ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 48, Number 4, 2018

EIGENVALUES FOR SYSTEMS OF FRACTIONAL p-LAPLACIANS

LEANDRO M. DEL PEZZO AND JULIO D. ROSSI

ABSTRACT. We study the eigenvalue problem for a system of fractional p-Laplacians, that is,

$$\begin{cases} (-\Delta_p)^r u = \lambda \frac{\alpha}{p} |u|^{\alpha-2} u |v|^{\beta} & \text{in } \Omega, \\ (-\Delta_p)^s v = \lambda \frac{\beta}{p} |u|^{\alpha} |v|^{\beta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{in } \Omega^c = \mathbb{R}^N \setminus \Omega. \end{cases}$$

We show that there is a first (smallest) eigenvalue that is simple and has associated eigenpairs composed of positive and bounded functions. Moreover, there is a sequence of eigenvalues λ_n such that $\lambda_n \to \infty$ as $n \to \infty$.

In addition, we study the limit as $p \to \infty$ of the first eigenvalue, $\lambda_{1,p}$, and we obtain $[\lambda_{1,p}]^{1/p} \to \Lambda_{1,\infty}$ as $p \to \infty$, where

$$\Lambda_{1,\infty} = \inf_{(u,v)} \left\{ \frac{\max\{[u]_{r,\infty}; [v]_{s,\infty}\}}{\| |u|^{\Gamma} |v|^{1-\Gamma} \|_{L^\infty(\Omega)}} \right\} = \left[\frac{1}{R(\Omega)} \right]^{(1-\Gamma)s+\Gamma r}.$$

Here,

$$R(\Omega) := \max_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) \text{ and } [w]_{t,\infty} := \sup_{(x,y) \in \overline{\Omega}} \frac{|w(y) - w(x)|}{|x-y|^t}.$$

Finally, we identify a PDE problem satisfied, in the viscosity sense, by any possible uniform limit along subsequences of the eigenpairs.

1. Introduction.

In this work, we deal with the nonlocal nonlinear eigenvalue problem

$$(1.1) \qquad \begin{cases} (-\Delta_p)^r u = \lambda \frac{\alpha}{p} |u|^{\alpha-2} u |v|^{\beta} & \text{in } \Omega, \\ (-\Delta_p)^s v = \lambda \frac{\beta}{p} |u|^{\alpha} |v|^{\beta-2} v & \text{in } \Omega, \\ u = v = 0 & \text{in } \Omega^c = \mathbb{R}^N \setminus \Omega, \end{cases}$$

2010 AMS *Mathematics subject classification.* Primary 35P30, 35R11, 45G05, 47G20.

Keywords and phrases. Eigenvalue problems, fractional operators, p-Laplacian.

The second author was partially supported by MTM2011-27998 (Spain).

Received by the editors on November 27, 2016, and in revised form on August 7, 2017.

DOI:10.1216/RMJ-2018-48-4-1077

where $p > 2$, $r, s \in (0,1)$ and $\alpha, \beta \in (0,p)$ are such that

$$\alpha + \beta = p, \quad \min\{\alpha; \beta\} \geq 1,$$

and λ is the eigenvalue. Here, and subsequently, Ω is a bounded smooth domain in $\mathbb{R}^N$ and $(-\Delta_p)^t$ denotes the fractional (p,t)-Laplacian, that is,

$$(-\Delta_p)^t u(x) := 2\mathrm{PV} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+tp}} \, dy, \quad x \in \Omega.$$

The natural functional space for our problem is

$$\mathcal{W}_p^{(r,s)}(\Omega) := \widetilde{W}^{r,p}(\Omega) \times \widetilde{W}^{s,p}(\Omega).$$

Here, $\widetilde{W}^{t,p}(\Omega)$ denotes the space of all u that belong to the fractional Sobolev space

$$W^{t,p}(\Omega) := \left\{ v \in L^p(\Omega) \colon \int_{\Omega^2} \frac{|v(x) - v(y)|^p}{|x - y|^{N+tp}} \, dx \, dy < \infty \right\}$$

such that $\widetilde{u} \in W^{t,p}(\mathbb{R}^N)$ where $\widetilde{u}$ is the extension by zero of u and $\Omega^2 = \Omega \times \Omega$. For a more detailed description of these spaces and some of their properties, see for instance, **[1, 15]**.

Note that, in our eigenvalue problem, we consider two different fractional operators (since we allow for $t \neq s$), and therefore, the natural space to consider here, that is, $\mathcal{W}_p^{(r,s)}(\Omega) = \widetilde{W}^{r,p}(\Omega) \times \widetilde{W}^{s,p}(\Omega)$ is asymmetric.

In this context, an eigenvalue is a real value λ for which there is a $(u,v) \in \mathcal{W}_p^{(r,s)}(\Omega)$ such that $uv \not\equiv 0$, and (u,v) is a weak solution of (1.1), i.e.,

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+rp}} \, dx \, dy$$
$$= \lambda \frac{\alpha}{p} \int_{\Omega} |u|^{\alpha-2} u |v|^{\beta} w \, dx$$

$$\int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(z(x)-z(y))}{|x-y|^{N+sp}}\,dx\,dy = \lambda\frac{\beta}{p}\int_\Omega |u|^\alpha|v|^{\beta-2}vz\,dx$$

for any $(w,z) \in \mathcal{W}_p^{(r,s)}(\Omega)$. The pair (u,v) is called a *corresponding eigenpair.*

Observe that, if λ is an eigenvalue with eigenpair (u,v), then $uv \not\equiv 0$ and

$$\lambda = \frac{[u]_{r,p}^p + [v]_{s,p}^p}{|(u,v)|_{\alpha,\beta}^p},$$

where

$$[w]_{t,p}^p := \int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^p}{|x-y|^{N+tp}}\,dx\,dy$$

and

$$|(u,v)|_{\alpha,\beta}^p := \int_\Omega |u|^\alpha|v|^\beta dx.$$

Thus,

$$\lambda \geq \lambda_{1,p},$$

where

$$\lambda_{1,p} := \inf\left\{\frac{[u]_{r,p}^p + [v]_{s,p}^p}{|(u,v)|_{\alpha,\beta}^p} : (u,v) \in \mathcal{W}_p^{(r,s)}(\Omega),\ uv \not\equiv 0\right\}. \tag{1.2}$$

Our first aim is to show that $\lambda_{1,p}$ is the first eigenvalue of our problem. In fact, in Section 3, we prove the following result.

Theorem 1.1. *There is a nontrivial minimizer* (u_p, v_p) *of* (1.2) *such that both components are positives,* $u_p, v_p > 0$ *in* Ω, *and* (u_p, v_p) *is a weak solution of* (1.1) *with* $\lambda = \lambda_{1,p}$. *Moreover,* $\lambda_{1,p}$ *is simple.*

Finally, there is a sequence of eigenvalues λ_n *such that* $\lambda_n \to \infty$ *as* $n \to \infty$.

It is yet to be ascertained whether or not the first eigenvalue is isolated. Here, our aim is to study $\lambda_{1,p}$ for large p. Towards this end, we look for the asymptotic behavior of $\lambda_{1,p}$ as $p \to \infty$. From now on, for any $p > 1$, (u_p, v_p) denotes the eigenpair associated to $\lambda_{1,p}$ such that $|(u,v)|_{\alpha,\beta} = 1$. In order to study the limit as $p \to \infty$, we need to assume that

$$p \min\{r, s\} \geq N, \tag{1.3}$$

and

$$\lim_{p\to\infty} \frac{\alpha_p}{p} = \Gamma, \quad 0 < \Gamma < 1. \tag{1.4}$$

Note that the last assumption and the fact that $\alpha_p + \beta_p = p$ implies

$$\lim_{p\to\infty} \frac{\beta_p}{p} = 1 - \Gamma, \quad 0 < 1 - \Gamma < 1.$$

In order to state our main theorem concerning the limit as $p \to \infty$, we need to introduce the following notation:

$$[w]_{t,\infty} := \sup_{(x,y)\in\overline{\Omega}} \frac{|w(y) - w(x)|}{|x-y|^t},$$

$$\widetilde{W}^{t,\infty}(\Omega) := \{w \in C_0(\overline{\Omega}) \colon [w]_{t,\infty} < \infty\},$$

$$\mathcal{W}^{(r,s)}_{\infty}(\Omega) := \widetilde{W}^{r,\infty}(\Omega) \times \widetilde{W}^{s,\infty}(\Omega),$$

and

$$R(\Omega) := \max_{x\in\Omega} \operatorname{dist}(x, \partial\Omega).$$

Now, we are ready to state our second result. It states that there is a limit for $[\lambda_{1,p}]^{1/p}$ and that this limit verifies both a variational characterization and a simple geometrical characterization. In addition, concerning eigenfunctions, there is a uniform limit (along with subsequences) that is a viscous solution to a limit PDE eigenvalue problem. The proofs of our results concerning limits as $p \to \infty$ are shown in Section 4.

Theorem 1.2. *Under assumptions* (1.3) *and* (1.4)*, we have that*

$$\lim_{p\to\infty} [\lambda_{1,p}]^{1/p} = \Lambda_{1,\infty},$$

where

$$\Lambda_{1,\infty} = \inf\left\{ \frac{\max\{[u]_{r,\infty}; [v]_{s,\infty}\}}{\| |u|^{\Gamma} |v|^{1-\Gamma} \|_{L^\infty(\Omega)}} \colon (u,v) \in \mathcal{W}^{(r,s)}_{\infty}(\Omega) \right\}.$$

Moreover, we have the following geometric characterization of the limit eigenvalue:

$$\Lambda_{1,\infty} = \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r}.$$

Lastly, there is a sequence $p_j \to \infty$ *such that* $(u_{p_j}, v_{p_j}) \to (u, v)$ *converges uniformly in* $\overline{\Omega}$, *where* (u_∞, v_∞) *is a minimizer of* $\Lambda_{1,\infty}$, *and a viscous solution to*

$$\begin{cases} \min\{\mathcal{L}_{r,\infty}u(x); \mathcal{L}^+_{r,\infty}u(x) - \Lambda_{1,\infty}u^\Gamma(x)v^{1-\Gamma}(x)\} = 0 & in\ \Omega, \\ \min\{\mathcal{L}_{s,\infty}u(x); \mathcal{L}^+_{s,\infty}u(x) - \Lambda_{1,\infty}u^\Gamma(x)v^{1-\Gamma}(x)\} = 0 & in\ \Omega, \\ u = v = 0 & in\ \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}_{t,\infty}w(x) &:= \mathcal{L}^+_{t,\infty}w(x) + \mathcal{L}^-_{r,\infty}w(x) \\ &= \sup_{y\in\mathbb{R}^N} \frac{w(x)-w(y)}{|x-y|^t} + \inf_{y\in\mathbb{R}^N} \frac{w(x)-w(y)}{|x-y|^t}. \end{aligned}$$

To end the introduction, we briefly refer to previous references on this subject. The limit of p-harmonic functions (solutions to the local p-Laplacian, that is, $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$) as $p \to \infty$, has been extensively studied in the literature (see **[3, 4]**) and naturally leads to solutions of the infinity Laplacian, given by $-\Delta_\infty u = -\nabla u D^2 u(\nabla u)^t = 0$. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see the survey **[3]**) and find applications in optimal transportation, image processing and tug-of-war games (see, e.g., **[10, 19, 25, 26]**, and the references therein). In addition, limits of the eigenvalue problem related to the p-Laplacian with various boundary conditions have been exhaustively examined **[18, 22, 23, 27, 28]** and naturally yield the infinity Laplacian eigenvalue problem (in the scalar case)

$$\min\{|\nabla u| - \lambda u,\ -\Delta_\infty u\} = 0. \tag{1.5}$$

In particular, the limit, as $p \to \infty$ of the first eigenvalue $\lambda_{p,D}$ of the p-Laplacian with Dirichlet boundary conditions and of its corresponding positive normalized eigenfunction u_p, has been studied in **[22, 23]**. It was proven there that, up to a subsequence, the eigenfunctions u_p uniformly converge to some Lipschitz function u_∞ satisfying $\|u_\infty\|_\infty = 1$

and

$$(1.6)\qquad (\lambda_{p,D})^{1/p} \longrightarrow \lambda_{\infty,D} = \inf_{u\in W^{1,\infty}(\Omega)} \frac{\|\nabla u\|_\infty}{\|u\|_\infty} = \frac{1}{R(\Omega)}.$$

Moreover, u_∞ is an extremal for this limit variational problem, and the pair u_∞, $\lambda_{\infty,D}$ is a nontrivial solution to (1.5). This problem has also been studied from an optimal mass-transport point of view **[11]**. Note that, here, the fact that we are dealing with two different operators in the system is reflected in that the limit is given by

$$\Lambda_{1,\infty} = \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r},$$

a quantity that depends upon s and t.

On the other hand, there is rich recent literature concerning eigenvalues for systems of p-Laplacian type (we refer, e.g., to **[5, 12, 14, 16, 29]**, and the references therein). The only known references concerning the asymptotic behavior as p goes to infinity of the eigenvalues for a system are **[6, 12]**, where the authors studied the behavior of the first eigenvalue for a system with the usual local p-Laplacian operator.

Finally, concerning limits as $p \to \infty$ in fractional eigenvalue problems (a single equation), we refer the interested reader to **[9, 17, 22]**. In **[22]**, the limit of the first eigenvalue for the fractional p-Laplacian was studied, while in **[17]**, higher eigenvalues were considered.

2. Preliminaries.

We begin with a review of the basic results that will be needed in subsequent sections. The known results are generally stated without proofs, but we provide references where the proofs may be found. In addition, we introduce some of our notational conventions.

2.1. Fractional Sobolev spaces.

Let $s \in (0,1)$ and $p \in (1,\infty)$. There are several choices for a norm for $W^{s,p}(\Omega)$. We choose the following:

$$\|u\|_{s,p}^p := \|u\|_{L^p(\Omega)}^p + |u|_{s,p}^p,$$

where

$$|u|_{s,p}^p = \int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dx\,dy.$$

Observe that, for any $u \in \widetilde{W}^{s,p}(\Omega)$, we obtain

$$|u|_{s,p} \le [u]_{s,p}.$$

Our first aim is to show a Poincaré-type inequality.

Lemma 2.1. *Let $s \in (0,1)$. For any $p > 1$, there is a positive constant C, independent of p, such that*

$$[u]_{s,p}^p \ge \frac{\omega_N}{sp}(\mathrm{diam}(\Omega)+1)^{-sp}\|u\|_{L^p(\Omega)}^p \quad \text{for all } u \in \widetilde{W}^{s,p}(\Omega),$$

where ω_N is the N-dimensional volume of an Euclidean ball of radius 1.

Proof. Let $u \in \widetilde{W}^{s,p}(\Omega)$. Then,

$$[u]_{s,p}^p \ge \int_\Omega |u(x)|^p \int_{\Omega_1} \frac{1}{|x-y|^{N+sp}}\, dy\, dx,$$

where $\Omega_1 = \{y \in \Omega^c\colon \mathrm{dist}(y,\Omega) \ge 1\}$. Now, we observe that, for any $x \in \Omega$, we have $B_{d+1}(x)^c \subset \Omega_1$ where $d = \mathrm{diam}(\Omega)$. Thus,

$$\begin{aligned}\int_{\Omega_1} \frac{dy}{|x-y|^{N+sp}} &\ge \int_{B_{d+1}(x)^c} \frac{dy}{|x-y|^{N+sp}} \\ &= \omega_N \int_{d+1}^\infty \frac{d\rho}{\rho^{sp+1}} = \frac{\omega_N}{sp}(d+1)^{-sp}\end{aligned}$$

for all $x \in \Omega$. Therefore, we conclude that

$$[u]_{s,p}^p \ge \frac{\omega_N}{sp}(d+1)^{-sp}\|u\|_{L^p(\Omega)}^p. \qquad \square$$

The next result will be one of the keys in the proof of Theorem 1.2.

Lemma 2.2. *Let $s \in (0,1)$ and $p > N/s$. If $q \in (N/s, p)$ and $t = s - N/q$, then*

$$\|u\|_{L^q(\Omega)} \le |\Omega|^{1/q-1/p}\|u\|_{L^p(\Omega)}$$

and

$$|u|_{t,q} \le \mathrm{diam}(\Omega)^{N/p}|\Omega|^{2/q-2/p}|u|_{s,p}$$

for all $u \in W^{s,p}(\Omega)$.

Proof. Since $q < p$, the first inequality is trivial. Therefore, we only need to prove the second one. Let $u \in W^{s,p}(\Omega)$. It follows from Hölder's inequality that

$$\begin{aligned} |u|_{t,q}^q &= \int_{\Omega^2} \frac{|u(x)-u(y)|^q}{|x-y|^{sq}}\,dx\,dy \\ &\le \left(\int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{sp}}\,dx\,dy\right)^{q/p} |\Omega|^{2-2q/p} \\ &\le \operatorname{diam}(\Omega)^{Nq/p}\left(\int_{\Omega^2} \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}}\,dx\,dy\right)^{q/p} |\Omega|^{2-2q/p}, \end{aligned}$$

as we wanted to show. □

2.2. Weak and viscous solutions. Here, we discuss the relation between the weak solutions of

$$\begin{cases} (-\Delta_p)^s u = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{2.1}$$

and the viscous solutions of the same problem.

We begin by introducing the precise definitions of weak and viscous solutions.

Definition 2.3. Let $f \in W^{-s,p}(\Omega)$ (the dual space of $\widetilde{W}^{s,p}(\Omega)$) and $u \in \widetilde{W}^{s,p}(\Omega)$. We say that u is a *weak solution* of (2.1) if and only if

$$\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+rp}}\,dx\,dy = \langle f, v\rangle$$

for every $v \in \widetilde{W}^{s,p}(\Omega)$. Here, $\langle \cdot,\cdot\rangle$ denotes the duality pairing of $\widetilde{W}^{s,p}(\Omega)$ with $W^{-s,p}(\Omega)$.

Definition 2.4. Let $p \ge 2$, $f \in C(\overline{\Omega})$ and $u \in C(\mathbb{R}^N)$ be such that $u = 0$ in Ω^c. We say that u is a *viscous subsolution* of (2.1) at a point $x_0 \in \Omega$ if and only if, for any test function $\varphi \in C_0^2(\mathbb{R}^N)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \le \varphi(x)$ for all $x \in \mathbb{R}^N$, we have that

$$2\int_{\mathbb{R}^N} \frac{|\varphi(x_0)-\varphi(y)|^{p-2}(\varphi(x_0)-\varphi(y))}{|x_0-y|^{N+sp}}\,dy \le f(x_0).$$

We say that u is a *viscous supersolution* of (2.1) at a point $x_0 \in \Omega$ if and only if, for any test function $\varphi \in C_0^2(\mathbb{R}^N)$ such that $u(x_0) = \varphi(x_0)$ and $u(x) \geq \varphi(x)$ for all $x \in \mathbb{R}^N$, we have that

$$2 \int_{\mathbb{R}^N} \frac{|\varphi(x_0) - \varphi(y)|^{p-2}(\varphi(x_0) - \varphi(y))}{|x_0 - y|^{N+sp}} \, dy \geq f(x_0).$$

Finally, u is called a *viscous solution* of (2.1) if it is both a viscous super and subsolution at x_0 for any $x_0 \in \Omega$.

By carefully following the proof of [**24,** Proposition 11], the next result is obtained.

Theorem 2.5. *Let $p \geq 2$ and $f \in C(\overline{\Omega})$. If u is a weak solution of* (2.1)*, then it is also a viscous solution.*

The next result is key in showing that every eigenpair associated to the first eigenvalue has a constant sign. For the proof, we refer to [**24,** Lemma 12].

Lemma 2.6. *Let $p \geq 2$. Assume that $u \geq 0$ and $u \equiv 0$ in Ω^c. If u is a viscous supersolution of $(-\Delta_p)^s u = 0$ in Ω, then either $u > 0$ in Ω or $u \equiv 0$ in $\mathbb{R}^N$.*

3. The eigenvalue problem.

We begin by showing that $\lambda_{1,p}$ is the first eigenvalue of our problem.

Lemma 3.1. *There is a nontrivial minimizer (u, v) of* (1.2) *such that $u, v > 0$ almost everywhere in Ω, and (u, v) is a weak solution of* (1.1) *with $\lambda = \lambda_{1,p}$.*

Proof. Since $C_0^\infty(\Omega) \times C_0^\infty(\Omega) \subset \mathcal{W}_p^{(r,s)}(\Omega)$, we have

$$0 \leq \inf \left\{ \frac{[u]_{r,p}^p + [v]_{s,p}^p}{|(u, v)|_{\alpha,\beta}^p} : (u, v) \in \mathcal{W}_p^{(r,s)}(\Omega), \; uv \not\equiv 0 \right\} < \infty. \tag{3.1}$$

Now, consider a minimizing sequence $\{(u_n, v_n)\}_{n \in \mathbb{N}}$ normalized according to $|(u_n, v_n)|_{(\alpha,\beta)} = 1$. It follows from (3.1) that $\{(u_n, v_n)\}$

is bounded in $\mathcal{W}_p^{(r,s)}(\Omega)$. Then, by the compactness of the Sobolev embedding theorem, there is a subsequence $\{(u_{n_j}, v_{n_j})\}_{j\in\mathbb{N}}$ such that

$$u_{n_j} \rightharpoonup u \text{ weakly in } \widetilde{\mathcal{W}}^{r,p}(\Omega), \qquad v_{n_j} \rightharpoonup v \text{ weakly in } \widetilde{\mathcal{W}}^{s,p}(\Omega),$$
$$u_{n_j} \to u \text{ strongly in } L^p(\Omega), \qquad v_{n_j} \to v \text{ strongly in } L^p(\Omega).$$

Thus, $|(u,v)|_{(\alpha,\beta)} = 1$, and

$$[u]_{r,p}^p + [v]_{s,p}^p \le \liminf_{j\to\infty}\{[u_{n_j}]_{r,p}^p + [v_{n_j}]_{s,p}^p\} = \lambda_{1,p}.$$

Therefore, (u,v) is a minimizer of (1.2). Moreover, since

$$[|u|]_{r,p}^p + [|v|]_{s,p}^p \le [u]_{r,p}^p + [v]_{r,p}^p,$$

we can assume that u and v are nonnegative functions.

The fact that this minimizer is a weak solution (1.1) with $\lambda = \lambda_{1,p}$ is straightforward and can be obtained from the arguments in [**24**].

Finally, since u and v are nonnegative functions and (u,v) is a weak solution of (1.1) with $\lambda = \lambda_{1,p}$, by [**7,** Theorem A.1], we obtain that u and v are positive functions almost everywhere in Ω. $\square$

The next result follows from the classical inequality

$$||a| - |b|| < |a - b| \quad \text{for all } ab < 0.$$

Corollary 3.2. *If (u,v) is an eigenpair corresponding to $\lambda_{1,p}$, then u and v have constant sign.*

Our next aim is to prove that all of the eigenpairs associated to $\lambda_{1,p}$ are bounded. For this, we follow ideas from [**8,** Theorem 3.2].

Lemma 3.3. *If (u,v) is an eigenpair associated to $\lambda_{1,p}$, then $u, v \in L^\infty(\mathbb{R}^N)$.*

Proof. Without loss of generality, we can assume that $r \le s$ and $u, v > 0$ almost everywhere in Ω.

It follows from the fractional Sobolev embedding theorem (see, e.g., [**13,** Corollary 4.53, Theorem 4.54]) that, if $r > N/p$, then the assertion holds.

Thus, we need to prove that the assertion also holds in the following cases:

Case 1. $r < N/p$;

Case 2. $r = N/p$.

Before we start analyzing the different cases, we will show two inequalities. For every $M > 0$, we define

$$u_M(x) := \min\{u(x), M\} \quad \text{and} \quad v_M(x) := \min\{v(x), M\}.$$

Since $(u, v) \in \mathcal{W}_p^{(r,s)}(\Omega)$, it is not difficult to verify that $(u_M, v_M) \in \mathcal{W}_p^{(r,s)}(\Omega)$. Moreover, if $q \geq 1$, then $(u_M^q, v_M^q) \in \mathcal{W}_p^{(r,s)}(\Omega)$. Hence, since (u, v) is an eigenpair associated to $\lambda_{1,p}$, $u_M \leq u$, $v_M \leq v$, and $\alpha, \beta \leq p$, we have

$$\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u_M^q(x) - u_M^q(y))}{|x-y|^{N+rp}}\, dx\, dy \leq \lambda_{1,p} \int_\Omega u^{\alpha+q-1} v^\beta dx,$$

$$\int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(v_M^q(x) - v_M^q(y))}{|x-y|^{N+sp}}\, dx\, dy \leq \lambda_{1,p} \int_\Omega u^{\alpha} v^{\beta+q-1} dx.$$

Hence, by using [**8,** Lemma C2], we obtain

$$\begin{aligned} &q\left(\frac{p}{q+p-1}\right)^p \int_{\mathbb{R}^{2N}} \frac{\left|u_M^{(q+p-1)/p}(x) - u_M^{(q+p-1)/p}(y)\right|^p}{|x-y|^{N+rp}}\, dx\, dy \leq \lambda_{1,p} \int_\Omega u^{\alpha+q-1} v^\beta dx, \\ &q\left(\frac{p}{q+p-1}\right)^p \int_{\mathbb{R}^{2N}} \frac{\left|v_M^{(q+p-1)/p}(x) - v_M^{(q+p-1)/p}(y)\right|^p}{|x-y|^{N+rp}}\, dx\, dy \leq \lambda_{1,p} \int_\Omega u^{\alpha} v^{\beta+q-1} dx. \end{aligned} \tag{3.2}$$

We now begin to analyze the different cases.

Case 1. $r < N/p$. Since $r \leq s$, then $p_r^\star \leq p_s^\star$. Therefore, by Sobolev's embedding theorem,

$$\begin{aligned}
&\left(\int_\Omega u_M^{[(q+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq C(N,r,p,\Omega)\int_{\mathbb{R}^{2N}} \frac{\left|u_M^{(q+p-1)/p}(x)-u_M^{(q+p-1)/p}(y)\right|^p}{|x-y|^{N+rp}}\,dx\,dy, \\
&\left(\int_\Omega v_M^{[(q+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq C(N,r,s,p,\Omega)\int_{\mathbb{R}^{2N}} \frac{\left|v_M^{(q+p-1)/p}(x)-v_M^{(q+p-1)/p}(y)\right|^p}{|x-y|^{N+rp}}\,dx\,dy.
\end{aligned}$$

Then, by (3.2), we obtain

$$\begin{aligned}
&\left(\int_\Omega u_M^{[(q+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq \frac{\lambda_{1,p}}{C(N,r,p,\Omega)}\left(\frac{q+p-1}{p}\right)^{p-1}\int_\Omega u^{\alpha+q-1}v^\beta dx, \\
&\left(\int_\Omega v_M^{[(q+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq \frac{\lambda_{1,p}}{C(N,r,s,p,\Omega)}\left(\frac{q+p-1}{p}\right)^{p-1}\int_\Omega u^{\alpha}v^{\beta+q-1} dx.
\end{aligned}$$

By using Fatou's lemma and Young's inequality, we obtain

$$\begin{aligned}
&\left(\int_\Omega u^{[(p+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq \frac{\lambda_{1,p}}{C(N,r,p,\Omega)}\left(\frac{p+q-1}{p}\right)^{p-1}\left(\int_\Omega u^{p+q-1}dx+\int_\Omega v^{p+q-1}dx\right), \\
&\left(\int_\Omega v^{[(q+p-1)/p]p_r^\star} dx\right)^{p/p_r^\star} \\
&\qquad \leq \frac{\lambda_{1,p}}{C(N,r,s,p,\Omega)}\left(\frac{q+p-1}{p}\right)^{p-1}\left(\int_\Omega u^{p+q-1}dx+\int_\Omega v^{p+q-1}dx\right).
\end{aligned}$$

Taking $\mathcal{Q} = (q+p-1)/p$, we obtain

$$\left(\int_\Omega u^{\mathcal{Q}[Np/(N-rp)]}dx\right)^{\mathcal{Q}(N-rp)/\mathcal{Q}N} \leq \frac{\lambda_{1,p}}{C(N,r,p,\Omega)}\mathcal{Q}^{p-1}\left(\int_\Omega u^{\mathcal{Q}p}dx + \int_\Omega v^{\mathcal{Q}p}dx\right),$$

$$\left(\int_\Omega v^{\mathcal{Q}[Np/(N-rp)]}dx\right)^{\mathcal{Q}(N-rp)/\mathcal{Q}N} \leq \frac{\lambda_{1,p}}{C(N,r,s,p,\Omega)}\mathcal{Q}^{p-1}\left(\int_\Omega u^{\mathcal{Q}p}dx + \int_\Omega v^{\mathcal{Q}p}dx\right).$$

Then,

$$\|u\|^{\mathcal{Q}p}_{L^{(\mathcal{Q}N/N-rp)p}(\Omega)} \leq \frac{\lambda_{1,p}}{C(N,r,p,\Omega)}\mathcal{Q}^{p-1}\left(\|u\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)} + \|v\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)}\right),$$

$$\|v\|^{\mathcal{Q}p}_{L^{(\mathcal{Q}N/N-rp)p}(\Omega)} \leq \frac{\lambda_{1,p}}{C(N,r,s,p,\Omega)}\mathcal{Q}^{p-1}\left(\|u\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)} + \|v\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)}\right).$$

Hence,

$$\left(\|u\|^{\mathcal{Q}p}_{L^{(\mathcal{Q}N/N-rp)p}(\Omega)} + \|v\|^{\mathcal{Q}p}_{L^{(\mathcal{Q}N/N-rp)p}(\Omega)}\right)^{1/\mathcal{Q}p} \leq \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{1/\mathcal{Q}}(\mathcal{Q}^{1/\mathcal{Q}})^{(p-1)/p}\left(\|u\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)} + \|v\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)}\right)^{1/\mathcal{Q}p}.$$

Now, taking the following sequence

$$\mathcal{Q}_0 = 1 \quad \text{and} \quad \mathcal{Q}_{n+1} = \mathcal{Q}_n\frac{N}{N-rp},$$

we have

$$\left(\|u\|^{\mathcal{Q}_np}_{L^{\mathcal{Q}_{n+1}p}(\Omega)} + \|v\|^{\mathcal{Q}_np}_{L^{\mathcal{Q}_{n+1}p}(\Omega)}\right)^{1/\mathcal{Q}_np} \leq \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{1/\mathcal{Q}_np}(\mathcal{Q}_n^{1/\mathcal{Q}_n})^{(p-1)/p}\left(\|u\|^{\mathcal{Q}_np}_{L^{\mathcal{Q}_np}(\Omega)} + \|v\|^{\mathcal{Q}_np}_{L^{\mathcal{Q}_np}(\Omega)}\right)^{1/\mathcal{Q}_np}$$

for all $n \in \mathbb{N}$. Moreover, since

$$\mathcal{Q}_{n+1} = \mathcal{Q}_nN/(N-rp),$$

we have that

$$\Big(\|u\|^{\mathcal{Q}_n p}_{L^{\mathcal{Q}_{n+1}p}(\Omega)}+\|v\|^{\mathcal{Q}_n p}_{L^{\mathcal{Q}_{n+1}p}(\Omega)}\Big)^{1/\mathcal{Q}_n p} \leq \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{1/\mathcal{Q}_n p}(\mathcal{Q}_n^{1/\mathcal{Q}_n})^{(p-1)/p}\Big(\|u\|^{\mathcal{Q}_{n-1} p}_{L^{\mathcal{Q}_n p}(\Omega)}+\|v\|^{\mathcal{Q}_{n-1} p}_{L^{\mathcal{Q}_n p}(\Omega)}\Big)^{1/\mathcal{Q}_{n-1} p}$$

for all $n \geq 2$. Then, iterating the last inequality, we obtain

(3.3)
$$\Big(\|u\|^{\mathcal{Q}_n p}_{L^{\mathcal{Q}_{n+1}p}(\Omega)}+\|v\|^{\mathcal{Q}_n p}_{L^{\mathcal{Q}_{n+1}p}(\Omega)}\Big)^{1/\mathcal{Q}_n p} \leq \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{1/p\sum_{i=0}^{n}1/\mathcal{Q}_i} \times \left(\prod_{i=0}^{n}\mathcal{Q}_i^{1/\mathcal{Q}_i}\right)^{(p-1)/p}\Big(\|u\|^p_{L^p(\Omega)}+\|v\|^p_{L^p(\Omega)}\Big)^{1/p}$$

for all $n \geq 2$.

Observe that $\mathcal{Q}_n \to \infty$ as $n \to \infty$ due to the fact that $N/(N-rp) > 1$. Moreover,

$$\sum_{i=0}^{\infty}\frac{1}{\mathcal{Q}_i}=\frac{N}{rp} \quad \text{and} \quad \prod_{i=0}^{\infty}\mathcal{Q}_i^{1/\mathcal{Q}_i}=\left(\frac{N}{N-rp}\right)^{N/rpp_r^\star}.$$

Hence, passing to the limit in (3.3), we deduce

$$\max\{\|u\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)}\} \leq \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{N/rp^2}\left(\frac{N}{N-rp}\right)^{(N/rpp_r^\star)(p-1/p)}\Big(\|u\|^p_{L^p(\Omega)}+\|v\|^p_{L^p(\Omega)}\Big)^{1/p},$$

that is, $u, v \in L^\infty(\Omega)$.

Case 2. $r = N/p$. In this case, $\mathcal{W}_p^{(r,s)}(\Omega) \hookrightarrow L^m(\Omega)\times L^m(\Omega)$ for all $m > 1$. Then,

$$\left(\int_\Omega u_M^{[(q+p-1)/p]2p}dx\right)^{1/2} \leq C(N,r,p,\Omega) \times \int_{\mathbb{R}^{2N}}\frac{|u_M^{(q+p-1)/p}(x)-u_M^{(q+p-1)/p}(y)|^p}{|x-y|^{N+rp}}\,dx\,dy,$$

$$\left(\int_\Omega v_M^{[(q+p-1)/p]2p}dx\right)^{1/2} \le C(N,r,s,p,\Omega) \times \int_{\mathbb{R}^{2N}} \frac{|v_M^{(q+p-1)/p}(x)-v_M^{(q+p-1)/p}(y)|^p}{|x-y|^{N+rp}}\,dx\,dy.$$

Applying the previous reasoning, we obtain

$$(\|u\|^{\mathcal{Q}p}_{L^{2\mathcal{Q}p}(\Omega)} + \|v\|^{\mathcal{Q}p}_{L^{2\mathcal{Q}p}(\Omega)})^{1/\mathcal{Q}p} \le \left(\frac{2\lambda_{1,p}}{C(N,r,s,p,\Omega)}\right)^{1/\mathcal{Q}} (\mathcal{Q}^{1/\mathcal{Q}})^{(p-1)/p}(\|u\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)}+\|v\|^{\mathcal{Q}p}_{L^{\mathcal{Q}p}(\Omega)})^{1/\mathcal{Q}p}.$$

Now, taking the sequence

$$\mathcal{Q}_0 = 1 \quad \text{and} \quad \mathcal{Q}_{n+1} = 2\mathcal{Q}_n,$$

the proof follows as in the previous case. □

In order to show that $\lambda_{1,p}$ is simple, we will first prove that $\lambda_{1,p}$ is the unique eigenvalue with the following property: any eigenpair associated to λ has a constant sign.

Theorem 3.4. *Let (u,v) be an eigenfunction associated to $\lambda_{1,p}$ such that $u,v \ge 0$ in Ω. If $\lambda > 0$ is such that there is an eigenpair (w,z) associated to λ such that $w,z > 0$, then $\lambda = \lambda_1(s,p)$, and there exist $k_1,k_2 \in \mathbb{R}$ such that $w = k_1u$ and $z = k_2v$ almost everywhere in $\mathbb{R}^N$.*

Proof. Since $\lambda_1(s,p)$ is the first eigenvalue, we have that $\lambda_1(s,p) \le \lambda$. Moreover, by **[7,** Theorem A.1], $u,v > 0$ almost everywhere in Ω since (u,v) is an eigenpair associated to $\lambda_{1,p}$ and $u,v \ge 0$.

Let $k \in \mathbb{N}$, and define $w_k := w+1/k$ and $z_k := z+1/k$. We begin by proving that $u^p/w_k^{p-1} \in \widetilde{\mathcal{W}}^{r,p}(\Omega)$. It is immediate that $u^p/w_k^{p-1} = 0$ in Ω^c and $w_k \in L^p(\Omega)$, due to the fact that $u \in L^\infty(\Omega)$, see Lemma 3.3.

On the other hand, for any $x,y \in \mathbb{R}^N$,

$$\left|\frac{u(x)}{w_k(x)^{p-1}(x)} - \frac{u(y)}{w_k(y)^{p-1}}\right| = \left|\frac{u(x)^p - u(y)^p}{w_k(x)^{p-1}} + \frac{u(y)^p(w_k(y)^{p-1} - w_k(x)^{p-1})}{w_k(x)^{p-1}w_k(y)^{p-1}}\right|s \le k^{p-1}|u(x)^p - u(y)^p| + \|u\|^p_{L^\infty(\Omega)}\frac{|w_k(x)^{p-1} - w_k(y)^{p-1}|}{w_k(x)^{p-1}w_k(y)^{p-1}}$$

$$\begin{aligned}
&\leq 2\|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} p|u(x)-u(y)| \\
&\quad + \|u\|_{L^\infty(\Omega)}^{p}(p-1)\frac{w_k(x)^{p-2}+w_k(y)^{p-2}}{w_k(x)^{p-1}w_k(y)^{p-1}}|w_k(x)-w_k(y)| \\
&\leq 2\|u\|_{L^\infty(\Omega)}^{p-1} k^{p-1} p|u(x)-u(y)| \\
&\quad + \|u\|_{L^\infty(\Omega)}^{p}(p-1)k^{p-1}\left(\frac{1}{w_k(x)}+\frac{1}{w_k(y)}\right)|w(y)-w(x)| \\
&\leq C(k,p,\|u\|_{L^\infty(\Omega)})(|u(x)-u(y)|+|w(x)-w(y)|).
\end{aligned}$$

Hence, we have that $u^p/w_k^{p-1} \in \widetilde{\mathcal{W}}^{r,p}(\Omega)$ for all $k \in \mathbb{N}$ since $u, w \in \widetilde{\mathcal{W}}^{r,p}(\Omega)$. Analogously, $v^p/z_k^{p-1} \in \widetilde{\mathcal{W}}^{s,p}(\Omega)$.

Set

$$L(\varphi,\psi)(x,y) = |\varphi(x)-\varphi(y)|^p - (\psi(x)-\psi(y))^{p-1}\left(\frac{\varphi(x)^p}{\psi(x)^{p-1}} - \frac{\varphi(y)^p}{\psi(y)^{p-1}}\right)$$

for all functions $\varphi \geq 0$ and $\psi > 0$. For [**2,** Lemma 6.2], for any $\varphi \geq 0$ and $\psi > 0$,

$$L(\varphi,\psi)(x,y) \geq 0 \quad \text{for all } (x,y).$$

Then,

$$\begin{aligned}
0 &\leq \int_{\Omega^2} \frac{L(u,w_k)(x,y)}{|x-y|^{N+rp}}\,dx\,dy + \int_{\Omega^2} \frac{L(v,z_k)(x,y)}{|x-y|^{N+sp}}\,dx\,dy \\
&\leq \int_{\mathbb{R}^{2N}} \frac{L(u,w_k)(x,y)}{|x-y|^{N+rp}}\,dx\,dy + \int_{\mathbb{R}^{2N}} \frac{L(v,z_k)(x,y)}{|x-y|^{N+sp}}\,dx\,dy \\
&= \lambda_{1,p}\int_\Omega |u|^\alpha |v|^\beta dx - \lambda\frac{\alpha}{p}\int_\Omega w^{\alpha-1}z^\beta \frac{u^p}{w_k^{p-1}}\,dx - \lambda\frac{\beta}{p}\int_\Omega w^\alpha z^{\beta-1}\frac{v^p}{z_k^{p-1}}\,dx
\end{aligned}$$

for all $k \in \mathbb{N}$, since $(u,v),(w,z)$ are eigenpairs associated to $\lambda_{1,p}$ and λ, respectively.

On the other hand, by Young's inequality,

$$\int_\Omega w^\alpha z^\beta \frac{u^\alpha v^\beta}{w_k^\alpha z_k^\beta}\,dx \leq \frac{\alpha}{p}\int_\Omega w^{\alpha-1}z^\beta\frac{u^p}{w_k^{p-1}}\,dx + \frac{\beta}{p}\int_\Omega w^\alpha z^{\beta-1}\frac{v^p}{z_k^{p-1}}\,dx$$

for all $k \in \mathbb{N}$. Then,

$$0 \leq \int_\Omega \frac{L(u,w_k)(x,y)}{|x-y|^{N+rp}}\,dx\,dy + \int_\Omega \frac{L(v,z_k)(x,y)}{|x-y|^{N+sp}}\,dx\,dy$$

$$\leq \lambda_{1,p} \int_\Omega |u|^\alpha |v|^\beta dx - \lambda \int_\Omega w^\alpha z^\beta \frac{u^\alpha v^\beta}{w_k^\alpha z_k^\beta}\, dx.$$

By Fatou's lemma and the dominated convergence theorem, we obtain

$$\begin{aligned} 0 &\leq \int_{\Omega^2} \frac{L(u,w)(x,y)}{|x-y|^{N+rp}}\, dx\, dy + \int_{\Omega^2} \frac{L(v,z)(x,y)}{|x-y|^{N+sp}}\, dx\, dy \\ &\leq (\lambda_{1,p} - \lambda) \int_\Omega |u|^\alpha |v|^\beta\, dx. \end{aligned}$$

Then, $\lambda = \lambda_{1,p}$, and $L(u,w) = 0$ and $L(v,z) = 0$ almost everywhere in Ω.

Finally, again by **[2,** Lemma 6.2], there exist $k_1, k_2 \in \mathbb{R}$ such that $w = k_1 u$ and $z = k_2 v$ almost everywhere in $\mathbb{R}^N$. □

Now, we show that $\lambda_{1,p}$ is simple.

Corollary 3.5. *Let (u_1, v_1) be an eigenpair associated to $\lambda_{1,p}$ normalized according to $|(u_1,v_1)|_{\alpha,\beta} = 1$. If (u,v) is an eigenpair associated to $\lambda_{1,p}$, then there is a constant k such that $(u,v) = k(u_1,v_1)$.*

Proof. By Theorem 3.4, there exist k_1 and k_2 such that $u = k_1 u_1$ and $v = k_2 v_2$. Without loss of generality, we can assume that $k_1 \leq k_2$. Then, since (u_1, v_1) and (u,v) are eigenpairs associated to the first eigenvalue $\lambda_{1,p}$ and $|(u,v)|_{\alpha,\beta} = 1$, we obtain

$$\left(\left(\frac{k_1}{k_2}\right)^\beta - 1\right)[u]_{r,p}^p + \left(\left(\frac{k_2}{k_1}\right)^\alpha - 1\right)[v]_{s,p}^p = 0.$$

Taking $x = k_1/k_2$, $a = [u]_{r,p}^p$ and $b = [v]_{s,p}^p$, we get

$$a(x^\beta - 1) + b\frac{1 - x^\alpha}{x^\alpha} = 0.$$

Multiplying by x^α and using that $\alpha + \beta = p$, we obtain

$$ax^p - (a+b)x^\alpha + b = 0.$$

In order to conclude the proof, we only need show that 1 is the unique zero of the function

$$f\colon [0,1] \longrightarrow \mathbb{R}, \quad f(x) = ax^p - (a+b)x^\alpha + b.$$

Observe that, for any $x \in (0,1)$, we have

$$f'(x) = pax^{\alpha-1}\left(x^{p-\alpha} - \frac{a+b}{a}\frac{\alpha}{p}\right) = pax^{\alpha-1}\left(x^{\alpha} - \frac{a+b}{a}\frac{\alpha}{p}\right).$$

On the other hand, since (u_1, v_1) is an eigenpair associated to $\lambda_{1,p}$ such that $|(u,v)|_{\alpha,\beta} = 1$, we have

$$a + b = \lambda_{1,p} \quad \text{and} \quad a = \frac{\alpha}{p}\lambda_{1,p}.$$

Then,

$$\frac{a+b}{a} = \frac{p}{\alpha},$$

that is,

$$\frac{a+b}{a}\frac{\alpha}{p} = 1.$$

Hence,

$$f'(x) < 0 \quad \text{for all } x \in (0,1).$$

that is, f is decreasing. Therefore, $x = 1$ is the unique zero of f. □

Recall that we made the assumption $\min\{\alpha,\beta\} \geq 1$. Now, if (u,v) is an eigenpair associated to $\lambda_{1,p}$, then $|u|^{\alpha-2}u|v|^{\beta}$ and $|u|^{\alpha}|v|^{\beta-2}v \in L^{\infty}(\Omega)$ due to Lemma 3.3. Thus, by **[21,** Theorem 1.1], we have the following result.

Lemma 3.6. *If (u,v) is an eigenpair associated to $\lambda_{1,p}$, then $\gamma_1 = \gamma_1(N,p,r) \in (0,r]$ and $\gamma_2 = \gamma_2(N,p,s) \in (0,s]$ exist such that $(u,v) \in C^{\gamma_1}(\overline{\Omega}) \times C^{\gamma_2}(\overline{\Omega})$.*

Thus, by Lemma 3.6 and Theorem 2.5, we have that

Corollary 3.7. *If (u,v) is an eigenpair associated to $\lambda_{1,p}$, then u is a viscous solution of*

$$\begin{cases} (-\Delta_p)^r u = \lambda_{1,p}\frac{\alpha}{p}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and v is a viscous solution of

$$\begin{cases} (-\Delta_p)^s v = \lambda_{1,p}\frac{\beta}{p}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

It follows by Corollary 3.7 and Lemma 2.6 that we obtain

Corollary 3.8. *If* (u, v) *is an eigenpair corresponding to the first eigenvalue* $\lambda_{1,p}$*, then* $|u|, |v| > 0$ *in* Ω.

Finally, we show a sequence of eigenvalues.

Lemma 3.9. *There is a sequence of eigenvalues* λ_n *such that* $\lambda_n \to \infty$ *as* $n \to \infty$.

Proof. We follow ideas from **[20]**, and hence, we omit the details. Let us consider

$$M_\tau = \{(u, v) \in \mathcal{W}_p^{(r,s)}(\Omega) \colon [u]_{r,p}^p + [v]_{s,p}^p = p\tau\}$$

and

$$\varphi(u, v) = \frac{1}{p} \int_\Omega |u|^\alpha |v|^\beta dx.$$

We look for critical points of φ restricted to the manifold M_τ using a minimax technique. We consider the class $\Sigma = \{A \subset \mathcal{W}_p^{(r,s)}(\Omega) \backslash \{0\} : A$ is closed, $A = -A\}$. Over this class, we define the genus $\gamma \colon \Sigma \to \mathbb{N} \cup \{\infty\}$ as $\gamma(A) = \min\{k \in \mathbb{N} \colon$ there exists a $\phi \in C(A, \mathbb{R}^k - \{0\})$, $\phi(x) = -\phi(-x)\}$. Now, we let $C_k = \{C \subset M_\tau \colon C$ is compact, symmetric and $\gamma(C) \leq k\}$, and

$$\beta_k = \sup_{C \in C_k} \min_{(u,v) \in C} \varphi(u, v).$$

Then, $\beta_k > 0$, and there is a $(u_k, v_k) \in M_\tau$ such that $\varphi(u_k, v_k) = \beta_k$, and (u_k, v_k) is a weak eigenpair with $\lambda_k = \tau/\beta_k$. □

4. The limit as $p \to \infty$. From now on, we assume that (1.3) and (1.4) hold. Recall that $\Lambda_{1,\infty}$ is defined by

$$\Lambda_{1,\infty} = \inf \left\{ \frac{\max\{[u]_{r,\infty}; [v]_{s,\infty}\}}{\| |u|^\Gamma |v|^{1-\Gamma} \|_{L^\infty(\Omega)}} \colon (u, v) \in \mathcal{W}_\infty^{(r,s)}(\Omega) \right\}.$$

First, we show the geometric characterization of $\Lambda_{1,\infty}$. Then, we prove that there exists a sequence of eigenpairs (u_p, v_p) associated to $\lambda_{1,p}$ such that $(u_p, v_p) \to (u_\infty, v_\infty)$ as $p \to \infty$ and (u_∞, v_∞) is a minimizer for $\Lambda_{1,\infty}$. Finally, we will show that (u_∞, v_∞) is a viscous solution of (4.3).

4.1. Geometric characterization. Observe that, from the Arzelà-Ascoli theorem, there exists a minimizer for $\Lambda_{1,\infty}$. Moreover, if (u,v) is a minimizer for $\Lambda_{1,\infty}$, then so is $(|u|,|v|)$. Now, we show the geometric characterization of $\Lambda_{1,\infty}$.

Lemma 4.1. *The following equality holds*:

$$\Lambda_{1,\infty} = \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r}.$$

Proof. We take (u,v) as a minimizer for $\Lambda_{1,\infty}$ with $u,v \geq 0$ normalized according to $\|u^\Gamma v^{1-\Gamma}\|_{L^\infty(\Omega)} = 1$. Therefore, there is a point $x_0 \in \Omega$ such that $u^\Gamma(x_0)v^{1-\Gamma}(x_0) = 1$. We call $a = u(x_0)$ and $b = v(x_0)$. Then, since $u,v = 0$ in Ω^c,

$$[u]_{r,\infty} = \sup_{(x,y)\in\overline{\Omega}} \frac{|u(y)-u(x)|}{|x-y|^r} \geq \frac{a}{[\text{dist}(x_0,\partial\Omega)]^r},$$

and

$$[v]_{s,\infty} = \sup_{(x,y)\in\overline{\Omega}} \frac{|v(y)-v(x)|}{|x-y|^s} \geq \frac{b}{[\text{dist}(x_0,\partial\Omega)]^s}.$$

Therefore, this yields

$$\Lambda_{1,\infty} \geq \inf_{(a,b,x_0)\in\mathcal{A}} \left\{ \max\left\{ \frac{a}{[\text{dist}(x_0,\partial\Omega)]^r}; \frac{b}{[\text{dist}(x_0,\partial\Omega)]^s} \right\}\right\},$$

where

$$\mathcal{A} := \{(0,\infty)\times(0,\infty)\times\overline{\Omega} \colon a^\Gamma b^{1-\Gamma} = 1\}.$$

In order to compute the infimum, we observe that we must have

$$\frac{a}{[\text{dist}(x_0,\partial\Omega)]^r} = \frac{b}{[\text{dist}(x_0,\partial\Omega)]^s},$$

that is,

$$a = b[\text{dist}(x_0,\partial\Omega)]^{r-s}.$$

Then, using $a^\Gamma b^{1-\Gamma} = 1$, we obtain

$$b\,[\text{dist}(x_0,\partial\Omega)]^{\Gamma(r-s)} = 1.$$

Hence,

$$b = [\mathrm{dist}(x_0, \partial\Omega)]^{\Gamma(s-r)}$$

and

$$a = [\mathrm{dist}(x_0, \partial\Omega)]^{(r-s)(1-\Gamma)}.$$

This yields

$$\inf_{x_0} [\mathrm{dist}(x_0, \partial\Omega)]^{-[(1-\Gamma)s+\Gamma r]},$$

which is attained at a point $x_0 \in \Omega$ that maximizes the distance to the boundary, that is, letting

$$R(\Omega) = \mathrm{dist}(x_0, \partial\Omega),$$

we obtain that

$$\Lambda_{1,\infty} \geq \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r}.$$

In order to conclude the proof, we need to show the reverse inequality. As before, let $x_0 \in \Omega$ be the point where the maximum distance to the boundary is attained. Set

$$u_0(x) = R(\Omega)^{(r-s)(1-\Gamma)} \left(1 - \frac{|x - x_0|}{R(\Omega)}\right)_+^r,$$
$$v_0(x) = R(\Omega)^{-(r-s)\Gamma} \left(1 - \frac{|x - x_0|}{R(\Omega)}\right)_+^s.$$

We observe that $(u_0, v_0) \in C^r(\mathbb{R}^N) \times C^s(\mathbb{R}^N)$, $\|u_0^\Gamma v_0^{1-\Gamma}\|_{L^\infty(\Omega)} = 1$ and

$$\max\{[u_0]_{r,\infty}; [v_0]_{s,\infty}\} \leq \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r}.$$

Therefore,

$$\Lambda_{1,\infty} = \inf\left\{\frac{\max\{[u]_{r,\infty}; [v]_{s,\infty}\}}{\||u|^\Gamma |v|^{1-\Gamma}\|_{L^\infty(\Omega)}} : (u,v) \in \mathcal{W}_\infty^{(r,s)}(\Omega)\right\} \leq \left[\frac{1}{R(\Omega)}\right]^{(1-\Gamma)s+\Gamma r}.$$

□

Remark 4.2. Observe that (u_0, v_0) is a minimizer of $\Lambda_{1,\infty}$.

4.2. Convergence. Now, we prove that there exists a sequence of eigenpairs (u_p, v_p) associated to $\lambda_{1,p}$ such that $(u_p, v_p) \to (u, v)$ as $p \to \infty$ and (u, v) is a minimizer for $\Lambda_{1,\infty}$.

Lemma 4.3. *Let (u_p, v_p) be an eigenpair for $\lambda_{1,p}$ such that u_p and v_p are positive and $|(u, v)|_{\alpha,\beta} = 1$. Then, there exists a sequence $p_j \to \infty$ such that*

$$(u_{p_j}, v_{p_j}) \longrightarrow (u_\infty, v_\infty)$$

uniformly in $\mathbb{R}^N$. The limit (u_∞, v_∞) belongs to the space $\mathcal{W}_\infty^{(r,s)}(\Omega)$ and is a minimizer of $\Lambda_{1,\infty}$. In addition, the following holds:

$$[\lambda_{1,p}]^{1/p} \longrightarrow \Lambda_{1,\infty}.$$

Proof. We begin by showing that

$$\limsup_{p\to\infty}[\lambda_{1,p}]^{1/p} \leq \Lambda_{1,\infty}. \tag{4.1}$$

Let $\gamma > 1$ be such that $\gamma \max\{r, s\} < 1$. Then, $(u_\gamma, v_\gamma) = (u_\infty^\gamma, v_\infty^\gamma) \in \mathcal{W}_p^{(r,s)}(\Omega) \cap \mathcal{W}_\infty^{(r,s)}(\Omega)$ for all $p > 1$. Thus,

$$[\lambda_{1,p}]^{1/p} \leq \frac{([u_\gamma]_{r,p}^p + [v_\gamma]_{s,p}^p)^{1/p}}{|(u_\gamma, v_\gamma)|_{\alpha,\beta}}$$

for all $p > 1$. In addition, we observe that $\|u_\gamma^\Gamma v_\gamma^{1-\Gamma}\|_{L^\infty(\Omega)} = 1$. Then,

$$\begin{aligned}\limsup_{p\to\infty}[\lambda_{1,p}]^{1/p} &\leq \max\{[u_\gamma]_{r,\infty}; [v_\gamma]_{s,\infty}\}\\ &\leq \max\Big\{2^{r(\gamma-1)}R(\Omega)^{\gamma(r-s)(1-\Gamma)-r}; 2^{s(\gamma-1)}R(\Omega)^{-\gamma(r-s)\Gamma-s}\Big\}.\end{aligned}$$

Therefore, passing to the limit as $\gamma \to 1$ in the previous inequality and using Lemma 4.1, we obtain (4.1).

Our next step is to show that

$$\Lambda_{1,\infty} \leq \liminf_{p\to\infty}[\lambda_{1,p}]^{1/p}.$$

Let $p_j > 1$ be such that

$$\liminf_{p\to\infty}[\lambda_{1,p}]^{1/p} = \lim_{j\to\infty}[\lambda_j]^{1/p_j},$$

where $\lambda_j = \lambda_{1,p_j}$. From (4.1), without of loss of generality, we can assume that

$$2\max\{N/r, N/s\} < p_1, \quad p_j \le p_{j+1},$$

and

$$[\lambda_j]^{1/p_j} = \big([u_j]_{r,p_j}^{p_j} + [v_j]_{s,p_j}^{p_j}\big)^{1/p_j} \le \Lambda_{1,\infty} + \varepsilon \tag{4.2}$$

for all $j \in \mathbb{N}$, where ε is any positive number and (u_j, v_j) is an eigenpair corresponding to λ_j normalized according to $|(u_j, v_j)|_{\alpha_j,\beta_j} = 1$ ($\alpha_j = \alpha_{p_j}$, $\beta_j = \beta_{p_j}$) and such that $u_j, v_j > 0$ in Ω.

Let $q \in (2\max\{N/r, N/s\}, p_1)$, $t_1 = r - N/q$ and $t_2 = s - N/q$. It follows from (4.2) and Lemmas 2.1 and 2.2 that $\{u_j\}$ and $\{v_j\}$ are bounded in $W^{t_1,q}(\Omega)$ and $W^{t_2,q}(\Omega)$, respectively. Since $q\min\{t_1, t_2\} \ge N$, taking a subsequence, if necessary, we get

$$u_j \longrightarrow u_\infty \text{ strongly in } C^{0,\gamma_1}(\overline{\Omega}),$$
$$v_j \longrightarrow v_\infty \text{ strongly in } C^{0,\gamma_2}(\overline{\Omega}),$$

due to the compact Sobolev embedding theorem. Here, $0 < \gamma_1 < t_1 - N/q = r - 2N/q$ and $0 < \gamma_1 < t_2 - N/q = s - 2N/q$. Therefore, $u_\infty = v_\infty = 0$ on $\partial\Omega$.

On the other hand, by Lemma 2.2,

$$\begin{aligned} |u_j|_{t_1,q} &\le \operatorname{diam}(\Omega)^{N/p_j}|\Omega|^{2/q-2/p_j}|u_j|_{r,p_j} \\ &\le \operatorname{diam}(\Omega)^{N/p_j}|\Omega|^{2/q-2/p_j}[\lambda_j]^{1/p_j}, \\ |v_j|_{t_2,q} &\le \operatorname{diam}(\Omega)^{N/p_j}|\Omega|^{2/q-2/p_j}|v_j|_{s,p_j} \\ &\le \operatorname{diam}(\Omega)^{N/p_j}|\Omega|^{2/q-2/p_j}[\lambda_j]^{1/p_j}. \end{aligned}$$

Then, passing to the limit as $j \to \infty$ and using Fatou's lemma, we obtain $(u_\infty, v_\infty) \in W^{t_1,q}(\Omega) \times W^{t_2,q}(\Omega)$ and

$$|u_\infty|_{t_1,q} \le |\Omega|^{2/q} \liminf_{p\to\infty}[\lambda_{1,p}]^{1/p},$$
$$|v_\infty|_{t_2,q} \le |\Omega|^{2/q} \liminf_{p\to\infty}[\lambda_{1,p}]^{1/p}.$$

Now, passing to the limit as $q \to \infty$, we obtain

$$[u_\infty]_{r,\infty} \le \liminf_{p\to\infty} [\lambda_{1,p}]^{1/p},$$
$$[v_\infty]_{s,\infty} \le \liminf_{p\to\infty} [\lambda_{1,p}]^{1/p},$$

that is, $(u_\infty, v_\infty) \in \mathcal{W}_\infty^{(r,s)}(\Omega)$ and

$$\max\{[u_\infty]_{r,\infty}; [v_\infty]_{r,\infty}\} \le \liminf_{p\to\infty} [\lambda_{1,p}]^{1/p}.$$

In order to conclude the proof, we need only show that

$$\|u_\infty^\Gamma v_\infty^{1-\Gamma}\|_{L^\infty(\Omega)} = 1.$$

For all $q > 1$, there exists a $j_0 \in \mathbb{N}$ such that $p_j > q$ if $j > j_0$, and therefore, by Fatou's lemma and Hölder's inequality, we get

$$\|u_\infty^\Gamma v_\infty^{1-\Gamma}\|_{L^q(\Omega)}^q \le \liminf_{j\to\infty} \int_\Omega u_j^{(\alpha_j/p_j)q} v_j^{(\beta_j/p_j)q} dx \le \liminf_{j\to\infty} |\Omega|^{1-(q/p_j)} = 1$$

due to $|(u_j, v_j)|_{\alpha_j,\beta_j} = 1$. Then, passing to the limit as $q \to \infty$, we have

$$\|u_\infty^\Gamma v_\infty^{1-\Gamma}\|_{L^\infty(\Omega)} \le 1.$$

On the other hand,

$$1 = |(u_j, v_j)|_{\alpha_j,\beta_j}^{1/p_j} \le \|u_j^{\alpha_j/p_j} v_j^{\beta_j/p_j}\|_{L^\infty(\Omega)} |\Omega|^{1/p_j} \longrightarrow \|u_\infty^\Gamma v_\infty^{1-\Gamma}\|_{L^\infty(\Omega)}.$$

Therefore, $\|u_\infty^\Gamma v_\infty^{1-\Gamma}\|_{L^\infty(\Omega)} = 1$. □

4.3. Viscous solution. Finally, we show that (u_∞, v_∞) is a viscous solution of

(4.3)
$$\begin{cases} \min\{\mathcal{L}_{r,\infty} u(x); \mathcal{L}_{r,\infty}^+ u(x) - \Lambda_{1,\infty} u^\Gamma(x) v^{1-\Gamma}(x)\} = 0 & \text{in } \Omega, \\ \min\{\mathcal{L}_{s,\infty} u(x); \mathcal{L}_{s,\infty}^+ u(x) - \Lambda_{1,\infty} u^\Gamma(x) v^{1-\Gamma}(x)\} = 0 & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$\begin{aligned} \mathcal{L}_{t,\infty} w(x) &= \mathcal{L}_{t,\infty}^+ w(x) + \mathcal{L}_{r,\infty}^- w(x) \\ &= \sup_{y\in\mathbb{R}^N} \frac{w(x) - w(y)}{|x-y|^t} + \inf_{y\in\mathbb{R}^N} \frac{w(x) - w(y)}{|x-y|^t}. \end{aligned}$$

Next, we the introduce the precise definition of the viscous solution of (4.3).

Definition 4.4. Let $(u,v) \in C(\mathbb{R}^N) \times C(\mathbb{R}^N)$ be such that $u,v \geq 0$ in Ω and $u = v = 0$ in Ω^c. We say that (u,v) is a *viscous subsolution* of (4.3) at a point $x_0 \in \Omega$ if and only if, for any test pair $(\varphi,\psi) \in C_0^2(\mathbb{R}^N) \times C_0^2(\mathbb{R}^N)$ such that $u(x_0) = \varphi(x_0)$, $v(x_0) = \psi(x_0)$, $u(x) \leq \varphi(x)$ and $v(x) \leq \psi(x)$ for all $x \in \mathbb{R}^N$, we have that

$$\min\{\mathcal{L}_{r,\infty}\varphi(x_0); \mathcal{L}_{r,\infty}^{+}\varphi(x_0) - \Lambda_{1,\infty}u^{\Gamma}(x_0)v^{1-\Gamma}(x_0)\} \leq 0,$$
$$\min\{\mathcal{L}_{r,\infty}\psi(x_0); \mathcal{L}_{r,\infty}^{+}\psi(x_0) - \Lambda_{1,\infty}u^{\Gamma}(x_0)v^{1-\Gamma}(x_0)\} \leq 0.$$

We say that (u,v) is a *viscous subsolution* of (4.3) at a point $x_0 \in \Omega$ if and only if, for any test pair $(\varphi,\psi) \in C_0^2(\mathbb{R}^N) \times C_0^2(\mathbb{R}^N)$ such that $u(x_0) = \varphi(x_0)$, $v(x_0) = \psi(x_0)$, $u(x) \geq \varphi(x)$ and $v(x) \geq \psi(x)$ for all $x \in \mathbb{R}^N$, we have that

$$\min\{\mathcal{L}_{r,\infty}\varphi(x_0); \mathcal{L}_{r,\infty}^{+}\varphi(x_0) - \Lambda_{1,\infty}u^{\Gamma}(x_0)v^{1-\Gamma}(x_0)\} \geq 0,$$
$$\min\{\mathcal{L}_{r,\infty}\psi(x_0); \mathcal{L}_{r,\infty}^{+}\psi(x_0) - \Lambda_{1,\infty}u^{\Gamma}(x_0)v^{1-\Gamma}(x_0)\} \geq 0.$$

Finally, u is a *viscous solution* of (4.3) at a point $x_0 \in \Omega$ viscous solution if it is both a viscous super and subsolution at every x_0.

Lemma 4.5. *(u_∞, v_∞) is a viscous solution of* (4.3).

Proof. The proof follows as in [**24,** Section 8]; we include a sketch here for completeness. We show that u_∞ is a viscous supersolution of the first equation in (4.3) (the fact that it is a viscous subsolution is similar). Assume that φ is a test function touching u_∞ strictly from below at a point $x_0 \in \Omega$. We have that $u_j - \varphi$ has a minimum at points $x_j \to x_0$. Since u_j is a weak solution (and hence, a viscous solution) to the first equation in our system, we have the inequality

$$-(-\Delta_{p_j})^r\varphi(x_j) + \lambda_{1,p_j}\frac{\alpha_j}{p_j}|\varphi|^{\alpha_j-2}\varphi|v|^{\beta_j}(x_j) \leq 0.$$

Writing, as in [**24**],

$$A_j^{p_j-1} = 2\int_{\mathbb{R}^N} \frac{|\varphi(x_j)-\varphi(y)|^{p_j-2}(\varphi(x_j)-\varphi(y))^+}{|x_j-y|^{N+sp_j}}\,dy,$$

$$B_j^{p_j-1} = 2\int_{\mathbb{R}^N} \frac{|\varphi(x_j)-\varphi(y)|^{p_j-2}(\varphi(x_j)-\varphi(y))^-}{|x_j-y|^{N+sp_j}}\,dy$$

and

$$C_j^{p_j-1} = \lambda_{1,p_j}\frac{\alpha_j}{p_j}|\varphi|^{\alpha_j-2}\varphi|v|^{\beta_j}(x_j),$$

we obtain

$$A_j^{p_j-1} + C_j^{p_j-1} \le B_j^{p_j-1}.$$

Using that

$$A_j \longrightarrow \mathcal{L}^+_{r,\infty}\varphi(x_0),$$

$$B_j \longrightarrow -\mathcal{L}^-_{r,\infty}\varphi(x_0)$$

and

$$C_j \longrightarrow \Lambda_{1,\infty}u^\Gamma(x_0)v^{1-\Gamma}(x_0),$$

we obtain

$$\min\{\mathcal{L}_{r,\infty}\varphi(x_0);\mathcal{L}^+_{r,\infty}\varphi(x_0)-\Lambda_{1,\infty}u^\Gamma(x_0)v^{1-\Gamma}(x_0)\}\le 0. \qquad \square$$

REFERENCES

1. R.A. Adams, *Sobolev spaces*, Pure Appl. Math. **65** (1975).

2. S. Amghibech, *On the discrete version of Picone's identity*, Discr. Appl. Math. **156** (2008), 1–10.

3. G. Aronsson, M.G. Crandall and P. Juutinen, *A tour of the theory of absolutely minimizing functions*, Bull. Amer. Math. Soc. **41** (2004), 439–505.

4. T. Bhattacharya, E. DiBenedetto and J.J. Manfredi, *Limits as $p\to\infty$ of $\Delta_p u_p = f$ and related extremal problems*, Rend. Sem. Mat. Univ. **1989** (1991), 15–68.

5. L. Boccardo and D.G. de Figueiredo, *Some remarks on a system of quasilinear elliptic equations*, Nonlin. Diff. Eqs. Appl. **9** (2002), 309–323.

6. D. Bonheure, J.D. Rossi and N. Saintier, *The limit as $p\to\infty$ in the eigenvalue problem for a system of p-Laplacians*, Ann. Mat. Pura Appl. **195** (2016), 1771–1785.

7. L. Brasco and G. Franzina, *Convexity properties of Dirichlet integrals and Picone-type inequalities*, Kodai Math. J. **37** (2014), 769–799.

8. L. Brasco, E. Lindgren and E. Parini, *The fractional Cheeger problem*, Interfaces Free Bound. **16** (2014), 419–458.

9. L. Brasco, E. Parini and M. Squassina, *Stability of variational eigenvalues for the fractional p-Laplacian*, Discr. Contin. Dynam. Syst. **36** (2016), 1813–1845.

10. V. Caselles, J.M. Morel and C. Sbert, *An axiomatic approach to image interpolation*, IEEE Trans. Image Proc. **7** (1998), 376–386.

11. T. Champion, L. De Pascale and C. Jimenez, *The ∞-eigenvalue problem and a problem of optimal transportation*, Comm. Appl. Anal. **13** (2009), 547–565.

12. L.M. Del Pezzo and J.D. Rossi, *The first nontrivial eigenvalue for a system of p-Laplacians with Neumann and Dirichlet boundary conditions*, Nonlin. Anal. **137** (2016), 381–401.

13. F. Demengel and G. Demengel, *Functional spaces for the theory of elliptic partial differential equations*, Universitext, Springer, London, 2012.

14. P.L. de Napoli and J.P. Pinasco, *Estimates for eigenvalues of quasilinear elliptic systems*, J. Diff. Eqs. **227** (2006), 102–115.

15. E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.

16. J. Fleckinger, R.F. Mansevich, N.M. Stavrakakis and F. de Thlin, *Principal eigenvalues for some quasilinear elliptic equations on* $\mathbb{R}^n$, Adv. Diff. Eqs. **2** (1997), 981–1003.

17. G. Franzina and G. Palatucci, *Fractional p-eigenvalues*, Riv. Math. Univ. Parma **5** (2014), 373–386.

18. J. Garcia-Azorero, J.J. Manfredi, I. Peral and J.D. Rossi, *Steklov eigenvlue for the ∞-Laplacian*, Rend. Lincei **17** (2006), 199–210.

19. ———, *The Neumann problem for the ∞-Laplacian and the Monge-Kantorovich mass transfer problem*, Nonlin. Anal. **66** (2007), 349–366.

20. J. Garcia-Azorero and I. Peral, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323**, (1991), 877–895.

21. A. Iannizzotto, S. Mosconi and M. Squassina, *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iber. **32** (2016), 1353–1392.

22. P. Juutinen and P. Lindqvist, *On the higher eigenvalues for the ∞-eigenvalue problem*, Calc. Var. Part. Diff. Eqs. **23** (2005), 169–192.

23. P. Juutinen, P. Lindqvist and J.J. Manfredi, *The ∞-eigenvalue problem*, Arch. Rat. Mech. Anal. **148** (1999), 89–105.

24. E. Lindgren and P. Lindqvist, *Fractional eigenvalues*, Calc. Var. Part. Diff. Eqs. **49** (2014), 795–826.

25. Y. Peres, O. Schramm, S. Sheffield and D.B. Wilson, *Tug-of-war and the infinity Laplacian*, J. Amer. Math. Soc. **22** (2009), 167–210.

26. Y. Peres and S. Sheffield, *Tug-of-war with noise*: *A game theoretic view of the p-Laplacian*, Duke Math. J. **145** (2008), 91–120.

27. J.D. Rossi and N. Saintier, *On the first nontrivial eigenvalue of the ∞-Laplacian with Neumann boundary conditions*, Houston J. Math. **42** (2016), 613–635.

28. ———, *The limit as $p \to +\infty$ of the first eigenvalue for the p-Laplacian with mixed Dirichlet and Robin boundary conditions*, Nonlin. Anal. **119** (2015), 167–178.

29. N. Zographopoulos, *p-Laplacian systems at resonance*, Appl. Anal. **83** (2004), 509–519.

Conicet–Utdt, Departamento de Matemáticas y Estadística, Av. Figueroa, Alcorta 7350 (C1428BCW) Buenos Aires, Argentina
Email address: **ldelpezzo@utd.edu**

Universidad de Buenos Aires, Conicet and Departamento de Matemática, FCEyN, Ciudad Universitaria (1428), Pabellon I, Buenos Aires, Argentina
Email address: **jrossi@dm.uba.ar**