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INTRODUCTION SUBSPACES OF THE DUALS OF MEASURE ALGEBRAS

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ABSTRACT. Let $\mathcal{G}$ be a locally compact group. In continuation of our studies on the first and second duals of measure algebras by the use of the theory of generalized functions, here we study the C*-subalgebra $GL_0(\mathcal{G})$ of $GL(\mathcal{G})$ as an introverted subspace of $M(\mathcal{G})^*$. In the case where $\mathcal{G}$ is non-compact, we show that any topological left invariant mean on $GL(\mathcal{G})$ lies in $GL_0(\mathcal{G})^\perp$. We then endow $GL_0(\mathcal{G})^*$ with an Arens-type product, which contains $M(\mathcal{G})$ as a closed subalgebra and $M_a(\mathcal{G})$ as a closed ideal, which is a solid set with respect to absolute continuity in $GL_0(\mathcal{G})^*$. Among other things, we prove that $\mathcal{G}$ is compact if and only if $GL_0(\mathcal{G})^*$ has a non-zero left (weakly) completely continuous element.

1. Introduction. Throughout this paper, $\mathcal{G}$ is a locally compact group with left Haar measure λ and identity element e, and the notations $C_c(\mathcal{G})$ and $C_0(\mathcal{G})$ refer to the space of all bounded complex-valued continuous functions with compact support and the space of all functions vanishing at infinity, respectively. Moreover, $M(\mathcal{G})$ refers to the measure algebra of $\mathcal{G}$ consisting of all complex regular Borel measures on $\mathcal{G}$ with the total variation norm and the convolution product $*$ defined by the formula

$$\langle \mu * \nu, g\rangle = \int_{\mathcal{G}} \int_{\mathcal{G}} g(xy)\, d\mu(x)\, d\nu(y)$$

for all $\mu, \nu \in M(\mathcal{G})$ and $g \in C_0(\mathcal{G})$. It is folklore that $M(\mathcal{G})$ is the first dual space of $C_0(\mathcal{G})$ for the pairing

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$$\langle \mu, g \rangle := \int_{\mathcal{G}} g(x)\, d\mu(x), \quad \mu \in M(\mathcal{G}),\ g \in C_0(\mathcal{G}).$$

In the last 30 years, research on the second duals of Banach algebras has mostly centered around the Banach algebras related to locally compact groups and has been dealt with by Lau, et al., in the works **[3, 4, 7, 12]**. In particular, **[7]** is the first important work devoted to the study of the second duals of measure algebras. Among other things, the authors of **[7]** have conjectured that the Banach algebra $M(\mathcal{G})$ is strongly Arens irregular, and its second dual $M(\mathcal{G})^{**}$ determines $\mathcal{G}$ in the category of all locally compact groups. Later, the second duals of measure algebras were studied in a series of papers. Here, we would like to mention that the Gharahmani-Lau conjecture on the strong Arens irregularity of $M(\mathcal{G})$, stated in **[7]** as well as earlier in **[11]**, has recently been solved **[14]**. Particularly, **[4]** is the second important work devoted to the study of $M(\mathcal{G})^{**}$, where most of the known results about these Banach algebras up to the year 2012 may be found. Recall from **[4]** that $M(\mathcal{G})^*$, the first dual space of $M(\mathcal{G})$, as the second dual of the C*-algebra $C_0(\mathcal{G})$, is a commutative unital C*-algebra, and therefore, if $\widetilde{\mathcal{G}}$ denotes the hyper-Stonean envelope of $\mathcal{G}$, then we can recognize $M(\mathcal{G})^*$ as $C(\widetilde{\mathcal{G}})$, the space of all bounded complex-valued continuous functions on $\widetilde{\mathcal{G}}$. It follows that

$$M(\mathcal{G})^{**} \cong M(\widetilde{\mathcal{G}}),$$

where $\cong$ denotes the isometric algebra isomorphism. Many authors, up until the year 2012, have used a type of this identification as a tool for the study of $M(\mathcal{G})^{**}$, see **[4]** and the references therein for more details.

Recently, in the works **[6, 10]**, we studied the first and second duals of measure algebras by the use of the theory of generalized functions, which were introduced and investigated by Šreĭdr **[16]** and Wong **[17, 18]**. In those papers, we observed that $GL_0(\mathcal{G})$, the space of all generalized functions which vanishes at infinity, plays a crucial role in our investigation. Motivated by this, here we study the C*-algebra $GL_0(\mathcal{G})$ as an introverted subspace of $M(\mathcal{G})^*$. In particular, in the case where $\mathcal{G}$ is non-compact, we show that any topological left invariant mean on $GL(\mathcal{G})$ lies in $GL_0(\mathcal{G})^\perp$, which demonstrates that the weak*-closed subspace $GL_0(\mathcal{G})^\perp$ of $M(\mathcal{G})^*$ is far from devoid of interest. We then endow $GL_0(\mathcal{G})^*$ with an Arens-type product which contain $M(\mathcal{G})$ and $M_a(\mathcal{G})$ as a closed subalgebra and a closed ideal, respectively. Among other things, we prove that the existence of a non-zero left (weakly) completely continuous element in $GL_0(\mathcal{G})^*$ is equivalent to the compactness of $\mathcal{G}$.

2. Generalized functions: An overview. In this section, we give a brief overview of generalized functions in the sense of Wong [**17**]. Nevertheless, we shall require some facts regarding the theory of C*-algebra. For background on this theory, we use [**15**] as a reference and adopt that book's notation. Moreover, our notation and terminology are standard and, concerning Banach algebras related to locally compact groups, they are in general those of Hewitt and Ross [**9**]. This section is mostly contained in the papers of Wong [**17, 18**].

For any complex regular Borel measure μ on $\mathcal{G}$, let $L^\infty(|\mu|)$ denotes the Banach space of all essentially bounded $|\mu|$–measurable complex functions f_μ on $\mathcal{G}$ with the essential supremum norm

$$\|f_\mu\|_{\mu,\infty} = \inf\{\alpha \geq 0 : |f_\mu| \leq \alpha,\ |\mu|\text{-almost everywhere}\}.$$

Consider the product linear space

$$\prod\{L^\infty(|\mu|) : \mu \in M(\mathcal{G})\}.$$

An element $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ in this product is called a *generalized function* if $f_\mu = f_\nu$ $|\mu|$-almost everywhere for any $\mu, \nu \in M(\mathcal{G})$ with $\mu \ll \nu$, where $\mu \ll \nu$ means that $|\mu|$ is absolutely continuous with respect to $|\nu|$. We note that this condition implies that, for given generalized functions $f = (f_\mu)_{\mu\in M(\mathcal{G})}$,

$$\sup\{\|f_\mu\|_{\mu,\infty} : \mu \in M(\mathcal{G})\} < \infty;$$

otherwise, there is a sequence (μ_n) in $M(\mathcal{G})$ for which $\|f_{\mu_n}\|_{\mu_n,\infty} \geq n$ for all $n \in \mathbb{N}$. Set

$$\mu = \sum_{n=1}^{\infty} 2^{-n}\|\mu_n\|^{-1}|\mu_n|.$$

Then, $\mu_n \ll \mu$, and hence, $\|f_\mu\|_{\mu,\infty} \geq \|f_{\mu_n}\|_{\mu_n,\infty} \geq n$ for all $n \in \mathbb{N}$, which is a contradiction.

Now, following Wong [**17**], we use the notation $GL(\mathcal{G})$ to denote the commutative unital C*-algebra of all generalized functions endowed with the coordinatewise operations, the involution $f \mapsto f^*$, where $f^* :=$

$(\overline{f_\mu})_{\mu\in M(\mathcal{G})}$, and the norm

$$\|f\|_\infty := \sup\{\|f_\mu\|_{\mu,\infty} : \mu \in M(\mathcal{G})\},$$

where $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ is in $GL(\mathcal{G})$. The identity element of $GL(\mathcal{G})$ is, of course, the generalized function $\mathbf{1} := (1_\mu)_{\mu\in M(\mathcal{G})}$, where 1_μ is the identity element of $L^\infty(|\mu|)$. Moreover, we write $f = (f_\mu)_{\mu\in M(\mathcal{G})} \geq 0$ to mean that the generalized function f is positive in the C*-algebra sense, and denote by $GL(\mathcal{G})^+$ the set of all positive elements of $GL(\mathcal{G})$.

Remark 2.1. It is not difficult to verify that a generalized function $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ is positive in the C*-algebra sense if and only if $f_\mu(x) \geq 0$ for all $x \in \mathcal{G}$ and all $\mu \in M(\mathcal{G})$; see [**15,** page 45], [**17,** page 85] for more information.

As a main result, Wong [**17**] has shown that, for each $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ in $GL(\mathcal{G})$, the equation

$$\langle \Psi(f), \zeta\rangle := \int_{\mathcal{G}} f_\zeta(x)\, d\zeta(x), \quad \zeta \in M(\mathcal{G}),$$

defines a linear functional $\Psi(f)$ on $M(\mathcal{G})$. In particular, the map $f \mapsto \Psi(f)$ is an isometric linear mapping from $GL(\mathcal{G})$ onto $M(\mathcal{G})^*$; see [**16**] and [**17,** Theorems 2.1, 2.2] for the same result on the special case where $\mathcal{G}$ is a certain locally compact abelian group. In particular, any $L \in M(\mathcal{G})^*$ can be considered as a generalized function $\Psi^{-1}(L)$, and we do not distinguish between a generalized function f and its unique corresponding linear functional $\Psi(f)$. In particular, this duality allows us to consider $GL(\mathcal{G})$ as a Banach $M(\mathcal{G})$-bimodule. In particular, if ζ and $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ are arbitrary elements of $M(\mathcal{G})$ and $GL(\mathcal{G})$, respectively, then, one can consider the linear functionals $f\zeta$ and ζf on $M(\mathcal{G})$ defined by

$$\langle f\zeta, \mu\rangle = \langle f, \zeta * \mu\rangle, \qquad \langle \zeta f, \mu\rangle = \langle f, \mu * \zeta\rangle, \quad \mu \in M(\mathcal{G}).$$

In order to find the generalized functions corresponding to these linear functionals, following Wong [**17**], [**18,** page 610], we define

$$\zeta \circ f \in \prod\{L^\infty(|\mu|) : \mu \in M(\mathcal{G})\}$$

as

$$(\zeta \circ f)_\mu = l_\zeta f_{\zeta*\mu}, \quad \mu \in M(\mathcal{G}),$$

where

$$l_\zeta f_{\zeta*\mu}(y) = \int_{\mathcal{G}} f_{\zeta*\mu}(xy)\, d\zeta(x) \quad \text{for } |\mu|\text{-almost everywhere } y \in \mathcal{G}.$$

Then, $\zeta \circ f$ is again a generalized function such that

$$\langle \zeta \circ f, \mu \rangle = \langle f, \zeta * \mu \rangle, \quad \mu \in M(\mathcal{G}),$$

see [**17,** pages 88, 89]. Thus, $\zeta \circ f$ is the generalized function corresponding to the functional $f\zeta \in M(\mathcal{G})^*$ such that $\Psi(\zeta \circ f) = f\zeta$. In addition, by using the right convolution notation, we can show that $f \circ \zeta$ is the generalized function corresponding to the functional $\zeta f \in M(\mathcal{G})^*$. In what follows, we do not distinguish between the linear functionals $f\zeta$ and ζf and their corresponding generalized functions. Later on, we will need the next remark in our present investigation.

Remark 2.2. Suppose that $BM(\mathcal{G})$ denotes the Banach space of all bounded Borel measurable functions on $\mathcal{G}$ with the supremum norm $\|\cdot\|_u$. Then, each $f \in BM(\mathcal{G})$ may be regarded as an element $(f_\mu)_{\mu \in M(\mathcal{G})}$ in $GL(\mathcal{G})$ where, for each $\mu \in M(\mathcal{G})$, the functions f_μ denote the equivalent class of f in $L^\infty(|\mu|)$. Hence, $BM(\mathcal{G})$ can be considered as a closed subspace of $GL(\mathcal{G})$ containing the space $C_b(\mathcal{G})$ of all complex-valued continuous bounded functions on $\mathcal{G}$. Moreover, each $f \in BM(\mathcal{G})$ may be regarded as an element in $M(\mathcal{G})^*$ by the pairing $\langle f, \mu \rangle = \int_{\mathcal{G}} f d\mu$, $\mu \in M(\mathcal{G})$. In this case, the restriction of the map Ψ to $BM(\mathcal{G})$ is precisely the embedding of $BM(\mathcal{G})$ into $M(\mathcal{G})^*$.

3. Generalized functions that vanish at infinity.

We commence this section by recalling the main object of the work which is introduced and studied by the authors in [**10**].

Definition 3.1. A generalized function $f = (f_\mu)_{\mu \in M(\mathcal{G})}$ vanishes at infinity if, for each $\varepsilon > 0$, there is a compact subset K_ε of $\mathcal{G}$ for which $\|f_\mu \chi_{\mathcal{G}\setminus K_\varepsilon}\|_{\mu,\infty} < \varepsilon$ for all $\mu \in M(\mathcal{G})$; formally, for all $\varepsilon > 0$, there exists a $K_\varepsilon \in \mathcal{K}(\mathcal{G})$ such that, for all $\mu \in M(\mathcal{G})$,

$$|f_\mu(x)| < \varepsilon \quad \text{for } |\mu|\text{-almost all } x \in \mathcal{G} \setminus K_\varepsilon,$$

where χ_{K_ε} denotes the characteristic function of K_ε on $\mathcal{G}$, and $\mathcal{K}(\mathcal{G})$ denotes the set of all compact subsets in $\mathcal{G}$.

We denote by $GL_0(\mathcal{G})$ the C*-subalgebra of $GL(\mathcal{G})$ consisting of all generalized functions that vanish at infinity.

The aim of the present section is to study some aspects of $GL_0(\mathcal{G})$ as a C*-subalgebra of $GL(\mathcal{G})$. We give a simple but important result whose proof involves nothing more than routine calculations.

Lemma 3.2. *Suppose that $\mathcal{K}(\mathcal{G})$ is directed downwards and, for each α, $u_{K_\alpha} \in C_c(\mathcal{G})$ is chosen such that $0 \leq u_{K_\alpha} \leq 1$ and $u_{K_\alpha}(x) = 1$ for all $x \in K_\alpha$. Then, (u_{K_α}) is a bounded approximate identity for $GL_0(\mathcal{G})$.*

Our next result shows that the subspaces

$$M(\mathcal{G}) \circ GL_0(\mathcal{G}) := \{\zeta \circ f : \zeta \in M(\mathcal{G}) \text{ and } f \in GL_0(\mathcal{G})\}$$

and

$$GL_0(\mathcal{G}) \circ M(\mathcal{G}) := \{f \circ \zeta : \zeta \in M(\mathcal{G}) \text{ and } f \in GL_0(\mathcal{G})\}$$

of $GL(\mathcal{G})$ coincide with $GL_0(\mathcal{G})$.

Lemma 3.3. *The following assertions hold.*

(i) $M(\mathcal{G}) \circ GL_0(\mathcal{G}) = GL_0(\mathcal{G})$;
(ii) $GL_0(\mathcal{G}) \circ M(\mathcal{G}) = GL_0(\mathcal{G})$.

Proof. We prove the first; the proof of the second is similar. Since the inclusion $GL_0(\mathcal{G}) \subseteq \delta_e \circ GL_0(\mathcal{G}) \subseteq M(\mathcal{G}) \circ GL_0(\mathcal{G})$ holds, it will be sufficient to prove the reverse inclusion. Toward this end, let $\zeta \in M(\mathcal{G})$, $f = (f_\mu)_{\mu \in M(\mathcal{G})} \in GL_0(\mathcal{G})$ and $\epsilon > 0$ be given. Without loss of generality, we may assume that ζ is non-zero and positive and that $f \neq 0$. By the regularity of ζ, we can choose a compact subset K_1 of $\mathcal{G}$ such that $0 < \zeta(\mathcal{G} \setminus K_1) < (\epsilon/2)\|f\|_\infty$. Also, since f vanishes at infinity, there is a compact subset K_2 in $\mathcal{G}$ with $\|f - \chi_{K_2} f\|_\infty < (\epsilon/2)\|\zeta\|$. Therefore,

(3.1)
$$\|\zeta \circ f - (\chi_{K_1}\zeta) \circ (\chi_{K_2} f)\|_\infty \leq \|\zeta - \chi_{K_1}\zeta\| \|f\|_\infty + \|\chi_{K_1}\zeta\| \|f - \chi_{K_2} f\|_\infty \leq \epsilon,$$

where $\chi_{K_1}\zeta$ is the measure in $M(\mathcal{G})$ defined on each Borel subset A of $\mathcal{G}$ by

$$\chi_{K_1}\zeta(A) = \int_A \chi_{K_1} d\zeta.$$

Now, suppose that μ is an arbitrary element of $M(\mathcal{G})$. Observe that

$$((\chi_{K_1}\zeta) \circ (\chi_{K_2}f))_\mu(x) = \int_{K_1} \chi_{K_2}(yx) f_{(\chi_{K_1}\zeta)*\mu}(yx)\, d\zeta(y).$$

For each $x \in \mathcal{G} \setminus {K_1}^{-1}K_2$, we get $K_1 x \subseteq \mathcal{G} \setminus K_2$, and hence, $((\chi_{K_1}\zeta) \circ (\chi_{K_2}f))(x) = 0$ for μ-almost all $x \in \mathcal{G} \setminus {K_1}^{-1}K_2$. Thus, inequality (3.1) implies that

$$|(\zeta \circ f)_\mu(x)| < \varepsilon \quad \text{for } \mu\text{-almost all } x \in \mathcal{G} \setminus {K_1}^{-1}K_2.$$

It follows that $\zeta \circ f \in GL_0(\mathcal{G})$. We have now completed the proof of Lemma 3.3. □

Now, let $M_a(\mathcal{G})$ be the closed ideal of $M(\mathcal{G})$ consisting of all absolutely continuous measures with respect to λ, and let $L^1(\mathcal{G})$ denote the group algebra of $\mathcal{G}$ as defined in [**9,** Theorems 14.17, 14.18]. Then, the Radon-Nikodym theorem can be interpreted as an identification of $M_a(\mathcal{G})$ with $\{\nu_\varphi : \varphi \in L^1(\mathcal{G})\}$, where ν_φ is the measure in $M(\mathcal{G})$ defined on each Borel subset A of $\mathcal{G}$ by

$$\nu_\varphi(A) = \int_A \varphi\, d\lambda.$$

This allows us to show that $M_a(\mathcal{G})^*$, the first dual space of $M_a(\mathcal{G})$, is $L^\infty(\mathcal{G})$, where $L^\infty(\mathcal{G})$ denotes the Lebesgue space as defined in [**9,** Definition 12.11] equipped with the essential supremum norm. Given any $\sigma \in M_a(\mathcal{G})$ and $g \in L^\infty(\mathcal{G})$, define the complex-valued functions $g \star \sigma$ and $\sigma \star g$ on $\mathcal{G}$ by

$$(g \star \sigma)(x) = \langle g, \delta_x * \sigma \rangle = \int_{\mathcal{G}} g(xy)\, d\sigma(y)$$

and

$$(\sigma \star g)(x) = \langle g, \sigma * \delta_x \rangle = \int_{\mathcal{G}} g(yx)\, d\sigma(y)$$

for all $x \in \mathcal{G}$, where δ_x denotes the Dirac measure at x. Then, it is easy to verify that the functions $g \star \sigma$ and $\sigma \star g$ are in $C_b(\mathcal{G})$; this is due to the fact that $M_a(\mathcal{G})$ can be identified with all $\nu \in M(\mathcal{G})$ such that the maps $x \mapsto \delta_x * |\nu|$ and $x \mapsto |\nu| * \delta_x$ from $\mathcal{G}$ into $M(\mathcal{G})$ are norm continuous, see, for example, [**9,** 19.27, 20.31]. In particular, if

$$\mathcal{P} : GL(\mathcal{G}) \longrightarrow L^\infty(\mathcal{G})$$

is the adjoint of the natural embedding from $M_a(\mathcal{G})$ into $M(\mathcal{G})$, then $\mathcal{P}$ is the restriction mapping, and hence, norm decreasing and onto.

For the formulation of the following statements, we recall Remark 2.2 which allows us to consider $C_b(\mathcal{G})$ as a closed subspace of $GL(\mathcal{G})$ containing $C_0(\mathcal{G})$.

Lemma 3.4. *If σ is an arbitrary element of $M_a(\mathcal{G})$, then, for a given $f = (f_\mu)_{\mu\in M(\mathcal{G})}$ in $GL(\mathcal{G})$ and all $\mu \in M(\mathcal{G})$, we have*

(i) $(\sigma \circ f)_\mu = \sigma \star \mathcal{P}(f)$, *$|\mu|$-almost everywhere,*
(ii) $(f \circ \sigma)_\mu = \mathcal{P}(f) \star \sigma$, *$|\mu|$-almost everywhere.*

In particular, $(\sigma \circ f)_\mu$ and $(f \circ \sigma)_\mu$ are in $C_b(\mathcal{G})$ for all $\mu \in M(\mathcal{G})$.

Proof. We prove assertion (i); the proof of (ii) is similar. First, note that $\sigma \circ f$ is the generalized function $h = (h_\mu)_{\mu\in M(\mathcal{G})}$, where, for each $\mu \in M(\mathcal{G})$, the following equality is satisfied:

$$h_\mu(x) = (\sigma \circ f)_\mu(x) = l_\sigma f_{\sigma*\mu}(x) = \int_{\mathcal{G}} f_{\sigma*\mu}(yx)\, d\sigma(y).$$

On the other hand, for an arbitrary μ in $M(\mathcal{G})$ and any Borel subset A of $\mathcal{G}$, since $\chi_A\mu \ll \mu$, we have

$$\begin{aligned}\int_{\mathcal{G}} \chi_A h_\mu\, d\mu &= \int_{\mathcal{G}} f_{\sigma*\chi_A\mu} d(\sigma * \chi_A\mu)\\ &= \langle \mathcal{P}(f), \sigma * \chi_A\mu\rangle\\ &= \int_{\mathcal{G}}\int_{\mathcal{G}} \mathcal{P}(f)(yx)\, d\sigma(y)\, d(\chi_A\mu)(x)\\ &= \int_{\mathcal{G}} \chi_A\ (\sigma \star \mathcal{P}(f))\, d\mu.\end{aligned}$$

Hence, $(\sigma \circ f)_\mu = \sigma \star \mathcal{P}(f)$ $|\mu|$-almost everywhere $\mu \in M(\mathcal{G})$. It follows that $\sigma \circ f \in C_b(\mathcal{G})$. □

Now, in light of Lemmas 3.3 and 3.4, the following proposition is immediate.

Proposition 3.5. *The following assertions hold.*

(i) $M_a(\mathcal{G}) \circ GL_0(\mathcal{G}) = C_0(\mathcal{G})$;
(ii) $GL_0(\mathcal{G}) \circ M_a(\mathcal{G}) = C_0(\mathcal{G})$.

Recall from [**17,** page 90] that a linear functional m in $GL(\mathcal{G})^*$ is called a mean if $\mathsf{m}(\mathbf{1}) = 1$ and $\mathsf{m}(f) \geq 0$ whenever $f \in GL(\mathcal{G})$ with $f \geq 0$, and it is topological left invariant if $\mathsf{m}(\zeta \circ f) = \mathsf{m}(f)$ for all $f \in GL(\mathcal{G})$ and

$$\zeta \in P(\mathcal{G}) = \{\nu \in M(\mathcal{G}) : \nu \geq 0 \text{ and } \|\nu\| = 1\}.$$

In [**17,** Theorem 4.1], Wong proved that $GL(\mathcal{G})$ has a topological left invariant mean if and only if $M(\mathcal{G})^*$ has a topological left invariant mean. In particular, he showed that Ψ^*, the adjoint of Ψ, maps the set of all topological left invariant means on $M(\mathcal{G})^*$ onto that of $GL(\mathcal{G})$. Related to this result, we have the following result which asserts that, in the case where $\mathcal{G}$ is non-compact, any topological left invariant mean on $GL(\mathcal{G})$ lies in $GL_0(\mathcal{G})^\perp$, where here and in the sequel, $GL_0(\mathcal{G})^\perp$ denotes the following weak*-closed subspace of $GL(\mathcal{G})^*$

$$\{\mathsf{m} \in GL(\mathcal{G})^* : \langle \mathsf{m}, f\rangle = 0 \text{ for all } f \in GL_0(\mathcal{G})\}.$$

In fact, the next result shows that $GL_0(\mathcal{G})^\perp$ is far from devoid of interest.

Proposition 3.6. *If $\mathcal{G}$ is non-compact, then any topological left invariant mean on $GL(\mathcal{G})$ lies in $GL_0(\mathcal{G})^\perp$.*

Proof. Suppose that m is a topological left invariant mean on $GL(\mathcal{G})$. First, note that the non-compactness of $\mathcal{G}$ implies that there exists a sequence (x_n) of disjoint elements of $\mathcal{G}$ and a compact symmetric neighborhood V of e such that the sets x_nV for all $n \in \mathbb{N}$ are pairwise disjoint, see [**9,** 11.43(e)]. Now, it is not difficult to verify that $\chi_{xV} = \delta_{x^{-1}} \circ \chi_V$, $|\mu|$-almost everywhere for all $x \in \mathcal{G}$ and $\mu \in M(\mathcal{G})$. Moreover, by Remark 2.2, for each $p \in \mathbb{N}$, the function

$$\sum_{n=1}^{p} \chi_{x_nV} = \sum_{n=1}^{p} \delta_{x_n{}^{-1}} \circ \chi_V$$

is in $GL(\mathcal{G})$ for which $\sum_{n=1}^{p} \chi_{x_nV} \leq \mathbf{1}$. It follows that

$$p\langle \mathsf{m}, \chi_V\rangle = \left\langle \mathsf{m}, \sum_{n=1}^{p} \chi_{x_nV} \right\rangle \leq 1, \quad p \in \mathbb{N}.$$

Thus, $\langle \mathsf{m}, \chi_V\rangle = 0$, and therefore, we have

$$\langle \mathsf{m}, \chi_{xV}\rangle = \langle \mathsf{m}, \delta_{x^{-1}} \circ \chi_V\rangle = 0, \quad x \in \mathcal{G}. \tag{3.2}$$

Now, suppose that $f = (f_\mu)_{\mu \in M(\mathcal{G})}$ is a non-zero element of $GL_0(\mathcal{G})$. The proof will be completed by showing that $\langle \mathsf{m}, f\rangle = 0$. Toward this end, without loss of generality, we may assume that $\|f\|_\infty = 1$. Then, since $f^*f = (|f_\mu|^2)_{\mu \in M(\mathcal{G})}$ vanishes at infinity, for a given $\varepsilon > 0$, one can choose $y_1, \ldots, y_q \in \mathcal{G}$ such that

$$|f_\mu|^2 \leq \sum_{i=1}^{q} \chi_{y_i V} + \varepsilon, \quad |\mu|\text{-almost everywhere } \mu \in M(\mathcal{G}).$$

Now, by considering

$$h = \sum_{i=1}^{q} \chi_{y_i V} + \varepsilon \in BM(\mathcal{G})$$

as an element of $GL(\mathcal{G})$, we have $f^*f \leq h$, see Remark 2.1. Hence, in light of [**15,** Theorem 3.3.2] and equality (3.2), we see that

$$|\langle \mathsf{m}, f\rangle|^2 \leq \langle \mathsf{m}, f^*f\rangle \leq \langle \mathsf{m}, h\rangle \leq \varepsilon.$$

It follows that $\langle \mathsf{m}, f\rangle = 0$. Hence, $\mathsf{m} \in GL_0(\mathcal{G})^\perp$. $\square$

4. $GL_0(\mathcal{G})^*$ as a subalgebra of $M(\mathcal{G})^{**}$. As is known, there exist two natural products on $M(\mathcal{G})^{**}$ extending the one on $M(\mathcal{G})$, known as the first and second Arens products of $M(\mathcal{G})^{**}$. The first Arens product on $M(\mathcal{G})^{**}$ is defined in three steps as follows. For m, n in $M(\mathcal{G})^{**}$, the element $\mathsf{m} \odot \mathsf{n}$ of $M(\mathcal{G})^{**}$ is defined by

$$\langle \mathsf{m} \odot \mathsf{n}, f\rangle = \langle \mathsf{m}, \mathsf{n}f\rangle, \quad f \in GL(\mathcal{G}),$$

where $\langle \mathsf{m}f, \zeta\rangle = \langle \mathsf{m}, f\zeta\rangle$ and $f\zeta = \zeta \circ f$ for all $\zeta \in M(\mathcal{G})$. Equipped with this product, $M(\mathcal{G})^{**}$ is a Banach algebra which contains $M(\mathcal{G})$ as a subalgebra. Moreover, by the duality relation between $M(\mathcal{G})$ and $GL(\mathcal{G})$, there exists a unique generalized function $h \in GL(\mathcal{G})$ such that $\mathsf{m}f = \Psi(h)$; in what follows, we denote the generalized function $\Psi^{-1}(\mathsf{m}f)$ corresponding to $\mathsf{m}f \in M(\mathcal{G})^*$ by $\mathsf{m}f$. Moreover, $M(\mathcal{G})$ and $GL_0(\mathcal{G})$ are in duality with respect to the natural bilinear map given for each $\eta \in M(\mathcal{G})$ and $f = (f_\mu)_{\mu \in M(\mathcal{G})}$ in $GL_0(\mathcal{G})$ by

$$\langle f, \eta\rangle = \int_{\mathcal{G}} f_\eta \, d\eta.$$

Therefore, $M(\mathcal{G})$ may be identified with a closed subspace of $GL_0(\mathcal{G})^*$. Furthermore, if $f = (f_\mu)_{\mu \in M(\mathcal{G})} \in GL_0(\mathcal{G})$ and $\zeta \in M(\mathcal{G})$, then, by

Lemma 3.3, $\zeta \circ f$ and $f \circ \zeta$ are also in $GL_0(\mathcal{G})$ and

$$\langle \zeta \circ f, \nu \rangle = \langle f, \zeta * \nu \rangle \quad \text{and} \quad \langle f \circ \zeta, \nu \rangle = \langle f, \nu * \zeta \rangle,$$

for all $\nu \in M(\mathcal{G})$. Hence, the product $\odot$ is well defined on $GL_0(\mathcal{G})^*$, and $GL_0(\mathcal{G})^*$ is a Banach algebra with this product, if we show that $GL_0(\mathcal{G})$ is a topologically introverted subspace of $GL(\mathcal{G})$. Toward this end, we have the following result.

Proposition 4.1. *The space $GL_0(\mathcal{G})$ is left (right) topologically introverted in $GL(\mathcal{G})$, that is, $\mathsf{m}f \in GL_0(\mathcal{G})$ ($f\mathsf{m} \in GL_0(\mathcal{G})$) for all $\mathsf{m} \in GL_0(\mathcal{G})^*$ and $f \in GL_0(\mathcal{G})$.*

Proof. We need only show that $GL_0(\mathcal{G})$ is a left topologically introverted subspace of $GL(\mathcal{G})$; the proof of the other assertion is similar. Toward this end, let $\mathsf{m} \in GL_0(\mathcal{G})^*$, $f = (f_\mu)_{\mu \in M(\mathcal{G})} \in GL_0(\mathcal{G})$ and $\varepsilon > 0$ be given. Since $GL_0(\mathcal{G})$ is spanned by its positive elements, we can suppose that $\mathsf{m} \geq 0$. Also, since f vanishes at infinity, there is a compact set B in $\mathcal{G}$ with $|f_\mu(x)| < \varepsilon$ for μ-almost all $x \in \mathcal{G} \setminus B$ ($\mu \in M(\mathcal{G})$).

Now, let ϱ denote the restriction of m to $C_0(\mathcal{G})$. Then, there exists a compact subset K of $\mathcal{G}$ such that $\varrho(\mathcal{G} \setminus K) < \varepsilon/2$. In particular, if m_K denotes the continuous linear functional on $GL_0(\mathcal{G})$, defined by

$$\langle \mathsf{m}_K, h \rangle := \langle \mathsf{m}, h - u_K h \rangle, \quad h \in GL_0(\mathcal{G}),$$

where u_K is a fixed function in $C_c(\mathcal{G})$ such that $0 \leq u_K \leq 1$, and $u_K(x) = 1$ for all $x \in K$. Then, the positivity of the linear functional m_K on $GL_0(\mathcal{G})$ implies that $\|\mathsf{m}_K\| = \lim_\alpha \langle \mathsf{m}_K, u_{K_\alpha} \rangle$, where (u_{K_α}) is the net introduced in Lemma 3.2. Hence, there exists an α_0 such that

$$\|\mathsf{m}_K\| - \frac{\varepsilon}{2} \leq \langle \mathsf{m}_K, u_{K_{\alpha_0}} \rangle \leq \|\mathsf{m}_K|_{C_0(\mathcal{G})}\|.$$

It follows that $\|\mathsf{m}_K\| < \varepsilon$. Indeed,

$$\begin{aligned} \|\mathsf{m}_K|_{C_0(\mathcal{G})}\| &= \sup\{|\langle \varrho, g - u_K g \rangle| : g \in C_0(\mathcal{G}) \text{ and } \|g\| \leq 1\} \\ &= \sup\{|\langle \chi_{\mathcal{G} \setminus K} \varrho, (g - u_K g) \rangle| : g \in C_0(\mathcal{G}) \text{ and } \|g\| \leq 1\} \\ &\leq \|\chi_{\mathcal{G} \setminus K} \varrho\| = \varrho(\mathcal{G} \setminus K). \end{aligned}$$

If, now, ν is an arbitrary probability measure in $M(\mathcal{G})$, then $\zeta := (\chi_{\mathcal{G} \setminus BK^{-1}})\nu$ is a measure in $M(\mathcal{G})$ for which $\operatorname{supp}(\zeta) \subseteq \mathcal{G} \setminus BK^{-1}$. Further, choose a compact subset D in $\mathcal{G}$ for which $D \subseteq \mathcal{G} \setminus BK^{-1}$ and

$|\zeta|(\mathcal{G}\setminus D)<\varepsilon$. Trivially, for each $x\in\mathcal{G}\setminus D^{-1}B$, we see that $Dx\subseteq\mathcal{G}\setminus B$, and therefore, for each $\mu\in M(\mathcal{G})$, we have

$$|(\zeta\circ f)_\mu(x)|\le\int_{\mathcal{G}\setminus D}|f_{\zeta*\mu}(yx)|\,d|\zeta|(y)+\int_D|f_{\zeta*\mu}(yx)|\,d|\zeta|(y)\le\varepsilon(\|f\|_\infty+1),$$

that is, $|(\zeta\circ f)_\mu(x)|\le\varepsilon(\|f\|_\infty+1)$ for μ-almost all $x\in\mathcal{G}\setminus D^{-1}B$. In particular, since $D^{-1}B\cap K=\emptyset$, we see that

$$\|(\zeta\circ f)\chi_K\|_\infty=\sup_{\mu\in M(\mathcal{G})}\|(\zeta\circ f)_\mu\chi_K\|_{\mu,\infty}\le\varepsilon(\|f\|_\infty+1).$$

Thus,

$$\begin{aligned}\int_{\mathcal{G}\setminus BK^{-1}}(\mathsf{m}f)_\zeta(x)\,d\zeta(x)&=\langle\mathsf{m}f,\zeta\rangle\\&=\langle\mathsf{m},u_K(\zeta\circ f)\rangle+\langle\mathsf{m}_K,\zeta\circ f\rangle\\&\le\varepsilon(\|f\|_\infty+1)\|\mathsf{m}\|+\varepsilon\|\zeta\|\|f\|_\infty.\end{aligned}$$

On the other hand, since $\zeta\ll\nu$, we have

$$\begin{aligned}\int_{\mathcal{G}\setminus BK^{-1}}(\mathsf{m}f)_\zeta(x)\,d\zeta(x)&=\int_{\mathcal{G}\setminus BK^{-1}}(\mathsf{m}f)_\nu(x)\,d\zeta(x)\\&=\int_{\mathcal{G}\setminus BK^{-1}}(\mathsf{m}f)_\nu(x)\,d\nu(x).\end{aligned}$$

This shows that, if $\nu\in M(\mathcal{G})$, then

$$(\mathsf{m}f)_\nu(x)\le\varepsilon[(\|f\|_\infty+1)\|\mathsf{m}\|+\|f\|_\infty]$$

for ν-almost all $x\in\mathcal{G}\setminus BK^{-1}$, and thus, $\mathsf{m}f\in GL_0(\mathcal{G})$. □

A linear functional m in $GL_0(\mathcal{G})^*$, respectively, $M(\mathcal{G})^{**}$, has *compact carrier* if there exists a compact set K in $\mathcal{G}$ such that $\langle\mathsf{m},f\rangle=\langle\mathsf{m},\chi_K f\rangle$ for all $f\in GL_0(\mathcal{G})$, respectively, $GL(\mathcal{G})$, such a compact set K is called a compact carrier for m. In the sequel, the notation $M_c(\mathcal{G})^{**}$ is used to denote the norm closure of functionals in $M(\mathcal{G})^{**}$ with compact carrier.

Now, with an argument similar to the proof of [**12,** Propositions 2.6, 2.7 and Theorems 2.8, 2.11], the following result can be proved which, in particular, shows that the restriction map is an isometric algebra isomorphism from $M_c(\mathcal{G})^{**}$ onto $GL_0(\mathcal{G})^*$. In other words, this result allows us to view $GL_0(\mathcal{G})^*$ as a subalgebra of $M(\mathcal{G})^{**}$.

Theorem 4.2. *The following assertions hold.*

(i) *Functionals in $GL_0(\mathcal{G})^*$ with compact carriers are norm dense in $GL_0(\mathcal{G})^*$.*
(ii) *If m and n are elements in $GL_0(\mathcal{G})^*$, respectively, $M_c(\mathcal{G})^{**}$, with compact carriers K and K' respectively, then $\mathsf{m} \odot \mathsf{n}$ has compact carrier KK'.*
(iii) *The restriction map is an isometry and an algebra isomorphism from $M_c(\mathcal{G})^{**}$ onto $GL_0(\mathcal{G})^*$.*
(iv) *$M(\mathcal{G})^{**} = GL_0(\mathcal{G})^* \oplus GL_0(\mathcal{G})^{\perp}$. In fact, any $\mathsf{m} \in M(\mathcal{G})^{**}$ has a unique decomposition $\mathsf{m} = \mathsf{m}_* + \mathsf{m}_\perp$, where $\mathsf{m}_* \in GL_0(\mathcal{G})^*$, $\mathsf{m}_\perp \in GL_0(\mathcal{G})^{\perp}$ and $\|\mathsf{m}\| = \|\mathsf{m}_*\| + \|\mathsf{m}_\perp\|$. Moreover, $\mathsf{m} \geq 0$ if and only if $\mathsf{m}_* \geq 0$ and $\mathsf{m}_\perp \geq 0$.*
(v) *$GL_0(\mathcal{G})^{\perp}$ is a weak*-closed ideal of $M(\mathcal{G})^{**}$.*
(vi) *$GL_0(\mathcal{G})^*$ is a left or right ideal of $M(\mathcal{G})^{**}$ if and only if $\mathcal{G}$ is compact.*
(vii) *$M(\mathcal{G})$ is a left or right ideal of $GL_0(\mathcal{G})^*$ if and only if $\mathcal{G}$ is discrete.*
(viii) *$M_a(\mathcal{G})$ is a two-sided ideal in $GL_0(\mathcal{G})^*$.*

Proof. The details are omitted, and we only give a proof for (ii) and (viii).

(ii) Suppose that m and n are elements in $GL_0(\mathcal{G})^*$, respectively, $M_c(\mathcal{G})^{**}$, with compact carriers K and K', respectively, and that f is an arbitrary element in $GL_0(\mathcal{G})$, respectively, $GL(\mathcal{G})$. First, observe that

$$\langle \mathsf{m} \odot \mathsf{n}, f\rangle = \langle \mathsf{m}, \mathsf{n}f\rangle = \langle \mathsf{m}, \chi_K(\mathsf{n}f)\rangle.$$

On one hand, for $\mu \in M(\mathcal{G})$, we have $\zeta := \chi_K\mu \ll \mu$, and thus,

$$\begin{aligned}\langle \chi_K(\mathsf{n}f), \mu\rangle &= \int_{\mathcal{G}} \chi_K(\mathsf{n}f)_\mu \, d\mu \\ &= \int_{\mathcal{G}} (\mathsf{n}f)_\zeta \, d\zeta = \langle \mathsf{n}, \chi_{K'}(\zeta \circ f)\rangle.\end{aligned}$$

On the other hand, $\chi_{K'}(\zeta \circ f) = \zeta \circ (\chi_{KK'} f)$. Indeed, for each $\nu \in M(\mathcal{G})$ and $h \in C_0(\mathcal{G})$, we have

$$\begin{aligned}(\chi_{KK'}(\zeta * \nu))(h) &= \int_{\mathcal{G}} \chi_{KK'} h \, d(\zeta * \nu) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} \chi_{KK'}(xy)\chi_K(x)h(xy) \, d\mu(x) \, d\nu(y) \\ &= \int_{\mathcal{G}} \int_{\mathcal{G}} \chi_{K'}(y)\chi_K(x)h(xy) \, d\mu(x) \, d\nu(y) \\ &= (\zeta * (\chi_{K'}\nu))(h),\end{aligned}$$

and this implies that

$$\langle \chi_{K'}(\zeta \circ f), \nu \rangle = \langle \zeta \circ f, \chi_{K'}\nu \rangle = \langle f, \chi_{KK'}(\zeta * \nu) \rangle = \langle \zeta \circ (\chi_{KK'}f), \nu \rangle.$$

Hence, by using these equalities, we have

$$\langle \chi_K(\mathsf{n}f), \mu \rangle = \langle \mathsf{n}, \zeta \circ (\chi_{KK'}f) \rangle = \langle \mathsf{n}(\chi_{KK'}f), \zeta \rangle = \langle \chi_K(\mathsf{n}(\chi_{KK'}f)), \mu \rangle.$$

Consequently,

$$\begin{aligned}\langle \mathsf{m} \odot \mathsf{n}, f \rangle &= \langle \mathsf{m}, \chi_K(\mathsf{n}f) \rangle = \langle \mathsf{m}, \chi_K(\mathsf{n}(\chi_{KK'}f)) \rangle \\ &= \langle \mathsf{m}, \mathsf{n}(\chi_{KK'}f) \rangle = \langle \mathsf{m} \odot \mathsf{n}, \chi_{KK'}f \rangle.\end{aligned}$$

It follows that $\mathsf{m} \odot \mathsf{n}$ has compact carrier KK'.

(viii) That $M_a(\mathcal{G})$ is a closed subalgebra of $GL_0(\mathcal{G})^*$ is trivial. Now, suppose that $\sigma \in M_a(\mathcal{G})$ and $\mathsf{m} \in GL_0(\mathcal{G})^*$. We show that $\mathsf{m} \odot \sigma \in GL_0(\mathcal{G})^*$; that $\sigma \odot \mathsf{m} \in GL_0(\mathcal{G})^*$ is similar. Let ζ denote the restriction of m to $C_0(\mathcal{G})$. Since $M_a(\mathcal{G})$ is an ideal in $M(\mathcal{G})$, we have $\zeta * \sigma \in M_a(\mathcal{G})$. We now invoke Proposition 3.5 to conclude that

$$\langle \mathsf{m} \odot \sigma, f \rangle = \langle \zeta, f \circ \sigma \rangle = \langle \zeta * \sigma, f \rangle, \quad f \in GL_0(\mathcal{G}),$$

whence $\mathsf{m} \odot \sigma = \zeta * \sigma \in M_a(\mathcal{G})$. □

As is standard, for a locally compact space X, we say that a subset $S \subseteq M(X)$, the Banach space of all complex regular Borel measures on X, is solid with respect to absolute continuity, if $t \in S$ wherever $t \ll s$, for some $s \in S$. Now, as an application of Theorem 4.2 above, by a method similar to that of [**8,** Lemma 5, Theorem 6], the following generalization of that theorem may be obtained. The reader will see that the compactness of $\mathcal{G}$ is assumed in that proof only to conclude that $M_a(\mathcal{G})$ is an ideal in $M(\mathcal{G})^{**}$, whereas $M_a(\mathcal{G})$ is always an ideal of $GL_0(\mathcal{G})^*$. The details are omitted.

Theorem 4.3. *$M_a(\mathcal{G})$ is the unique minimal proper closed subset of $GL_0(\mathcal{G})^*$ which is an algebraic ideal and a solid set with respect to absolute continuity in $GL_0(\mathcal{G})^*$.*

Next, we turn our attention to the study of left (weakly) completely continuous elements of $GL_0(\mathcal{G})^*$. Toward this end, recall that, if $\mathcal{A}$ is a Banach algebra, then $a \in \mathcal{A}$ is said to be a left (weakly) completely continuous element of $\mathcal{A}$ whenever the operator $\ell_a : b \mapsto ab$ is a (weakly) compact operator on $\mathcal{A}$.

In what follows, for $I \subseteq GL_0(\mathcal{G})^*$, the left annihilator of I is denoted by $\mathrm{lan}(I)$ and defined by

$$\mathrm{lan}(I) = \{\mathsf{l} \in GL_0(\mathcal{G})^* : \mathsf{l} \odot I = \{0\}\};$$

also, the right annihilator of I is denoted by $\mathrm{ran}(I)$ and defined by

$$\mathrm{ran}(I) = \{\mathsf{r} \in GL_0(\mathcal{G})^* : I \odot \mathsf{r} = \{0\}\}.$$

Moreover, $L_0^\infty(\mathcal{G})$ stands for the C*-subalgebra of $L^\infty(\mathcal{G})$ consisting of all functions g on $\mathcal{G}$ such that, for each $\varepsilon > 0$, there is a compact subset K of $\mathcal{G}$ for which $|g(x)| < \varepsilon$ for all $x \in \mathcal{G} \setminus K$.

Theorem 4.4. *The following assertions hold.*

(i) *If $\sigma \in M_a(\mathcal{G})$, then σ is a left (weakly) completely continuous element of $M_a(\mathcal{G})$ if and only if σ is a left (weakly) completely continuous element of $GL_0(\mathcal{G})^*$.*

(ii) *Any left (weakly) completely continuous element m of $GL_0(\mathcal{G})^*$ has the form $\mathsf{m} = \sigma + \mathsf{r}$ for some $\sigma \in M_a(\mathcal{G})$ and $\mathsf{r} \in \mathrm{ran} \times (\mathcal{P}^*(L_0^\infty(\mathcal{G})^*))$.*

Proof. We only give the proof for the left completely continuous element.

(i) The direct implication being trivial, we give the proof of the backward implication only. Toward this end, suppose that $\sigma \in M_a(\mathcal{G})$ is a left completely continuous element of $M_a(\mathcal{G})$. Then, the closure of the following set is compact in $M_a(\mathcal{G})$

$$\{\sigma * \upsilon : \upsilon \in M_a(\mathcal{G}),\ \|\upsilon\| \le 1\}.$$

On the other hand, if $(e_\alpha)_\alpha$ is an approximate identity for $M_a(\mathcal{G})$ bounded by one, then for each α and $\mathsf{m} \in GL_0(\mathcal{G})^*$ with $\|\mathsf{m}\| \leq 1$, we have
$$\|\sigma \odot \mathsf{m} - \sigma * (e_\alpha \odot \mathsf{m})\| \leq \|\sigma - \sigma * e_\alpha\|.$$
This, together with the fact that $M_a(\mathcal{G})$ is an ideal in $GL_0(\mathcal{G})^*$, implies that
$$\{\sigma \odot \mathsf{m} : \mathsf{m} \in GL_0(\mathcal{G})^*, \|\mathsf{m}\| \leq 1\} \subseteq \overline{\{\sigma * \upsilon : \upsilon \in M_a(\mathcal{G}), \|\upsilon\| \leq 1\}}^{M_a(\mathcal{G})}.$$
Thus, the operator $\ell_\sigma : GL_0(\mathcal{G})^* \to GL_0(\mathcal{G})^*$ is compact.

(ii) Suppose that m is a left completely continuous element of $GL_0(\mathcal{G})^*$. Then, since $M_a(\mathcal{G})$ is an ideal in $GL_0(\mathcal{G})^*$, the operator $\ell_{\mathsf{m}}|_{M_a(\mathcal{G})}$ is a compact operator on $M_a(\mathcal{G})$. From this, we can conclude that there exists a $\sigma \in M_a(\mathcal{G})$ such that $\ell_{\mathsf{m}} = \ell_\sigma$ on $M_a(\mathcal{G})$, see [**1**]. In particular, Proposition 3.5 implies that $\langle \mathsf{m}, f\rangle = \langle \sigma, f\rangle$ for all $f \in C_0(\mathcal{G})$, and thus, we have $\upsilon \odot \mathsf{m} = \upsilon * \sigma$ $(\upsilon \in M_a(\mathcal{G}))$. We now invoke the weak*-density of $M_a(\mathcal{G})$ in $\mathcal{P}^*(L_0^\infty(\mathcal{G})^*)$ to conclude that $\mathcal{P}^*(L_0^\infty(\mathcal{G})^*) \odot \mathsf{r} = 0$, where $\mathsf{r} = \mathsf{m} - \sigma$, that is, $\mathsf{r} \in \mathrm{ran}(\mathcal{P}^*(L_0^\infty(\mathcal{G})^*))$. $\square$

In [**13,** page 467], Losert, using the C*-algebra structure of $M(\mathcal{G})^*$, proved that $M(\mathcal{G})^{**}$ has a non-zero left (weakly) completely continuous element if and only if $\mathcal{G}$ is compact. Related to this result, we have the following result for $GL_0(\mathcal{G})^*$ where our approach in its proof is totally different from Losert's result and relies on the theory of generalized functions.

Theorem 4.5. *The following conditions are equivalent.*

(i) *$\mathcal{G}$ is compact.*
(ii) *$GL_0(\mathcal{G})^*$ has a non-zero left completely continuous element.*
(iii) *$GL_0(\mathcal{G})^*$ has a non-zero left weakly completely continuous element.*

Proof. We need only show that (iii) implies (i). Indeed, if $\mathcal{G}$ is compact, then $M(\mathcal{G})^{**} = GL_0(\mathcal{G})^*$, and the normalized Haar measure m on $\mathcal{G}$ is a left (weakly) completely continuous element of $GL_0(\mathcal{G})^*$ and (ii) $\Rightarrow$ (iii) is trivial. Toward this end, suppose that m is a nonzero left weakly completely continuous element of $GL_0(\mathcal{G})^*$. Then, the set $\{\mathsf{m} \odot \delta_x : x \in \mathcal{G}\}$ is weakly compact, and therefore, $\{|\mathsf{m} \odot \delta_x| : x \in \mathcal{G}\}$ is weakly compact in $GL_0(\mathcal{G})^*$ by Dieudonne's characterization of weakly compact subsets; see [**5,** Theorem 4.22.1]. It follows that $\mathcal{E} := \{|\mathsf{m}| \odot \delta_x : x \in \mathcal{G}\}$ is weakly compact. This is due to the fact that $|\mathsf{m} \odot \delta_x| = |\mathsf{m}| \odot \delta_x$ for all $x \in \mathcal{G}$. Now, we apply the Kerin-Smulyan theorem [**2**] to infer that the closed convex hull $\mathcal{K}$ of $\mathcal{E}$ is weakly compact in $GL_0(\mathcal{G})^*$.

On the other hand, it is easy to see that the map $T_x : \mathcal{K} \to \mathcal{K}$ defined by $T_x(\mathsf{n}) = \mathsf{n} \odot \delta_x$ is affine for all $x \in \mathcal{G}$. Moreover, we have

$$\begin{aligned}\|\mathsf{n} \odot \delta_x\| &= \sup\{|\langle \mathsf{n} \odot \delta_x, f\rangle| : f \in GL_0(\mathcal{G})^*,\ \|f\|_\infty \le 1\}\\ &= \sup\{|\langle \mathsf{n}, f \circ \delta_x\rangle| : f \in GL_0(\mathcal{G})^*,\ \|f\|_\infty \le 1\}\\ &= \sup\{|\langle \mathsf{n}, h\rangle| : h \in GL_0(\mathcal{G})^*,\ \|h\|_\infty \le 1\}\\ &= \|\mathsf{n}\|,\end{aligned}$$

for all $\mathsf{n} \in GL_0(\mathcal{G})^*$. It follows that the map T_x is distal for all $x \in \mathcal{G}$. Thus, there exists a fixed point $\mathsf{q} \in \mathcal{K}$ for the maps T_x $(x \in \mathcal{G})$, that is, $\mathsf{q} \odot \delta_x = \mathsf{q}$ for all $x \in \mathcal{G}$ by the Ryll-Nardzewski fixed point theorem; see [**2,** Theorem 10.8]. In particular,

$$\mathsf{q} = \sum_{i=1}^{t} a_i |\mathsf{m}| \odot \delta_{x_i}$$

for some $x_1, \ldots, x_t \in \mathcal{G}$ and $a_1, \ldots, a_t$ with $\sum_{i=1}^t a_i = 1$. Now, if (K_α) denotes the family of compact subsets of $\mathcal{G}$ ordered by the upward inclusion, then $(\chi_{K_\alpha x^{-1}})$ is a bounded approximate identity for $GL_0(\mathcal{G})$ for all $x \in \mathcal{G}$. Thus,

$$\begin{aligned}\|\mathsf{q} \odot \delta_x\| &= \left\| \sum_{i=1}^{t} a_i |\mathsf{m}| \odot \delta_{x_i x} \right\| = \lim_\alpha \sum_{i=1}^{t} a_i \langle |\mathsf{m}| \odot \delta_{x_i x}, \chi_{K_\alpha}\rangle\\ &= \sum_{i=1}^{t} a_i \lim_\alpha \langle |\mathsf{m}|, \chi_{K_\alpha x_i x^{-1}}\rangle = \|\mathsf{m}\|.\end{aligned}$$

Therefore, $\|\mathsf{q}\| = \|\mathsf{m}\|$; since $\mathsf{m} \neq 0$, it follows that $\mathsf{q} \neq 0$.

In order to prove (i), suppose on the contrary that $\mathcal{G}$ is not compact and that $\overline{\mathsf{q}}$ is an extension of q from $GL_0(\mathcal{G})$ to a positive functional with the same norm on $GL(\mathcal{G})$, see for example, [**15,** Theorem 3.3.8]. Then, in the same manner as in the proof of Proposition 3.6, it may be shown that $\overline{\mathsf{q}}|_{GL_0(\mathcal{G})} = 0$. This implies that $\mathsf{q} = 0$, a contradiction. The proof of Theorem 4.5 is now complete. □

We conclude this work with the following result which is of interest in its own right. In this proposition, the notation $C_0(\mathcal{G})^\perp$ is used to denote the set of all $m \in GL_0(\mathcal{G})^*$ such that $\mathsf{m}|_{C_0(\mathcal{G})} = 0$ and

$$\mathcal{E}_1(\mathcal{G}) = \{E \in L_0^\infty(\mathcal{G})^* : \|E\| = 1 \text{ and } E \text{ is a right identity for } L_0^\infty(\mathcal{G})^*\}.$$

It should be noted that $E \in \mathcal{E}_1(\mathcal{G})$ if and only if it is a weak*-cluster point of an approximate identity in $M_a(\mathcal{G})$ bounded by one, see **[12]**.

Proposition 4.6. *$GL_0(\mathcal{G})^*$ is commutative if and only if $\mathcal{G}$ is discrete and abelian.*

Proof. The necessity of the condition "$GL_0(\mathcal{G})^*$ is commutative" is clear. We prove its sufficiency. Toward this end, suppose that $GL_0(\mathcal{G})^*$ is commutative. That $\mathcal{G}$ is abelian follows trivially. In order to prove that $\mathcal{G}$ is discrete, we note that Proposition 3.5, together with the fact that the right translations on $GL_0(\mathcal{G})^*$ are weak*-continuous, implies that

$$\operatorname{ran}(\mathcal{P}^*(L_0^\infty(\mathcal{G})^*)) = C_0(\mathcal{G})^\perp.$$

Moreover, from another application of Proposition 3.5, we can obtain that $\mathcal{P}^*(E) - \delta_e \in C_0(\mathcal{G})^\perp$ for all $E \in \mathcal{E}_1(\mathcal{G})$.

On the other hand, from the commutativity of $GL_0(\mathcal{G})^*$, we get that $C_0(\mathcal{G})^\perp = \operatorname{lan}(\mathcal{P}^*(L_0^\infty(\mathcal{G})^*))$. We, therefore, have

$$\mathcal{P}^*(E) - \delta_e \in \operatorname{lan}(\mathcal{P}^*(L_0^\infty(\mathcal{G})^*)) \quad \text{for all } E \in \mathcal{E}_1(\mathcal{G}).$$

It follows that each element of $\mathcal{P}^*(\mathcal{E}_1(\mathcal{G}))$ is also a left identity for $\mathcal{P}^*(L_0^\infty(\mathcal{G})^*)$. We now invoke parts (ii) and (iii) of **[12,** Theorem 2.11] to conclude that $M_a(\mathcal{G}) = M(\mathcal{G})$. This implies that $\mathcal{G}$ is discrete. ☐

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