad \text{almost everywhere on } \Omega,$$

then $H = \mathrm{Im}\widetilde{S_+} \bigoplus \mathrm{Im}\widetilde{S_-}$ (topologically), $\widetilde{S_\pm}(\mathrm{D}(L)) \subset \mathrm{D}(L)$, $\widetilde{S_+} - \widetilde{S_-}$ is a linear homeomorphism on H with $(\widetilde{S_+}-\widetilde{S_-})^{-1} = \widetilde{S_+}-\widetilde{S_-}$ and, on $\mathrm{D}(L)$, we have $L\widetilde{S_\pm} = \widetilde{S_\pm}L$. Consequently, if we set, in equation (4.5),

$$u = (\widetilde{S_+}-\widetilde{S_-})v$$

so that $v = (\widetilde{S_+}-\widetilde{S_-})u$, then we obtain the equivalent equation $L(\widetilde{S_+}-\widetilde{S_-})v - N((\widetilde{S_+}-\widetilde{S_-})v) = \overline{h}$. Moreover, $u_j \in H_j$ is a solution of (4.7) if and only if $v_j = (\widetilde{S_+}-\widetilde{S_-})u_j \in H_j$ satisfies the equation

$$\widetilde{L}v_j - P_j\widetilde{N}v_j = P_j\overline{h},$$

where $\widetilde{L} = L(\widetilde{S_+}-\widetilde{S_-})$, $\widetilde{N} = N(\widetilde{S_+}-\widetilde{S_-})$. Now, $\widetilde{L}$ has the same domain, kernel, range and spectrum as L. For every $w \in H, \widetilde{N}$ is also of class C^1 at w, and

$$\widetilde{N}'(w) = N'((\widetilde{S_+} - \widetilde{S_-})w)(\widetilde{S_+} - \widetilde{S_-}).$$

Next, we will show that $\widetilde{N}$ is Lipschizian and monotonic. From condition (3.3) and the above property, i.e., $(\widetilde{S_+} - \widetilde{S_-})^{-1} = \widetilde{S_+} - \widetilde{S_-}$, of $\widetilde{S_+} - \widetilde{S_-}$, we have (note that $\rho(A) = \|A\|$) that

$$\begin{aligned}\|\widetilde{N}'(w)\| &= \|N'((\widetilde{S_+} - \widetilde{S_-})w)(\widetilde{S_+} - \widetilde{S_-})\| \\ &\leqslant \|N'((\widetilde{S_+} - \widetilde{S_-})w)\| = \|{f_u}'(t,x,(\widetilde{S_+} - \widetilde{S_-})w)\| \\ &\leqslant \max\{\|A\|,\|B\|\} \leqslant \max_{1\leqslant k\leqslant n}\{|\nu_{1k}|,|\nu_{2k}|\}.\end{aligned}$$

This means that $\widetilde{N}$ is Lipschitzian.

For every $w, v \in H$, using the symmetry of $N'(u)$, we can obtain

$$\begin{aligned}(\widetilde{N}'(w)v,v) &= (N'((\widetilde{S_+} - \widetilde{S_-})w)(\widetilde{S_+} - \widetilde{S_-})v, (\widetilde{S_+} - \widetilde{S_-})v) \\ &= (N'((\widetilde{S_+} - \widetilde{S_-})w)\widetilde{S_+}v, \widetilde{S_+}v) \\ &\quad - (N'((\widetilde{S_+} - \widetilde{S_-})w)\widetilde{S_-}v, \widetilde{S_-}v) \\ &\geqslant ((A + \alpha(\|w\|))\widetilde{S_+}v, \widetilde{S_+}v) \\ &\quad - ((B - \beta(\|w\|))\widetilde{S_-}v, \widetilde{S_-}v) \geqslant 0.\end{aligned}$$

This implies that $\widetilde{N}$ is monotone and takes bounded sets into bounded sets. Since $\sigma(\widetilde{L}) \setminus \{0\}$ is made of eigenvalues with finite multiplicity with no finite accumulation point, its right inverse

$$\widetilde{K} = \left(\widetilde{L}\mid_{\mathrm{D}(\widetilde{L})\bigcap \mathrm{Im}\widetilde{L}}\right)^{-1}$$

is compact, and we can apply Lemmas 4.1 and 4.2 to obtain the existence of $v \in \mathrm{D}(\widetilde{L})$ such that $\widetilde{L}v - \widetilde{N}v = \overline{h}$, and hence, we have the existence of the solution $u = (\widetilde{S_+} - \widetilde{S_-})v$ for (4.5), i.e., the existence of the solution u to (1.1)–(1.3).

For the uniqueness, let u^1 and u^2 be two periodic solutions to (1.1)–(1.3), and set

$$u_j^i = P_j u^i, \qquad u_j^i = x_j^i + z_j^i, \qquad x_j^i \in X_j, \qquad z_j^i \in Z_j,$$
$$v_j = x_j^1 - x_j^2, \qquad w_j = z_j^1 - z_j^2,$$

so that $u_j^1 - u_j^2 = v_j + w_j$, $j \in \mathbf{N}$. Therefore, using the notation of Lemma 4.1, we have

$$\begin{aligned}
0 &= \langle L(u^1 - u^2) - N(u^1 - u^2), v_j - w_j \rangle \\
&= \langle L(u_j^1 - u_j^2), v_j - w_j \rangle \\
&\quad - \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))(u^1 - u^2)\, ds, v_j - w_j \right\rangle \\
&= \langle L(v_j + w_j), v_j - w_j \rangle \\
&\quad - \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))(v_j + w_j)\, ds, v_j - w_j \right\rangle \\
&\quad - \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))(u^1 - u_j^1 + u_j^2 - u^2)\, ds, v_j - w_j \right\rangle
\end{aligned}$$

$$= \langle Lv_j, v_j \rangle - \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))v_j\, ds, v_j \right\rangle \tag{4.16}$$

$$- \langle Lw_j, w_j \rangle + \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))w_j\, ds, w_j \right\rangle \tag{4.17}$$

$$\begin{aligned}
&- \left\langle \int_0^1 N'(u^2 + s(u^1 - u^2))(u^1 - u_j^1 + u_j^2 - u^2)\, ds, v_j - w_j \right\rangle \qquad (4.18)\\
&\geqslant \int_0^1 \beta(\|u^2 + s(u^1 - u^2)\|)\, ds \|v_j\|^2 \\
&\quad + \int_0^1 \alpha(\|u^2 + s(u^1 - u^2)\|)\, ds \|w_j\|^2 \\
&\quad - C(\|u^1 - u_j^1\| + \|u^2 - u_j^2\|),
\end{aligned}$$

where the first two terms of the last inequality are due to (4.16), (4.17) and Lemma 4.1, and the last term is due to (4.18) and condition (3.3), with $C > 0$, some constant depending only upon ν_{1k}, ν_{2k}, $k = 1, 2, \ldots, n$, and r (see Lemma 4.2). Consequently, $v_j \to 0$ and $w_j \to 0$ as $j \to \infty$, so that

$$u^1 - u^2 = \lim_{j\to\infty} (u_j^1 - u_j^2) = \lim_{j\to\infty} (v_j + w_j) = 0,$$

and the proof of the theorem is complete. $\square$

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