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NONTRIVIAL SOLUTIONS FOR KIRCHHOFF-TYPE PROBLEMS INVOLVING THE $p(x)$-LAPLACE OPERATOR

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ABSTRACT. In this article, we study the existence of nontrivial solutions for the following $p(x)$ Kirchhoff-type problem

$$\begin{cases} -M\big(\int_\Omega A(x,\nabla u)\,dx\big)\operatorname{div}(a(x,\nabla u)) \\ \qquad = \lambda h(x)(\partial F/\partial u)(x,u) & \text{in } \Omega \\ u=0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a smooth bounded domain, $\lambda > 0$, $h \in C(\Omega)$, $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $a, A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous. The proof is based on variational arguments and the theory of variable exponent Sobolev spaces.

1. Introduction. In this paper, we consider the following $p(x)$ Kirchhoff-type problem

$$(\mathbf{P}_\lambda)\ \begin{cases} -M\big(\int_\Omega A(x,\nabla u)\,dx\big)\operatorname{div}(a(x,\nabla u)) = \lambda h(x)(\partial F/\partial u)(x,u) & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n, n \geq 3$, is a smooth bounded domain, $\lambda > 0$, $h \in C(\Omega)$, $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and it is a derivative with respect to the second variable of the mapping $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$, i.e., $a(x,\xi) = \nabla_\xi A(x,\xi)$.

Problem $(\mathbf{P}_\lambda)$ is related to the stationary version of a model, the so-called Kirchhoff equation, introduced by Kirchhoff **[16]**. To be more precise, Kirchhoff established a model given by the equation

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$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0. \tag{1.1}$$

This equation is an extension of the classical D'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.1) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension. Recently, studies of Kirchhoff type problems have been used in variational methods in the case involving the p-Laplacian operator **[2, 6, 7, 11, 18, 21]**. Moreover, due to the increase in attention towards partial differential equations with nonstandard growth conditions, it was further extended to the $p(x)$-Laplacian operator, defined by $\triangle_{p(x)} := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ (see, for example, **[3, 4, 8, 9, 14, 22, 23]**).

The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; hence it is inhomogeneous. This fact implies some difficulties. For example, we cannot use the theory of Sobolev spaces in many problems involving this operator. Some of the nonlinear problems involving $p(x)$-growth conditions are extremely attractive because those problems can be used to model dynamical phenomena that arise from the study of electrorheological fluids or elastic mechanics **[15, 24]**. Moreover, problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermorheological viscous flows of non-Newtonian fluids, in the mathematical description of the processes of filtration of an ideal barotropic gas through a porous medium and image processing **[1, 6]**. The detailed application backgrounds of the $p(x)$-Laplacian operator may be found in **[12, 13, 14, 17, 20]**. In the present paper, motivated by the above works, we give a very simple variational method to prove the existence of a nontrivial solution of problem $(\mathbf{P}_\lambda)$.

For any continuous and bounded function $\omega \in C_+(\overline{\Omega})$ we define ω^- and ω^+ as follows:

$$\omega^- := \min_{x\in\overline{\Omega}} \omega(x) \quad \text{and} \quad \omega^+ = \max_{x\in\overline{\Omega}} \omega(x).$$

Throughout this paper, we fix a nonnegative continuous function p on $\overline{\Omega}$ such that $p^- > 1$, and we will make the following assumptions:

$(\mathbf{A}_1)$ there exists a nonnegative measurable function $b \in L^{p'(x)}(\Omega)$ such that, for all $x \in \Omega$ and $y \in \mathbb{R}^n$,

$$|a(x,y)| \le c_1(b(x) + |y|^{p(x)-1})$$

for some $c_1 > 0$, where p' is the conjugate exponent of p defined by $1/p + 1/p' = 1$.

($\mathbf{A}_2$) There exists a $\sigma > 0$ such that

$$A\left(x, \frac{y+z}{2}\right) \le \frac{1}{2}A(x,y) + \frac{1}{2}A(x,z) - \sigma|y-z|^{p(x)}$$

for all $x \in \Omega$ and $y, z \in \mathbb{R}^n$.

($\mathbf{A}_3$) For all $x \in \Omega$ and $y \in \mathbb{R}^n$, we have $A(x,0) = 0$, $A(x,-y) = A(x,y)$ and

$$|y|^{p(x)} \le a(x,y) \cdot y \le p(x)A(x,y).$$

($\mathbf{H}_1$) $F : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that

$$F(x,tu) = t^{q(x)}F(x,u)(t > 0) \quad \text{for all } x \in \overline{\Omega},\ u \in \mathbb{R},$$

and

$$\int_\Omega h(x)F(x,t)\, dx \ge 0, \quad t \in \mathbb{R},$$

where $q \in C_+(\overline{\Omega})$.

($\mathbf{H}_2$) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that, for all $t > 0$, we have

$$\frac{1}{c_2}t^{r-1} \le M(t) \le c_2 t^{r-1},$$

for some $c_2 > 1$ and $r \ge 1$.

($\mathbf{H}_3$) h is a positive function such that $h \in L^{\alpha(x)}(\Omega)$ for some nonnegative continuous function α on Ω satisfying

$$p(x) < \frac{\alpha(x)-1}{\alpha(x)}\, p^*(x) \quad \text{and} \quad 1 < q(x) < \frac{\alpha(x)}{\alpha(x)-1}\, p^*(x),$$

where $p^*(x) = np(x)/(n-p(x))$ if $n > p(x)$ or $p^*(x) = \infty$ if $n \le p(x)$.

($\mathbf{H}_4$) $1 < q^- \le q^+ < rp^- < rp^+ < (p^*)^-$, where $(p^*)^- = np^-/(n-p^-)$.

Remark 1.1.

(1) Using assumption $(\mathbf{H}_1)$, for all $x \in \overline{\Omega}$, $u \in \mathbb{R}$, we have the so-called Euler identity

$$u\,\frac{\partial F(x,u)}{\partial u} = q(x)F(x,u),$$

and

$$|F(x,u)| \le K|u|^{q(x)} \tag{1.2}$$

for some constant $K > 0$.

(2) If $A(x,\xi) = |\xi|^{p(x)}/p(x)$, then $a(x,\xi) = |\xi|^{p(x)-2}\xi$, and we obtain the $p(x)$-Laplacian operator.

(3) If

$$A(x,\xi) = \frac{(1+|\xi|^2)^{p(x)/2} - 1}{p(x)},$$

then $a(x,\xi) = (1+|\xi|^2)^{(p(x)-2)/2}\xi$, and we obtain the generalized mean curvature operator

$$\operatorname{div}((1+|\nabla u|^2)^{(p(x)-2)/2}\nabla u).$$

Our main result is the following.

Theorem 1.2. *Assume that conditions* $(\mathbf{A}_1)$–$(\mathbf{A}_3)$ *and* $(\mathbf{H}_1)$–$(\mathbf{H}_4)$ *are satisfied. Then, there exists a* $\lambda_0 > 0$ *such that, for any* $\lambda \in (0, \lambda_0)$, *problem* $(\mathbf{P}_\lambda)$ *has a nontrivial weak solution.*

This paper is organized as follows. In Section 2, we will recall some basic facts about the variable exponent Lebesgue and Sobolev spaces which we will use later. The proof of our main result will be presented in Section 3.

2. Preliminaries. In this section, we recall some definitions and basic properties of the generalized Lebesgue Sobolev spaces $L^{\omega(x)}(\Omega)$, $W^{1,\omega(x)}(\Omega)$, $W_0^{1,\omega(x)}(\Omega)$ and $L^{\omega(x)}_{c(x)}(\Omega)$ (for details, see [**4, 5, 8, 13, 14**]). Set

$$C_+(\overline{\Omega}) := \{\omega : \omega \in C(\overline{\Omega}), \omega(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $\omega \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{\omega(x)}=\left\{u : u \text{ a measurable real-valued function s.t.} \int_\Omega |u(x)|^{\omega(x)} dx < \infty\right\}.$$

We recall the following so-called Luxemburg norm on this space, defined by the formula

$$|u|_{\omega(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{\omega(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < \omega^- \leq \omega^+ < \infty$ and continuous functions are dense if $\omega^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ almost everywhere $x \in \Omega$, then there exists a continuous embedding $L^{p_2}(x)(\Omega) \hookrightarrow L^{p_1}(x)(\Omega)$.

We denote by $L^{\omega'(x)}(\Omega)$ the conjugate space of $L^{\omega(x)}(\Omega)$, where $1/\omega(x) + 1/\omega'(x) = 1$. For any $u \in L^{\omega(x)}(\Omega)$ and $v \in L^{\omega'(x)}(\Omega)$, the Hölder inequality

$$\left|\int_\Omega uv\, dx\right| \leq \left(\frac{1}{\omega^-} + \frac{1}{(\omega')^-}\right)|u|_{\omega(x)}|v|_{\omega'(x)} \tag{2.1}$$

holds (see **[8, 13]**).

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the modular of the $L^{\omega(x)}(\Omega)$ space, which is the mapping $\rho_{\omega(x)} : L^{\omega(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{\omega(x)}(u) := \int_\Omega |u|^{\omega(x)} dx.$$

Proposition 2.1 (**[8, 13]**). *If $u, u_k \in L^{\omega(x)}(\Omega)$, $k = 1, 2, \ldots$, and $\omega^+ < \infty$, then we have:*

(i) $|u|_{\omega(x)} < 1$ (*respectively,* $= 1, > 1$) $\Leftrightarrow \rho_{\omega(x)}(u) < 1$ (*respectively,* $= 1, > 1$).
(ii) $|u|_{\omega(x)} > 1 \Rightarrow |u|^{\omega^-}_{\omega(x)} \leq \rho_{\omega(x)}(u) \leq |u|^{\omega^+}_{\omega(x)}$.
(iii) $|u|_{\omega(x)} < 1 \Rightarrow |u|^{\omega^+}_{\omega(x)} \leq \rho_{\omega(x)}(u) \leq |u|^{\omega^-}_{\omega(x)}$.
(iv) $\lim_{k\to\infty} |u_k|_{\omega(x)} = 0 \Leftrightarrow \lim_{k\to\infty} \rho_{\omega(x)}(u_k) = 0$.

Next, we define the Lebesgue-Sobolev space $W^{1,\omega(x)}(\Omega)$ by

$$W^{1,\omega(x)}(\Omega) = \{u \in L^{\omega(x)}(\Omega) : |\nabla u| \in L^{\omega(x)}(\Omega)\},$$

equipped with the norm

$$\|u\|_{1,\omega(x)} = |u|_{\omega(x)} + |\nabla u|_{\omega(x)}.$$

The space $W_0^{1,\omega(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,\omega(x)}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\omega(x)}$. Since the well-known Poincaré inequality holds, see [**8**], we can define an equivalent norm in $W_0^{1,\omega(x)}(\Omega)$ by

$$\|u\| = |\nabla u|_{\omega(x)}.$$

Proposition 2.2 ([**10**]). *Let p and q be measurable functions such that $p \in L^\infty(\Omega)$, $1 \leq p(x)$, $q(x) \leq \infty$ for almost every $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then:*

(i) $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$.

(ii) $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}$.

In particular, if $p(x) = p$ is constant, then $||u|^p|_{q(x)} = |u|^p_{pq(x)}$.

We also consider the weighted variable exponent Lebesgue space $L^{p(x)}_{c(x)}(\Omega)$. Let $c : \Omega \to \mathbb{R}$ be a measurable function such that $c(x) > 0$ almost everywhere $x \in \Omega$. We define

$$L^{p(x)}_{c(x)}(\Omega) = \Big\{u : u \text{ is a measurable real-valued function:} \\ \int_\Omega c(x)|u(x)|^{p(x)}dx < \infty\Big\},$$

with the norm

$$|u|_{(c(x),\omega(x))} := \inf\left\{\mu > 0 : \int_\Omega \left|c(x)\frac{u(x)}{\mu}\right|^{\omega(x)} dx \leq 1\right\}.$$

Then, $L^{p(x)}_{c(x)}$ is a Banach space which has similar properties with the usual variable exponent Lebesgue spaces. The modular of this space is $\rho_{(c(x),p(x))} : L^{p(x)}_{c(x)} \to \mathbb{R}$, defined by

$$\rho_{(c(x),p(x))}(u) = \int_\Omega c(x)|u(x)|^{p(x)}dx.$$

Proposition 2.3 (**[8, 13]**). *If $p^+ < \infty$ and $u, u_k \in L^{p(x)}_{c(x)}(\Omega)$, $k = 1, 2, \ldots$, then, we have*:

(i) $|u|_{(c(x),p(x))} < 1 \Rightarrow |u|^{p^+}_{(c(x),p(x))} \le \rho_{(c(x),p(x))}(u) \le |u|^{p^-}_{(c(x),p(x))}$.

(ii) $|u|_{(c(x),p(x))} > 1 \Rightarrow |u|^{p^-}_{(c(x),p(x))} \le \rho_{(c(x),p(x))}(u) \le |u|^{p^+}_{(c(x),p(x))}$.

(iii) $\lim_{k\to\infty} |u_k|_{(c(x),p(x))} = 0 \Leftrightarrow \lim_{k\to\infty} \rho_{(c(x),p(x))}(u_k) = 0$.

Proposition 2.4 (**[8, 13]**). *The following statements hold*:

(i) *if $1 < p^- \le p^+ < \infty$, then the spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *Let $q \in C_+(\overline{\Omega})$. If $q(x) < p^*(x)$, for all $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous. In addition, there is a constant $c_q > 0$ such that*

$$|u|_{q(x)} \le c_q||u|| \quad \textit{for all } u \in W_0^{1,p(x)}(\Omega).$$

Problem $(\mathbf{P}_\lambda)$ is posed in the framework of the Sobolev space

$$E = W_0^{1,p(x)}(\Omega).$$

Moreover, a function u in E is said to be a weak solution of problem $(\mathbf{P}_\lambda)$ if, for all $v \in E$, we have

$$M\bigg(\int_\Omega A(x, \nabla u)\bigg) \int_\Omega a(x, \nabla u)\nabla v\, dx = \lambda \int_\Omega h(x) \frac{\partial F(x,u)}{\partial u} v\, dx.$$

Thus, the corresponding energy functional of problem $(\mathbf{P}_\lambda)$ is defined as $J_\lambda : E \to \mathbb{R}$,

$$\begin{aligned} J_\lambda(u) &= \widehat{M}\bigg(\int_\Omega A(x, \nabla u)\bigg) - \lambda \int_\Omega \frac{h(x)F(x,u)}{q(x)}\, dx \\ &:= \widehat{M}(\Lambda(u)) - \lambda I(u), \end{aligned}$$

where $\widehat{M}(t) = \int_0^t M(s)\, ds$, $\Lambda(u) = \int_\Omega A(x, \nabla u)$ and $I(u) = \int_\Omega (h(x) F(x,u)/q(x))\, dx$.

Lemma 2.5 (**[19]**). *The function A verifies the following conditions*:

(i) *for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$, we have*

$$|A(x, \xi)| \le c_0(b(x)|\xi| + |\xi|^{p(x)}).$$

(ii) *For all* $x \in \Omega$, $\xi \in \mathbb{R}^n$ *and* $z \geq 1$, *we have*

$$A(x, z\xi) \leq A(x, \xi) z^{p(x)}.$$

Lemma 2.6. *The following statements hold*:

(i) *the functional* Λ *is well defined on* E.
(ii) *The functional* Λ *is of class* $C^1(E, \mathbb{R})$, *and*

$$\prec \Lambda'(u), v \succ = \int_\Omega a(x, \nabla u) \cdot \nabla v \, dx \quad \text{for all } u, v \in E.$$

(iii) *The functional* Λ *is weakly semi-continuous on* E.
(iv) *For all* $u, v \in E$

$$\Lambda\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Lambda(u) + \frac{1}{2}\Lambda(v) - k||u||^p.$$

(v) *For all* $u, v \in E$

$$\Lambda(u) - \Lambda(v) \geq \prec \Lambda'(v), u - v \succ$$

(vi) J_λ *is weakly lower semi-continuous on* E.
(vii) J_λ *is well defined on* E. *Moreover*, $J_\lambda \in C^1(E, \mathbb{R})$ *with the following derivative*:

$$\begin{aligned} \prec J'_\lambda(u), v \succ = M\left(\int_\Omega A(x, \nabla u)\right) \int_\Omega a(x, \nabla u) \nabla v \, dx \\ - \lambda \int_\Omega h(x) \frac{\partial F(x, u)}{\partial u} v \, dx. \end{aligned}$$

Thus, the weak solutions of $(\boldsymbol{P}_\lambda)$ *are precisely the critical points of* J_λ.

Proof. Since the proof of Lemma 2.6 is very similar to that of [**19,** Lemmas 2.2, 2.7], we omit it. $\square$

Lemma 2.7. *Assume that* $\mathbf{(H_1)}$–$\mathbf{(H_4)}$ *hold. Then, there exist real numbers* $\delta > 0$, $\gamma \in (0, 1)$ *and* $\lambda_0 > 0$ *such that, for any* $\lambda \in (0, \lambda_0)$, *we have*

$$J_\lambda(u) \geq \delta > 0 \quad \text{for all } u \in E \text{ with } ||u|| = \gamma.$$

Proof. Let $\gamma \in (0,1)$ and $u \in E$ be such that $||u|| = \gamma$. Then, from (1.2), we obtain

$$\int_\Omega \frac{h(x)F(x,u)}{q(x)}\,dx \le \frac{K}{q^-}\int_\Omega h(x)|u|^{q(x)}dx. \tag{2.2}$$

On the other hand, from **(H**$_3$**)**–**(H**$_4$**)** and the arguments developed in [**19,** Theorem 2.8], we have

$$\int_\Omega \frac{h(x)F(x,u)}{q(x)}\,dx \le \frac{CK}{q^-}(||u||^{q^-} + ||u||^{q^+}) \quad \text{for all } u \in E. \tag{2.3}$$

Since $||u|| = \gamma < 1$, then, using (1.2), (2.3) and Proposition 2.1 yields

$$\begin{aligned} J_\lambda(u) &\ge \frac{1}{c_1 r}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)^r - \lambda\frac{CK}{q^-}(||u||^{q^-} + ||u||^{q^+}) \\ &\ge \frac{1}{c_1 r(p^+)^r}||u||^{p^+ r} - 2\lambda\frac{CK}{q^-}||u||^{q^-} \\ &= \left(\frac{1}{c_1 r(p^+)^r}||u||^{p^+ r - q^-} - 2\lambda\frac{CK}{q^-}\right)||u||^{q^-}. \end{aligned} \tag{2.4}$$

Set

$$\lambda_0 = \frac{q^-\gamma^{p^+ r - q^-}}{2CKc_1 r(p^+)^r} \quad \text{and} \quad \delta = \lambda_0\gamma^{q^-}. \tag{2.5}$$

Then, it follows from (2.4) that, for all $\lambda \in (0, \lambda_0)$, we have

$$J_\lambda(u) \ge \delta > 0 \quad \text{for all } u \in E \text{ with } ||u|| = \gamma.$$

The proof of Lemma 2.7 is now complete. □

Lemma 2.8. *Assume that* **(H**$_1$**)**–**(H**$_4$**)** *hold. Then, there exists a* $\varphi \in E$ *with* $\varphi \ne 0$ *such that* $J_\lambda(t\varphi) < 0$ *for all* $t > 0$ *small enough.*

Proof. From assumption **(H**$_4$**)**, we know that $q^- < rp^-$. Then, we can choose $\varepsilon > 0$ such that

$$q^- + \varepsilon < rp^-. \tag{2.6}$$

On the other hand, since $q \in C(\overline{\Omega})$, then there exists an open set $\Omega_0 \subset \Omega$ such that

$$|q(x) - q^-| < \varepsilon \quad \text{for all } x \in \Omega_0. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$q(x) < q^- + \varepsilon < rp^- \quad \text{for all } x \in \Omega_0. \tag{2.8}$$

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\overline{\Omega}_0 \subset \operatorname{supp}(\varphi)$, $\varphi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \le \varphi(x) \le 1$ in Ω. Then, from the above facts for any $t \in (0,1)$, it follows that:

$$\begin{aligned}
J_\lambda(t\varphi) &= \widehat{M}\bigg(\int_\Omega A(x,\nabla t\varphi)\,dx\bigg) - \lambda\int_\Omega \frac{h(x)}{q(x)}F(x,t\varphi)\,dx \\
&\le \frac{c_1}{r}\bigg(\int_\Omega A(x,\nabla t\varphi)\,dx\bigg)^r - \lambda\int_\Omega t^{q(x)}\frac{h(x)}{q(x)}F(x,\varphi)\,dx \\
&\le \frac{c}{r}\bigg(\int_\Omega t^{p(x)}A(x,\nabla\varphi)\,dx\bigg)^r - \lambda t^{q^-+\varepsilon}\int_\Omega \frac{h(x)}{q(x)}F(x,\varphi)\,dx \\
&\le \frac{ct^{p^-r}}{r}\bigg(\int_\Omega A(x,\nabla\varphi)\,dx\bigg)^r - \frac{\lambda t^{q^-+\varepsilon}}{q^+}\int_\Omega h(x)F(x,\varphi)\,dx \\
&= t^{q^-+\varepsilon}\bigg[\frac{ct^{p^-r-q^--\varepsilon}}{r}\bigg(\int_\Omega A(x,\nabla\varphi)\,dx\bigg)^r - \frac{\lambda}{q^+}\int_\Omega h(x)F(x,\varphi)\,dx\bigg].
\end{aligned}$$

Therefore,

$$J_\lambda(t\varphi) < 0 \quad \text{for all } 0 < t < \theta^{1/rp^- - q^- - \varepsilon},$$

with

$$0 < \theta < \min\bigg(1, \frac{\lambda r\int_\Omega h(x)F(x,\varphi)\,dx}{cq^+(\int_\Omega A(x,\nabla\varphi)\,dx)^r}\bigg),$$

and the proof is now complete. □

3. Proof of Theorem 1.1. In this section, we prove our main result.

Existence of the nontrivial solution of $(\mathbf{P}_\lambda)$ follows from a minimization argument and Ekeland's variational principle. Indeed, from Lemma 2.7, we know that there exists a $\gamma \in (0,1)$ such that the open ball centered at the origin and of radius γ denoted by $B(0,\gamma)$ is such that

$$\overline{B(0,\gamma)} \subset E \quad \text{and} \quad \inf_{v\in\partial B(0,\gamma)} J_\lambda(v) > 0.$$

Moreover, from Lemma 2.8, there exists an $\omega \in E$ such that $J_\lambda(t\omega) < 0$ for small enough $t > 0$. Thus, from Lemma 2.7, we have

$$-\infty < \tau := \inf_{v \in \overline{B(0,\gamma)}} J_\lambda(v) < 0.$$

Let $\varepsilon > 0$ be small enough such that

$$0 < \varepsilon < \inf_{v \in \partial B(0,\gamma)} J_\lambda(v) - \inf_{v \in \overline{B(0,\gamma)}} J_\lambda(v). \tag{3.1}$$

Applying Ekeland's variational principle **[11]** to the functional $J_\lambda : \overline{B(0,\gamma)} \to \mathbb{R}$, we can find $u_\varepsilon \in \overline{B(0,\gamma)}$ such that

$$J_\lambda(u_\varepsilon) < \inf_{v \in \overline{B(0,\gamma)}} J_\lambda(v) + \varepsilon \tag{3.2}$$

and

$$J_\lambda(u_\varepsilon) < J_\lambda(v) + \varepsilon||v - u_\varepsilon||, \quad v \neq u_\varepsilon.$$

Combining (3.1) and (3.2), we obtain

$$J_\lambda(u_\varepsilon) < \inf_{v \in \overline{B(0,\gamma)}} J_\lambda(v) + \varepsilon < \inf_{v \in \partial B(0,\gamma)} J_\lambda(v).$$

Thus, $u_\varepsilon \in B(0,\gamma)$.

Now, we define $\phi_\lambda : \overline{B(0,\gamma)} \to \mathbb{R}$ by

$$\phi_\lambda(u) := J_\lambda(u) + \varepsilon||u - u_\varepsilon||.$$

It is clear that u_ε is a minimum point of ϕ_λ. Moreover, for $t > 0$ small enough and for all $v \in B(0,1)$, we have

$$\frac{\phi_\lambda(u_\varepsilon + tv) - \phi_\lambda(u_\varepsilon)}{t} \geq 0. \tag{3.3}$$

Consequently,

$$\frac{J_\lambda(u_\varepsilon + tv) - J_\lambda(u_\varepsilon)}{t} + \varepsilon||v|| \geq 0.$$

Letting t tend to zero, we obtain

$$\prec J'_\lambda(u_\varepsilon), v \succ + \varepsilon||v|| \geq 0,$$

and we infer that $||J'_\lambda(u_\varepsilon)|| \leq \varepsilon$. Thus, there exists a sequence $\{u_k\} \subset B(0,\gamma)$ such that

$$J_\lambda(u_k) \longrightarrow \alpha := \inf_{v\in \overline{B(0,\gamma)}} J_\lambda(v) < 0 \quad \text{and} \quad J'_\lambda(u_k) \longrightarrow 0. \tag{3.4}$$

Since the sequence $\{u_k\}$ is bound in E, up to subsequence again denoted by $\{u_k\}$, there exists a $u \in E$ such that u_k converges weakly to u in E, that is,

$$\prec J'_\lambda(u_k), u_k - u \succ \longrightarrow 0 \quad \text{as } k \to \infty.$$

On the other hand, using $(\mathbf{H}_3)$, (2.1) and Proposition 2.4, E is a compact embedding in $L^{q(x)}_{h(x)}(\Omega)$, see [**19**]. Therefore,

$$\lim_{k\to\infty} \int_\Omega h(x)F(x,u)(u_k - u)\, dx = 0$$

since

$$\prec J'_\lambda(u_k), u_k - u \succ = M\bigg(\int_\Omega A(x,\nabla u_k)\, dx\bigg)\int_\Omega a(x,\nabla u_k)(\nabla u_k - \nabla u)\, dx \\ - \lambda \int_\Omega h(x)F(x,u_k)(u_k - u)\, dx.$$

Then, it follows that

$$\lim_{k\to\infty} \int_\Omega a(x,\nabla u_k)(\nabla u_k - \nabla u)\, dx = 0,$$

that is,

$$\lim_{k\to\infty} \prec \Lambda'(u_k), u_k - u \succ = 0.$$

Moreover, from Lemma 2.6 (v), we have

$$0 = \lim_{k\to\infty} \prec \Lambda'(u_k), u_k - u \succ \leq \lim_{k\to\infty} (\Lambda(u) - \Lambda(u_k)) = \Lambda(u) - \lim_{k\to\infty} \Lambda(u_k).$$

Therefore, $\lim_{k\to\infty} \Lambda(u_k) \leq \Lambda(u)$. Using this fact and Lemma 2.6 (iii), we obtain

$$\lim_{k\to\infty} \Lambda(u_k) = \Lambda(u).$$

Now, we aim to prove that $\{u_k\}$ strongly converges to u in E. Supposing otherwise, then, there exist $\varepsilon > 0$ and a subsequence of $\{u_k\}$, also denoted $\{u_k\}$, such that $||u_k - u|| \geq \varepsilon$. Moreover, from

Lemma 2.6 (iv), we have

$$\frac{1}{2}\Lambda(u)+\frac{1}{2}\Lambda(u_k)-\Lambda\left(\frac{u_k+u}{2}\right)\geq\sigma||u_k-u||^{p^-}\geq\sigma\varepsilon^{p^-}.$$

Consequently,

$$\limsup_{k\to\infty}\Lambda\left(\frac{u_k+u}{2}\right)\leq\Lambda(u)-\sigma\varepsilon^{p^-}. \tag{3.5}$$

On the other hand, $(u_k+u)/2$ weakly converges to u in E. Then, by Lemma 2.6 (iii), we obtain

$$\Lambda(u)\leq\liminf_{k\to\infty}\Lambda\left(\frac{u_k+u}{2}\right). \tag{3.6}$$

Combining (3.5) and (3.6), we obtain a contradiction. Therefore, $\{u_k\}$ strongly converges to u in E. Finally, we conclude that u is a nontrivial weak solution of problem $(\mathbf{P}_\lambda)$. The proof is complete. $\square$

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