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SOME RESULTS ON A CLASS OF MIXED VAN DER WAERDEN NUMBERS

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ABSTRACT. The mixed van der Waerden number $w(k_1, \ldots, k_r; r)$ is the least positive integer n such that every r-coloring of $[1,n]$ admits a monochromatic arithmetic progression of length k_i, for at least one i. We denote by $w_2(k;r)$ the case in which $k_1 = \cdots = k_{r-1} = 2$ and $k_r = k$. For $k \leq r$, we give upper and lower bounds for $w_2(k;r)$, also indicating cases when these bounds are achieved. We determine exact values in the cases where $(k,r) \in \{(p,p),(p,p+1),(p+1,p+1)\}$ and give bounds in the cases where $(k,r) \in \{(p,p+2),(p+2,p+2)\}$, for primes p. We provide a table of values for the cases $k \leq r$ with $3 \leq k \leq 10$ and for several values of r, correcting some known values.

1. Introduction. The theorem of van der Waerden **[19]** in 1927 is one of the foremost and important results in Ramsey theory. For any pair of positive integers k and r, there exists a least positive integer $w(k;r)$ such that any r-coloring of the integers in the interval $[1, w(k;r)]$ must contain a monochromatic k-term arithmetic progression. Equivalent forms of van der Waerden's theorem may be found in **[15]**. The original proof gave enormous bounds; the interested reader is referred to **[11]**. Shelah **[17]** gave an elementary proof with primitive recursive bounds, and Gowers **[10]** gave a non-elementary proof of bounds that can actually be written down. As easy application of the Pigeonhole principle shows that $w(k;r) = r+1$ for $k \in \{1,2\}$ and $w(k;1) = k+1$. For $k \geq 3$ and $r \geq 2$, the only known exact values are

$$w(3;2) = 9, \qquad w(3;3) = 27,$$
$$w(4;2) = 35, \qquad w(3;4) = 76,$$

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$$w(5;2) = 178, \qquad w(4;3) = 293,$$
$$w(6;2) = 1132.$$

For any set of r positive integers, $k_1, \ldots, k_r$, van der Waerden's theorem ensures that there is a least positive integer $w(k_1, \ldots, k_r; r)$ such that any r-coloring of the integers in the interval $[1, w(k_1, \ldots, k_r; r)]$ must contain a k_i-term arithmetic progression in color i, for at least one i. The numbers $w(k_1, \ldots, k_r; r)$ are called *mixed* van der Waerden numbers and may be seen as generalizations of the van der Waerden numbers $w(k; r)$. Results on mixed van der Waerden numbers were found by Chvátal [**9**], Brown [**8**], Stevens and Shantaram [**18**], Beeler and O'Neil [**7**], Beeler [**6**], Landman, Robertson and Culver [**16**], Ahmed [**1, 2, 3, 4**] and Ahmed, Kullmann and Snelivy [**5**]. In 2005, the article by Khodkar and Landman [**12**] reported recent progress on Ramsey theory, and in particular, on mixed van der Waerden numbers. A table of all exact values of mixed van der Waerden numbers known through December 2012 may be found in [**3**].

From the definition of the van der Waerden number $w(k_1, \ldots, k_r; r)$, we see that there must be at least one r-coloring of the integers in the interval $[1, w(k_1, \ldots, k_r; r) - 1]$ that contains no k_i-term arithmetic progression in color i, for each i. We call such a coloring a *valid* r-coloring of the integers in the interval $[1, w(k_1, \ldots, k_r; r) - 1]$. Thus, a valid r-coloring of the integers in the interval $[1, n - 1]$ implies that the van der Waerden is at least n. If each k_i equals 2, the Pigeonhole principle implies $w(k_1, \ldots, k_r; r) = w(2; r)$ equals $r + 1$. Therefore, the first non-trivial mixed van der Waerden number in general is the case where each k_i, except one, equals 2. Following the notation introduced in [**16**], we denote the case $k_1 = \cdots = k_{r-1} = 2$ and $k_r = k$ by $w_2(k; r)$. Landman, Robertson and Culver [**16**] considered the case $k > r \geq 2$, obtaining lower bounds in all cases and exact values in some. Khodkar and Landman [**12**] made some in roads into the case $k \leq r$, viz., $w_2(k; r) \leq r(k - 1)$ when $k < r < 3(k - 1)/2$. They also provided a computer-generated table of values of $w_2(k, r)$ for $3 \leq k \leq 5$ and $k < r \leq 13$.

The purpose of this work is to determine $w_2(k; r)$ in the case $k \leq r$. In Section 2, in Theorems 2.1 and 2.3, we give upper and lower bounds for $w_2(k; r)$, also indicating cases when these bounds are achieved. From the results in Section 2, in Theorem 2.9 and in

Corollary 2.6 (i), (ii) we determine exact values in the cases where $(k,r) \in \{(p,p),(p,p+1),(p+1,p+1)\}$, and in Theorems 4.2 and in Corollary 2.6 (iii) we give bounds for $w_2(k;r)$, in the case of prime p. The general upper bound we obtain improves those obtained in **[12]** for the case $k < r < 3(k-1)/2$. The upper and lower bounds for the difference $w_2(k;r+1) - w_2(k;r)$ given in Theorem 2.8 can be used to provide general bounds in terms of $w_2(k;k)$. In Section 3, we provide a table of values for the cases $k \le r$ with $3 \le k \le 10$ and for several values of r, correcting some values provided in [**12,** page 8, Table 1]. We also include some conjectures based upon limited computer-generated values for $w_2(k;r)$. A summary of our results is given in Table 1.

TABLE 1. Summary of results on $\mathtt{w}_2(k;r)$, $k \le r$. Note that p is the largest prime $\le k$.

$\mathtt{w}_2(k;r)$	(k,r)	Result
$[(k-1)p+3, k(k-1)+1]$	(k,k), k composite	Theorems 2.1, 2.3
p^2-p+1	(p,p), p prime	Corollary 2.6 (i)
$[(p-1)^2+r, k(r-2)+2]$	$k<r$	Theorems 2.1, 2.3
p^2-p+2	$(p,p+1)$, p prime	Corollary 2.6 (ii)
p^2+3	$(p+1,p+1)$, p prime	Theorem 2.9
$[p(p-1)+3, p(p-1)+5]$	$(p,p+2)$, p prime	Theorem 4.2

2. Results for $w_2(k;r)$ when $k \le r$. Throughout this section, we use the notation $w_2(k;r)$ to denote the mixed van der Waerden number $w(2,\dots,2,k;r)$. By a *valid coloring* of $[1, w_2(k;r)-1]$, we mean an r-coloring of the integers in $[1, w_2(k;r)-1]$ that (i) contains at most one integer in each of $r-1$ singleton color classes, and (ii) does not contain a k-term arithmetic progression in the remaining color class.

Theorem 2.1. *Let $r \ge k \ge 3$. Then,*

$$w_2(k;r) \ge \begin{cases} (k-1)p+3 & \text{if } k=r,\ k \text{ composite;} \\ (p-1)^2+r & \text{otherwise,} \end{cases}$$

where p is the largest prime not exceeding k. Moreover, the lower bound in the second case is attained when $k=p$ and $r \in \{p,p+1\}$, where p is prime.

Proof. We divide the proof into two cases.

Case I ($k = r$, k composite). In order to prove the lower bound, we must provide a valid k-coloring of $[1, (k-1)p + 2]$. We color the integers in

$$A = \{ip + 2 : 1 \leq i \leq k - 2\} \cup \{1\}$$

with distinct colors and observe that $|A| = k - 1$.

It remains to show that the set

$$B = [1, (k-1)p + 2] \setminus A$$

has no arithmetic progression of length k. Suppose $a, a + d, \dots, a + (k-1)d$ is a k-term AP in B. Since $a + (k-1)d \leq 2 + (k-1)p$, we must have $d \leq p$. If $d = p$, then $a+(k-1)d = a+(k-1)p \leq 2+(k-1)p$ such that $a = 1$ or 2. However, each is impossible since neither 1 nor $p + 2$ belong to B.

Hence, $d < p$, and thus, the integers $p + 2, 2p + 2, \dots, dp + 2$ form a complete residue system modulo d. Thus, exactly one of these integers must be congruent to a modulo d. Suppose $ip + 2 = a + jd$. Since $a \neq 1$,

$$j = (ip + 2 - a)/d \leq ip/d \leq p \leq k - 1.$$

Hence, $a + jd \in A \cap B$, which is impossible.

Therefore, no arithmetic progression of length k can lie entirely in B.

Case II ($k < r$ or $k = r$, k prime). In order to prove the lower bound, we must provide a valid r-coloring of $[1, N]$, where $N = (p-1)^2 + r - 1 = p(p-1) + r - p$. We color the integers in $A = \{ip : 1 \leq i \leq p - 1\} \cup [p(p-1) + 1, N]$ with distinct colors and observe that

$$|A| = (p - 1) + (N - p(p - 1)) = N - (p - 1)^2 = r - 1.$$

It remains to show that the set $B = [1, N] \setminus A$ has no arithmetic progression of length k. Suppose $a, a + d, \dots, a + (k-1)d$ is a k-term arithmetic progression in B. If $p \nmid d$, the integers $a, a+d, \dots, a+(p-1)d$ form a complete residue system modulo p. Since $k \geq p$, at least one of the terms of the arithmetic progression must be a multiple of p and thus must belong to A. If $p \mid d$, then $a + (k-1)d \geq 1 + (p-1)p$ such that $a + (k-1)d \in A$. Therefore, no arithmetic progression of length k can lie entirely in B.

The lower bound in the second case is attained for $k = p$ and $r \in \{p, p+1\}$ for prime p, by Theorem 2.3. $\square$

TABLE 2.

1	2	...	r-p	...	**p**
p+1	p+2	...	p+(r-p)	...	**2p**
⋮	⋮	⋮	⋮	⋮	**⋮**
(p-2)p+1	(p-2)p+2	...	(p-2)p+(r-p)	...	**(p-1)p**
(p-1)p+1	**(p-1)p+2**	**...**	**(p-1)p+(r-p)**		

Remark 2.2. The integers in the set A in the second case of Theorem 2.1 are represented in boldface in Table 2, providing a valid coloring of $[1, N]$, where $N = (p-1)p + (r-p)$. This array is drawn under the assumption $r - p < p$; when $r - p \geq p$, integers from A fill up one or more full row in the array. Unless $p \mid r$, the last row is only partially occupied by integers in A. If $r \equiv j \mod p$ and $j \neq 0$, this coloring cannot always be extended to $[1, N+1]$ since $N+1$ must belong to B, resulting in an arithmetic progression of length at least p in B consisting of all integers in column $(j+1)$ up to and including $N+1$. If either $p \mid r$ or $p < k$, the given valid coloring of $[1, N]$ can be extended to a valid coloring of $[1, M]$ for some $M > N$. However, if $k = p$, p prime, and $r \in \{p, p+1\}$, we shall show that there is no valid coloring of $[1, N+1]$. This proves that the lower bound of Theorem 2.1 in the second case is sharp.

Theorem 2.3. *Let $r \geq k \geq 3$. Then*

$$w_2(k; r) \leq \begin{cases} k(k-1)+1 & \text{if } k = r; \\ k(r-2)+2 & \text{if } k < r. \end{cases}$$

Moreover, equality holds in the first case if and only if $k = r$, k prime, and in the second case if and only if $k = p$ and $r = p+1$, for prime p.

Proof. We divide the proof into two cases.

Case I ($k = r$). We first show that in any valid k-coloring of $[1, k(k-1)]$, any two integers that are in singleton color classes must be congruent modulo k. Consider a valid k-coloring of $[1, k(k-1)]$, and partition this interval into the $k-1$ intervals $[(i-1)k+1, ik]$, $1 \leq i \leq k-1$. Let the $k-1$ integers in the singleton color classes be $a_1, \ldots, a_{k-1}$, written in increasing order. None of these intervals can contain successive terms a_i, a_{i+1} since then some interval will contain all k integers from one color class, and this is not possible in any valid coloring. Hence, each interval contains exactly one term of the sequence $a_1, \ldots, a_{k-1}$ such that $a_i = (i-1)k + r_i$ with $1 \leq r_i \leq k$, for each i. Moreover, $a_{i+1} - a_i \leq k$, since otherwise there would be at least k consecutive integers from one color class. Suppose $a_{i+1} - a_i = d < k$ for at least one i. Then, both a_i and a_{i+1} belong to the interval $[(i-1)k+1, (i+d-1)k]$ when $i + d \leq k$, and to the interval $[(k-d-1)k+1, (k-1)k]$ when $i + d > k$. If we partition this interval into d classes of residues modulo d, then a_i and a_{i+1} belong to the same class. Since this interval contains exactly the d integers among the a_is and each class of residues has k elements, there must be one class that contains no a_i. This class is therefore monochromatic, contradicting the valid coloring of the interval $[1, k(k-1)]$. Therefore, $a_{i+1} - a_i = k$ for each i, so that

$$a_i = (i-1)k + j \tag{2.1}$$

with $1 \leq j \leq k$, for each i.

Consider any k-coloring of $[1, k(k-1)+1]$. Any non-valid coloring of $[1, k(k-1)]$ is automatically a non-valid coloring of $[1, k(k-1)+1]$. Therefore, we may assume that the k-coloring is a valid coloring of $[1, k(k-1)]$. Hence, the integers in the singleton color classes are

$$j,\ j+k,\ j+2k, \ldots, j+(k-2)k$$

for some $j \in \{1, \ldots, k\}$ by (2.1). If $j = 1$, the k consecutive terms

$$(k-2)k+2,\ (k-2)k+3, \ldots, (k-1)k+1$$

are monochromatic. If $j > 1$, the k terms in arithmetic progression

$$1,\ 1+k,\ 1+2k, \ldots, 1+k(k-1)$$

are monochromatic. Thus, any k-coloring of $[1, k(k-1)+1]$ must contain either a monochromatic k-term arithmetic progression, or a monochromatic 2-term arithmetic progression. This proves the upper bound in Case I.

In order to prove the necessary and sufficient condition for equality, observe that equality holds when $k = r$ is a prime by Theorem 2.1. If $k = r$ is not prime, let q denote the smallest prime divisor of k. In any valid k-coloring of $[1, k(k-1)]$, any two integers that are in singleton color classes must be congruent modulo k; say, $i, i+k, \ldots, i+(k-2)k$ with $1 \le i \le k$. Then,

$$i+1,\ i+1+q, \ldots, i+1+(k-1)q$$

is a k-term arithmetic progression in $[1, k(k-1)]$. This proves the necessary and sufficient condition for equality.

Case II ($k < r$). Suppose that there is a valid r-coloring of $[1, k(r-2)+2]$. Partition this interval into the $r-2$ intervals $[(i-1)k+1, ik]$, $1 \le i \le r-2$. Let the $r-1$ integers in the singleton color classes be $a_1, \ldots, a_{r-1}$, written in increasing order.

Each of these intervals must contain at least one a_i since, otherwise, some interval will contain all k integers from one color class, and this is impossible in any valid coloring. Two cases arise:

(i) $a_{r-1} \in \{k(r-2)+1, k(r-2)+2\}$;

(ii) $a_{r-1} \le k(r-2)$.

Suppose that $a_{r-1} \in \{k(r-2)+1, k(r-2)+2\}$. Then, each of the $r-2$ intervals $[(i-1)k+1, ik]$, $1 \le i \le r-2$, contains exactly one of the terms $a_1, \ldots, a_{r-2}$. Arguing as in Case I, we obtain that $a_{i+1} - a_i = k$ such that $a_i = (i-1)k + j$, $i \in \{1, \ldots, r-3\}$, for some $j \in \{1, \ldots, k\}$. If $k < r-1$, the numbers $j_0, j_0+k, \ldots, j_0+(k-1)k$, $j_0 \ne j$, form a k-term arithmetic progression from one color class. If $k = r-1$, the only possibility for a valid coloring is when $j = 2$ and $a_{r-1} = k(r-2)+1$. In this case, the k-term arithmetic progression with common difference $k-1$ and last term $k(r-2)+2$ are from the same color class, leading to a contradiction to the valid coloring.

The case $a_{r-1} \le k(r-2)$ remains for consideration. Since $a_{i+1} - a_i \le k$ for each i, each interval $[(i-1)k+1, ik]$ for $i \in \{1, \ldots, r-2\}$ contains exactly one of the numbers $a_1, \ldots, a_{r-1}$, except the interval $[(j-1)k+1, jk]$. Thus, $a_i = (i-1)k + b_i$ for $i \in \{1, \ldots, j\}$ and $a_i = (i-2)k + b_i$ for $i \in \{j+1, \ldots, r-1\}$, with $1 \le b_i \le k$. Moreover, $b_1 \ge b_2 \ge \cdots \ge b_j$ and $b_{j+1} \ge b_{j+2} \ge \cdots \ge b_{r-1} \ge 3$. Observe that $b_{r-1} = 1$ or 2 leads to at least k consecutive integers $a_{r-1}+1, \ldots, k(r-2)+2$. If none of the b_is is 1, we have an arithmetic progression with common difference k and initial term 1; the same argument applies to the case when none of the b_is is 2. Hence, there exists an integer $t \leq j-1$ such that $b_t = 2$ and $b_{t+1} = 1$.

First, suppose that k is prime. If $a_1 = 2$ and $a_2 = 1$, then there exists a monochromatic p-term arithmetic progression with common difference $a_3 - a_2 = a_3 - 1$ in the interval $[3, (a_3-1)p+2]$. Otherwise, we write $a_t - a_{t-1} = d_1$ and $a_{t+2} - a_{t+1} = d_2$. Since $a_{t+1} - a_t = k-1$ is even, $a_{t+2} - a_{t-1}$ cannot exceed $2k$ (for otherwise, there are at least k integers of the same parity between a_{t-1} and a_{t+2} of the same color). Hence, $d_1 + d_2 \leq 2k - (k-1) = k+1$. If $d_1 \leq t$, the interval $[(t-d_1+1)k+1, (t+1)k]$ contains exactly d_1 of the a_is, of which two differ by d_1, leaving a choice of a k-term arithmetic progression of the same color. Hence, $d_1 \geq t+2$. Similarly, if $d_2 \leq r-t-1$, the interval $[a_t+1, a_t+d_2k]$ contains exactly d_2 of the a_is, of which two differ by d_2, leaving a choice of a k-term arithmetic progression of the same color. Hence, $d_2 \geq r-t-1$. However, then $d_1+d_2 \geq (t+2)+(r-t-1) = r+1 > k+1$, leading to a contradiction.

Now, suppose k is composite. Assume, without loss of generality, $j > \lfloor (r-2)/2 \rfloor$. We claim that $a_{m+1} - a_m \neq a_{n+1} - a_n$ for $1 \leq n < m < j$. Suppose otherwise; let $a_{m+1} - a_m = a_{n+1} - a_n = d$. If $d < k$, there must exist an interval of length dk containing a k-term arithmetic progression with common difference d since the interval either contains $a_n, a_{n+1}, a_m, a_{m+1}, a_j, a_{j+1}$ or a_n, a_{n+1} and not a_j, a_{j+1}. If $d = k$ is composite, the above argument applies to any factor of d. This proves the claim. Hence,

$$\sum_{i=1}^{\lceil (r-2)/2 \rceil} (a_{i+1} - a_i) \geq \sum_{i=1}^{\lceil (r-2)/2 \rceil} i \geq \frac{1}{8} r(r-2) \geq \frac{1}{8}(k+1)(k-1) > k$$

for $k > 8$, contradicting $j > \lfloor (r-2)/2 \rfloor$ when $k > 8$. For $k \leq 8$, the result follows from Table 3 and Theorem 2.8. This proves the upper bound in Case II. Moreover, we have $w_2(k;r) \leq k(r-2)$ when k is composite.

The necessary and sufficient condition for equality directly follows from the upper bound $w_2(k;r) \leq k(r-2)$ for composite k and Theorem 4.2. $\square$

Remark 2.4. Suppose $a_1, a_2, \ldots, a_{k-1}$ are the integers in the singleton color classes in any valid coloring of $[1, k(k-1)]$. Equation (2.1) in the proof of Theorem 2.3 shows that $a_i \equiv a_j \mod k$ for $1 \leq i < j \leq k-1$.

Remark 2.5. The upper bounds in Theorem 2.3 are an improvement on the upper bound $w_2(k;r) \leq r(k-1)$ when $k < r < 3(k-1)/2$, obtained by Khodkar and Landman **[12]**.

Corollary 2.6.

(i) *For any prime p,*

$$w_2(p;p) = p(p-1) + 1.$$

(ii) *For any odd prime p,*

$$w_2(p;p+1) = p(p-1) + 2.$$

(iii) *For any odd prime p such that $p+2$ is composite,*

$$p(p+1) + 3 \leq w_2(p+2;p+2) \leq p(p+3) + 3.$$

Proof. This is a direct consequence of Theorems 2.1 and 2.3. □

Remark 2.7. Corollary 2.6 (i), (ii) show that the lower bound in the second case of Theorem 2.1 is sharp.

Theorem 2.8. *Let $r \geq k \geq 3$. Then,*

$$1 \leq w_2(k;r+1) - w_2(k;r) \leq k.$$

Proof. Consider a valid $(r+1)$-coloring of the interval $[1, w_2(k;r+1) - 1]$, and let $a_1, \ldots, a_r$ be the integers, arranged in increasing order, in the singleton color classes. Since a valid coloring cannot contain a monochromatic set of k consecutive integers,

$$a_r \in \{w_2(k;r+1) - k, \ldots, w_2(k;r+1) - 1\}.$$

Since the same coloring on $[1, a_r - 1]$ is a valid r-coloring, there is always a valid r-coloring of $[1, w_2(k;r+1) - k - 1]$. Therefore, $w_2(k;r) \geq w_2(k;r+1) - k$, as desired. □

Theorem 2.9. *For any prime* p,

$$w_2(p+1;p+1) = p^2+3.$$

Proof. When $p=2$, Corollary 2.6 (i) applies with $p=3$ to give $w_2(3;3) = 7 = 2^2+3$. When p is odd, the lower bound $w_2(p+1;p+1) \geq p^2+3$ is a special case of Theorem 2.1.

In order to prove the upper bound, assume by way of contradiction that there is a valid $(p+1)$-coloring of $[1, p^2+3]$. Let the integers in the singleton color class be $S = \{a_1, \ldots, a_p\}$, arranged in increasing order. We clearly must have $a_{i+1} - a_i \leq p+1$ for $i \in \{1, \ldots, p-1\}$ and $a_p \geq (p^2+3) - p = p(p-1)+3$. Let $j \in \{1,2,3\}$. Since $\{j + ip : 0 \leq i \leq p\}$ constitutes a $(p+1)$-term arithmetic progression, at least one member of S must correspond congruent to j modulo p. Let $s_i \equiv i \mod p$ for $i \in \{1,2,3\}$ and $s_i \in S$.

We claim that, if $[ip+1, (i+1)p] \cap S = \emptyset$ for some $i \in \{1, \ldots, p-1\}$, then each of the intervals $[(i-1)p+1, ip]$, $[(i+1)p+1, (i+2)p]$ must contain at least two elements of S, and these must be of opposite parity. Suppose that $[ip+1, (i+1)p] \cap S = \emptyset$ for some i. This forces both ip and $(i+1)p+1$ to belong to S. Moreover, if s', s'' are the elements of S immediately preceding ip and immediately succeeding $(i+1)p+1$, respectively, then $s'' - s' \leq 2p+2$. Otherwise, since ip, $(i+1)p+1$ have the same parity, the $(p+1)$-term arithmetic progression $\{(i-1)p, (i-1)p+2, \ldots, (i+1)p\}$ is monochromatic. This proves that the interval $[(i-1)p+1, ip]$ contains at least two elements of S. The proof that the interval $[(i+1)p+1, (i+2)p]$ contains at least two elements of S is similar and is omitted. This proves the claim.

We also claim that, if $[(i-1)p+1, ip] \cap S = \emptyset$ and $[(i+1)p+1, (i+2)p] \cap S = \emptyset$, then the interval $[ip+1, (i+1)p]$ contains at least three elements of S. Suppose $[(i-1)p+1, ip] \cap S = \emptyset$ and $[(i+1)p+1, (i+2)p] \cap S = \emptyset$. Then, $ip+1 \in S$ and $(i+1)p \in S$. Since $ip+1$ and $(i+1)p$ are of the same parity, there would be a monochromatic $(p+1)$-term arithmetic progression with common difference 2 within the interval $[(i-1)p+1, (i+2)p]$ if there is no other element of S in the interval $[ip+1, (i+1)p]$. This proves our claim.

We next claim that there is at most one interval of the form $[ip+1, (i+1)p]$, $i \in \{1, \ldots, p-2\}$, which contains no element of S. Suppose, to the contrary, that there are at least two such intervals;

let the first of these be $I_s = [sp+1,(s+1)p]$ and the last of these be $I_t = [tp+1,(t+1)p]$. Any interval $I = [ip+1,(i+1)p]$ with $s-1 \le i \le t+1$ is one of four types: (i) empty intersection with S; (ii) flanked by two intervals, each having empty intersection with S, for $i \neq \{s-1,t+1\}$; (iii) flanked by one interval with empty intersection with S and one with non-empty intersection with S; and (iv) non-empty intersection with S and flanked by intervals with non-empty intersection with S. Then, $|I \cap S|$ must be at least 3 in case (ii), at least 2 in case (iii), and at least 1 in case (iv). This implies that there must be at least $t-s+5$ elements of S within the interval

$$[(s-1)p+1,(t+2)p] = I_{s-1} \cup I_s \cup \cdots \cup I_{t+1},$$

since each of the intervals I_j $(j \neq s,t)$ contains at least one element of S and an additional four elements of S come from $I_{s-1}, I_{s+1}, I_{t-1}, I_{t+1}$.

There remain $(p-2)-(t-s+3) = p-t+s-5$ intervals of the form $[ip+1,(i+1)p]$, $i \in \{1,\ldots,p-2\}$. If there is an element of S in the interval $[p(p-1)+1,p^2]$, then $p-t+s-5$ intervals of the form $[ip+1,(i+1)p]$, $i \in \{1,\ldots,p-2\}$ contain $p-t+s-6$ elements, which contradicts our assumption. If there is no element of S in the interval $[p(p-1)+1,p^2]$, then $a_p > p^2$ and $a_{p-2} > p(p-2)$. This implies $p-t+s-7$ intervals of the form $[ip+1,(i+1)p]$, $i \in \{1,\ldots,p-3\}$ contain $p-t+s-8$ elements, which contradicts our assumption. This proves our claim.

We next claim that every interval of the form $[ip+1,(i+1)p]$, $i \in \{1,\ldots,p-2\}$ contains at least one element of S. Suppose, to the contrary, that $[jp+1,(j+1)p] \cap S = \emptyset$. This implies the intervals I_{j-1} and I_j, I_{j+1} together contain at least four elements of S. If $I_{p-1} \cap S = \emptyset$, then $a_p > p^2$, which implies $I_0 \cap S = \emptyset$. On the other hand, if $I_{p-1} \cap S \neq \emptyset$, then the previous argument implies $I_0 \cap S = \emptyset$. Let i be such that $a_i \equiv 1 \mod p$ and $a_{i+1} \equiv 2 \mod p$. Such an i exists since $a_1 = p+1$, and there exists $a_n \equiv 2 \mod p$ with $a_n < jp$. If $a_i > (j+1)p$ and $a_{i+1} - a_i = p+1$, then there must exist a monochromatic $(p+1)$-term arithmetic progression with common difference 2 within the interval $[a_n - 2p - 2, a_n]$, where $a_n \equiv 3 \mod p$ and $a_{n-1} \equiv 2 \mod p$. If $a_{i+1} - a_i = 1$ and $a_i \notin I_{j-1} \cup I_{j+1}$, then $I_t \cap S = \emptyset$ for some $t \neq j$, which is a contradiction. If $a_i \in I_{j-1}$, then I_{j-1} must contain at least three elements of S, leading to the contradiction that some interval I_t has empty intersection with S. If

$a_i \in I_{j+1}$, the only possible $(p+1)$-coloring that does not contain a monochromatic $(p+1)$-term arithmetic progression with common difference 2 has $a_s \equiv 3 \mod p$ for all $s > i+1$. This induces a monochromatic $(p+1)$-term arithmetic progression with common difference $p-i+1$ ending at p^2+3. Hence, the claim.

The argument in the preceding paragraph also shows that $[p(p-1)+1, p^2] \cap S \neq \emptyset$. Two cases remain: (i) $I_j \cap S \neq \emptyset$ for $j \in \{0,\ldots,p-1\}$ and (ii) $I_0 \cap S = \emptyset$ and $|I_j \cap S| = 2$. In case (i), $a_{i+1} - a_i = p$ for $i \in \{0,\ldots,p-1\}$, by Remark 2.4. This gives a monochromatic $(p+1)$-term arithmetic progression with common difference p starting at 1, 2 or 3. In case (ii), let j be such that $a_j \equiv 1 \mod p$ and $a_{j+1} \equiv 2 \mod p$, which leads to monochromatic $(p+1)$-term arithmetic progression with common difference either 2 or $p-j+1$, as in the preceding paragraph, which proves the theorem. $\square$

Remark 2.10. Let p be a prime, $p > 5$. Suppose $a_1, a_2, \ldots, a_p$, not necessarily in increasing order, are the integers in the singleton color classes in any valid coloring of $[1, w_2(p; p+1) - 1]$. Computer based evidence suggests that $a_i \equiv a_j \mod p$ for $1 \leq i < j \leq p-1$ and $a_p \equiv 1 \mod p$. Our proof would have been considerably shortened if we were able to prove this.

Theorem 2.11. *For odd prime p,*

$$p(p-1)+3 \leq w_2(p; p+2) \leq p(p-1)+5.$$

Proof. For $p \in \{3,5\}$, Table 1 gives $w_2(3;5) = 10$ and $w_2(5;7) = 23$. We assume $p > 5$ for the rest of this proof. The lower bound follows directly from Theorem 2.1.

In order to prove the upper bound, assume by way of contradiction that there is a valid $(p+2)$-coloring of $[1, p(p-1)+5]$. Let the integers in the singleton color class be $S = \{a_1, \ldots, a_{p+1}\}$, arranged in increasing order. We clearly must have $a_{i+1} - a_i \leq p$ for $i \in \{1,\ldots,p\}$ and $a_{p+1} \geq p(p-1)+5-(p-1) = p(p-2)+6$. Moreover, corresponding to each $j \in \{1,\ldots,5\}$, there must correspond at least one member of S congruent to j modulo p. Let $s_i \equiv i \mod p$ for $i \in \{1,\ldots,5\}$ and $s_i \in S$.

Note that $a_{p-1} \geq p(p-1)+1$ implies at most $p-2$ of the a_is in the interval $[1, p(p-1)]$. This is impossible since some interval of the form $[ip+1, (i+1)p]$, $i \in \{0, 1, \ldots, p-2\}$, would contain no element of S, thus contradicting the valid coloring. Suppose $a_p \geq p(p-1)+1$. This provides a valid $(p-1)$-coloring of the interval $[1, p(p-1)]$. From Remark 2.4, $a_i \equiv a_j \mod p$ for $1 \leq i < j \leq p-1$. However, since a_p and a_{p+1} are members of only two of the five p-term arithmetic progressions with the common difference p and starting with $1, 2, \ldots, 5$, there must be at least one p-term arithmetic progression with common differences p all of the same color. Hence, $a_p \leq p(p-1)$.

We claim that no interval of the form $[ip+1, (i+1)p$, $i \in \{0, 1, \ldots, p-2\}$ can contain three consecutive terms a_{j-1}, a_j, a_{j+1} of S. By way of contradiction, suppose some such interval contains at least three consecutive terms of S. If $a_{p+1} \geq p(p-1)+1$, some interval of the form $[ip+1, (i+1)p]$, $i \in \{0, 1, \ldots, p-2\}$, would contain no element of S, thus contradicting the valid coloring. Therefore, $a_{p+1} \leq p(p-1)$ in this case. Since exactly i elements of S belong to the interval $[1, ip]$, the three consecutive elements of S in the interval $[ip+1, (i+1)p]$ are $a_{i+1}, a_{i+2}, a_{i+3}$. Observe that, for $\ell \geq i+3$, we must have $a_{p+1} - a_\ell \leq (p-\ell+1)p$. Since $a_{p+1} \geq p(p-2)+6$, we therefore have $a_\ell \geq \big(p(p-2)+6\big) - (p-\ell+1)p = p(\ell-3)+6$. Thus, the elements of S congruent to $1, \ldots, 5$ modulo p must be one of $a_1, \ldots, a_{i+2}$. Hence, at least three among the elements of S congruent to $1, \ldots, 5$ must be in the interval $[1, ip]$. Since each of the intervals $[1, p], [p+1, 2p], \ldots, [(i-1)p+1, ip]$ contains exactly one element of S, it follows that two consecutive such intervals contain two elements of S that differ by $p-1$. However, this leads to a contradiction to the assumption of a valid coloring since these two elements of S have the same parity, leading to a p-term arithmetic progression with common difference 2 and of the same color contained within these two consecutive intervals. Hence, the claim follows.

Two possibilities remain: (i) there must be exactly two intervals of the form $[ip+1, (i+1)p]$, $i \in \{0, 1, \ldots, p-2\}$, each of which has exactly two elements of S; and (ii) exactly one interval of the form $[ip+1, (i+1)p]$, $i \in \{0, 1, \ldots, p-2\}$ contains two elements of S and $a_{p+1} \geq p(p-1)+1$. Let $s^\star$ denote the largest of the four elements of S in case (i), and let $s^\star = a_{p+1}$ in case (ii). Following the same arguments as in the previous claim, the elements of S congruent to $1, \dots, 5$ must each be less than or equal to $s^\star$.

We observe that at most two pairs of consecutive elements of S may differ by $p-1$. Having three such pairs would lead to a contradiction since the $p+1$ elements of S must be placed in the $p-1$ residue classes modulo $p-1$ and the pairs belong to the same residue class modulo $p-1$.

Our next claim is that at most two pairs among $s_1, \dots, s_5$ may differ by $p-2$. Suppose, by way of contradiction, that there are three or more such pairs. The least among these pairs must be less than p, and the largest must be greater than $p(p-2)$, for otherwise, considering one of two intervals $[1, p(p-2)]$, $[p+1, p(p-1)]$ would give a monochromatic p-term arithmetic progression with the common difference $p-2$ by the previous argument. We next observe that $a_{p+1} \leq p(p-1)$, since at least one of the intervals $[1, p(p-2)]$, $[p+1, p(p-1)]$ must contain two pairs that differ by $p-2$ and must contain at most $p-1$ elements of S. Now, following an argument similar to that in the previous paragraph, we arrive at a contradiction. Hence,

$$a_p = a_{p+1} - (p-2) \geq (p(p-2)+6) - (p-2) = p(p-3) + 8.$$

Thus, the three or more pairs that differ by $p-2$ belong to the interval $[1, p(p-2)]$, which again leads to a contradiction by the same argument as in the previous paragraph. This completes the claim that at most two pairs among $s_1, \dots, s_5$ may differ by $p-2$.

Consider the first interval of the form $[ip+1, (i+1)p]$ that contains two elements of S, viz., a_i and a_{i+1}. Let S_1 consist of the elements of S that are congruent to one of $1, \dots, 5$ modulo p and are at most a_i, and let S_2 consist of elements of S that are congruent to one of $1, \dots, 5$ modulo p and are at least a_{i+1}. Observe that neither S_1 nor S_2 can contain consecutive s_i, s_j with i, j of opposite parity except when one of s_i, s_j belongs to an interval of the form $[ip+1, (i+1)p]$ that contains two elements of S; otherwise, there would be a monochromatic p-term arithmetic progression with common difference 2. The only ways to place $s_1, \dots, s_5$ into either S_1 or S_2 and not arrive at a contradiction to this argument are the following possibilities for the set S_1:

$$\{5,4\}, \{5,3,2\}, \{5,3,1\}, \{5,3\}, \{5,2\}, \{5,1\}, \{4,3\}, \{4,2,1\}, \{4,2\}, \{3,2\}, \{3\}, \{2,1\}.$$

When $S_1 = \{5, 3, 1\}$ or $\{4, 2\}$, there are three pairs that differ by $p-2$. Let

$$a_s = \max\{a_i \in S_1 : a_i \equiv 1, 2, 3, 4 \text{ or } 5 \bmod p, a_{i-1} \not\equiv a_i \bmod p\}$$

and

$$a_t = \max\{a_i \in S_2 : a_i \equiv 1, 2, 3, 4 \text{ or } 5 \bmod p, a_{i-1} \not\equiv a_i \bmod p\}.$$

When S_1 is one of $\{5, 4\}, \{5, 3, 2\}, \{5, 2\}, \{4, 3\}, \{4, 2, 1\}, \{2, 1\}$, the interval of length $2p$ ending either at a_s+4 or at a_s+5 contains exactly two members of S, and these are of the same parity, thus giving a monochromatic p-term arithmetic progression with common difference 2. When S_1 is one of $\{5, 3\}, \{5, 1\}, \{3, 2\}, \{3\}$, the interval of length $2p$ ending either at a_t+4 or at a_t+5 contains exactly two members of S, and these are of the same parity, thus giving a monochromatic p-term arithmetic progression with common difference 2. This completes the proof. $\square$

3. Tables of $w_2(k; r)$ for $k \le r$. In this section, we provide in Tables 3 and 4 values of $w_2(k; r)$ for $k \le r$, obtained by a computer program. These greatly extend [**12,** Table 1, Corollary 5]. The entries in blue are validated by the results in Section 2. The entries in red correspond to $k = 5$ and $7 \le r \le 13$ and correct the corresponding entries in [**12,** Table 1, Corollary 5]. Tables 5 and 6 provide a valid coloring of $[1, w_2(k; r) - 1]$ in all cases where $k \le r$ is not covered by the results in Theorem 2.9 and in Corollary 2.6 (i), (ii).

4. Concluding remarks. As indicated in Section 1, an almost complete study of the mixed van der Waerden numbers $w_2(k, r)$ for the case $k > r \ge 2$ was done by Landman, Robertson and Culver [**16**], who obtained lower bounds in all cases and exact values in some. The case $k \le r$ was studied by Khodkar and Landman [**12**], who provided the upper bound $w_2(k; r) \le r(k-1)$ when $k < r < 3(k-1)/2$, and provided a computer-generated table of values of $w_2(k, r)$ for $3 \le k \le 5$ and $k < r \le 13$. The purpose of our work was to take this forward to some extent, as indicated in Section 1. The computer-assisted tables of values of $w_2(k, r)$ have not only extended (and, on some occasions, corrected previously computed values), but have also served as the motivation for all of our results and conjectures. We conclude this paper with a few conjectures based on our computer-generated tables.

TABLE 3. Values of $\mathtt{w}_2(k;r)$ for $k=3$ and $19 \le r \le 25$.

k \ r	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
3	**7**	**8**	10	12	15	16	17	18	19	21	22	23	25	27	28	29
4		**12**	14	16	18	20	22	24	26	29	31	32	35	36	38	39
5			**21**	**22**	**23**	**26**	**30**	**32**	**35**	**40**	**45**	46	47			
6				**28**	30	31	34	38	42	43	47					
7					**43**	**44**	45	47	49	54	58					
8						**52**	53	55	57							
9							59	62	66							
10								69								

TABLE 4. Values of $\mathtt{w}_2(k;r)$ for $k=3$ and $19 \le r \le 25$.

k \ r	**19**	**20**	**21**	**22**	**23**	**24**	**25**
3	31	33	34	35	37	38	39

Conjecture 4.1. *For any odd prime p such that $p+2$ is composite,*

$$w_2(p+2;p+2) = p(p+1)+3.$$

Conjecture 4.2. *For prime $p>3$,*

$$w_2(p;p+2) = p(p-1)+3.$$

Conjecture 4.3. *Let $r \ge k \ge 3$. If p is the largest prime not exceeding k, then*

$$p-1 \le w_2(k+1;r)-w_2(k;r) \le k \text{ and } p \le w_2(k+1;r+1)-w_2(k;r) \le k.$$

Conjecture 4.3 is analogous to Theorem 2.8, and, if true, would provide upper and lower bounds for $w_2(k;r)$.

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TABLE 5. Values of $[1, \mathtt{w}_2(3; r) - 1]$ for $3 \leq r \leq 25$.

r	$w_2(3; r)$	**Valid coloring of** $[1, w_2(3; r) - 1]$ **(Integers in singleton color classes)**
3	7	3, 6
4	8	3, 6, 7
5	10	3, 4, 5, 8
6	12	3, 5, 6, 7, 10
7	15	3, 6, 7, 8, 9, 12
8	16	3, 6, 7, 8, 9, 12, 15
9	17	3, 6, 7, 8, 9, 12, 15, 16
10	18	3, 6, 7, 8, 9, 12, 15, 16, 17
11	19	3, 6, 7, 8, 9, 12, 15, 16, 17, 18
12	21	3, 4, 5, 8, 10, 11, 12, 13, 16, 17, 19
13	22	3, 4, 5, 8, 10, 11, 12, 13, 16, 17, 19, 21
14	23	2, 3, 6, 7, 9, 11, 12, 13, 14, 15, 16, 19, 22
15	25	3, 4, 6, 8, 9, 10, 12, 13, 14, 15, 17, 20, 21, 22
16	27	3, 4, 6, 8, 9, 10, 12, 13, 14, 15, 17, 20, 21, 22, 25
17	28	2, 5, 6, 7, 10, 12, 13, 14, 15, 17, 18, 19, 21, 23, 24, 27
18	29	1, 3, 6, 7, 8, 11, 13, 14, 15, 16, 18, 19, 20, 22, 24, 25, 28
19	31	2, 5, 6, 7, 10, 12, 13, 14, 15, 16, 17, 18, 19, 21, 24, 25, 26, 29
20	33	3, 4, 6, 8, 9, 12, 13, 15, 16, 17, 18, 19, 20, 21, 23, 26, 27, 28, 30
21	34	1, 4, 5, 7, 9, 10, 13, 14, 16, 17, 18, 19, 20, 21, 22, 24, 27, 28, 29, 31
22	35	1, 2, 5, 7, 8, 9, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, 26, 27, 29, 31, 32
23	37	3, 4, 6, 8, 9, 12, 13, 15, 17, 18, 19, 20, 21, 22, 23, 25, 26, 27, 30, 31, 32, 34
24	38	1, 4, 5, 7, 9, 10, 13, 14, 16, 18, 19, 20, 21, 22, 23, 24, 26, 27, 28, 31, 32, 33, 35
25	39	3, 5, 6, 7, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 24, 25, 28, 29, 31, 33, 34, 37, 38

TABLE 6. Values of $[1, w_2(k;r)-1]$ for $4 \le k \le 10$.

k	**r**	$\mathtt{w}_2(k;r)$	**Valid coloring of** $[1, \mathtt{w}_2(k;r)-1]$ **(Integers in singleton color classes)**
4	4	12	4, 7, 11
4	5	14	4, 5, 9, 12
4	6	16	3, 6, 10, 11, 13
4	7	18	4, 7, 9, 10, 11, 15
4	8	20	4, 7, 11, 12, 13, 14, 18
4	9	22	4, 7, 9, 10, 11, 15, 18, 20
4	10	24	3, 7, 8, 9, 11, 14, 15, 19, 22
4	11	26	4, 7, 9, 10, 11, 15, 18, 20, 21, 22
4	12	29	4, 7, 9, 10, 11, 15, 18, 20, 21, 22, 26
4	13	31	2, 5, 9, 10, 11, 13, 16, 20, 21, 22, 24, 27
4	14	32	4, 7, 9, 10, 11, 14, 18, 19, 20, 21, 22, 26, 29
4	15	35	4, 7, 11, 12, 13, 14, 15, 18, 22, 23, 24, 25, 28, 31
4	16	36	4, 7, 10, 11, 12, 13, 17, 20, 21, 22, 23, 24, 28, 31, 35
4	17	38	4, 7, 11, 12, 13, 14, 15, 18, 22, 23, 24, 25, 28, 31, 35, 36
4	18	39	4, 7, 11, 12, 13, 14, 15, 18, 22, 23, 24, 25, 28, 31, 35, 36, 37
5	5	21	5, 10, 15, 20
5	6	22	5, 10, 15, 20, 21
5	7	23	5, 10, 15, 20, 21, 22
5	8	26	5, 10, 15, 16, 17, 19, 23
5	9	30	5, 10, 15, 16, 17, 19, 23, 26
5	10	32	5, 10, 15, 16, 17, 19, 23, 26, 30
5	11	35	5, 10, 15, 20, 21, 22, 23, 24, 25, 30
5	12	40	5, 10, 15, 20, 21, 22, 23, 24, 25, 30, 35
5	13	45	5, 10, 15, 20, 21, 22, 23, 24, 25, 30, 35, 40
5	14	46	5, 10, 15, 20, 21, 22, 23, 24, 25, 30, 35, 40, 41
5	15	47	5, 10, 15, 20, 21, 22, 23, 24, 25, 30, 35, 40, 41, 42
6	6	28	6, 11, 16, 21, 27
6	7	30	6, 11, 16, 17, 23, 24
6	8	31	6, 11, 16, 17, 23, 24, 30
6	9	34	6, 7, 13, 14, 20, 21, 26, 31
6	10	38	6, 9, 15, 16, 17, 23, 26, 31, 33
6	11	42	4, 9, 14, 20, 21, 22, 23, 26, 31, 36
6	12	43	4, 9, 14, 20, 21, 22, 23, 26, 31, 36, 37
6	13	47	6, 11, 16, 18, 19, 20, 21, 22, 28, 33, 38, 43
7	7	43	7, 14, 21, 28, 35, 42
7	8	44	7, 14, 21, 28, 35, 42, 43
7	9	45	7, 14, 21, 28, 35, 42, 43, 44
7	10	47	7, 14, 18, 21, 22, 24, 30, 35, 40
7	11	49	7, 14, 21, 28, 30, 31, 32, 36, 41, 47
7	12	54	7, 14, 21, 25, 27, 28, 29, 31, 37, 42, 47
7	13	58	7, 14, 21, 24, 25, 29, 33, 34, 35, 37, 44, 51
8	8	52	8, 15, 22, 29, 36, 43, 51
8	9	53	8, 15, 22, 29, 36, 43, 51, 52
8	10	55	8, 9, 17, 18, 19, 26, 33, 40, 47
8	11	57	8, 13, 17, 18, 26, 31, 39, 40, 44, 49
9	9	59	8, 15, 22, 29, 36, 43, 50, 58
9	10	62	9, 18, 19, 26, 27, 36, 45, 46, 53
9	11	66	7, 13, 18, 23, 32, 39, 40, 41, 50, 59
10	10	69	10, 19, 24, 29, 38, 46, 47, 51, 60

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