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NONLINEAR SPECTRAL RADIUS PRESERVERS BETWEEN CERTAIN NON-UNITAL BANACH FUNCTION ALGEBRAS

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ABSTRACT. Let $\alpha_0 \in \mathbb{C} \setminus \{0\}$, A and B be Banach function algebras. Also, let $\rho_1 : \Omega_1 \to A$, $\rho_2 : \Omega_2 \to A$, $\tau_1 : \Omega_1 \to B$ and $\tau_2 : \Omega_2 \to B$ be surjections such that $\|\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2) + \alpha_0\|_\infty$ for all $\omega_1 \in \Omega_1, \omega_2 \in \Omega_2$, where Ω_1, Ω_2 are two non-empty sets. Motivated by recent investigations on such maps between unital Banach function algebras, in this paper we characterize these maps for certain non-unital Banach function algebras including pointed Lipschitz algebras and abstract Segal algebras of the Figà-Talamanca-Herz algebras when the underlying groups are first countable. Moreover, sufficient conditions are given to guarantee such maps induce weighted composition operators.

1. Introduction. The study of characterizing maps between Banach function algebras which preserve some specific properties connected with the norm, range, spectrum, and spectral-radius of algebra elements is an active research area. One important result concerning norm-preserving maps is the famous Banach-Stone theorem which gives a complete description of norm-preserving linear maps between $C(X)$-spaces, where $C(X)$ is the Banach algebra of all continuous complex-valued functions on a compact Hausdorff space X endowed with the supremum norm $\|\cdot\|_\infty$. This theorem has been extended in many directions. In particular, during recent years, several authors have worked on different preservers between function algebras without the linearity assumption. We refer to **[7]** for a survey of the study in this area. Specifically, a great deal of work has been done on maps preserving

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certain norm conditions which make them weighted composition operators or algebra isomorphisms. In this relation, the authors in **[9, 21]** independently studied non-symmetric multiplicatively norm-preserving maps between uniform algebras; more precisely, they characterized surjective maps $T : A \to B$ satisfying $\|TfTg + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in A$ and some fixed scalar $\alpha_0 \in \mathbb{C} \setminus \{0\}$. In **[12]**, surjections $T : C(X) \to C(X)$ were studied such that, for each $f, g \in C(X)$, $\|Tf\overline{Tg} - 1\|_\infty = \|f\overline{g} - 1\|_\infty$, and it is shown that, if $T(\pm 1) = \pm 1$ and $T(\pm i) = \pm i$, then T must be an algebra isomorphism. In **[1]**, the authors obtained a complete description of similar maps for pointed Lipschitz algebras. In order to unify these results, we put them into a general framework as follows. Given two non-empty sets Ω_1, Ω_2 and function algebras A, B, it is natural to consider surjective maps

$$\rho_1 : \Omega_1 \longrightarrow A, \qquad \rho_2 : \Omega_2 \longrightarrow A, \qquad \tau_1 : \Omega_1 \longrightarrow B, \qquad \tau_2 : \Omega_2 \longrightarrow B$$

satisfying

$$\|\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2) + \alpha_0\|_\infty$$

for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, where α_0 is a fixed non-zero scalar. Most recently, Hirasawa, Miura and Takagi **[11]** have considered $\Omega := \Omega_1 = \Omega_2$ and analyzed two pairs of surjections

$$S_1, S_2 : \Omega \longrightarrow A \quad \text{and} \quad T_1, T_2 : \Omega \longrightarrow B$$

between unital Banach function algebras A and B. Some related results (involving pairs of maps) between either unital Banach function algebras or particular subsets of the group of invertible elements of the algebras have also been obtained in **[6, 8, 11, 14, 15, 16, 23, 27, 28]**. Moreover, it should be noted that their proofs rely on the existence of unit elements, and they cannot be adopted for non-unital case.

As observed, while there is extensive study on characterizing the above maps defined between unital Banach function algebras, there are far fewer results on non-unital Banach function algebras (only **[1]**). In this paper, we provide a complete description of such norm preserving maps defined between certain non-unital Banach function algebras, including pointed Lipschitz algebras and abstract Segal algebras of the Figà-Talamanca-Herz algebras on first countable, noncompact locally compact groups (also see Remark 3.15 (1)). We first show that such maps induce surjective maps

$$T_1, T_2 : A \longrightarrow B$$

such that

$$\|T_1 f T_2 g + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$$

for all $f, g \in A$, and then, with an approach similar to [**1**], characterize them to obtain the form of original maps. In particular, an extension of [**1,** Theorem 4.8] is derived. We also give sufficient conditions for maps to be weighted composition operators.

2. Preliminaries. Throughout this paper, all topological spaces are Hausdorff. Let X be a locally compact space, and let $C_0(X)$ denote the algebra of all continuous complex-valued functions on X vanishing at infinity, equipped with the supremum norm $\|\cdot\|_\infty$. A subalgebra A of $C_0(X)$ is called a *function algebra* on X if A strongly separates the points of X in the sense that, for each $x, x' \in X$ with $x \neq x'$, there exists an $f \in A$ with $f(x) \neq f(x')$ and $f(x) \neq 0$. A *Banach function algebra* on X is a function algebra on X which is a Banach algebra with respect to a norm. A uniformly closed function algebra A on X is called a *uniform algebra* if X is compact and A contains the constant functions.

Let A be a Banach function algebra on a locally compact space X. For any $f \in A$, we denote the *maximizing set* of f by $M_f := \{x \in X : |f(x)| = \|f\|_\infty\}$. A point $x \in X$ is called a *strong boundary point* for A if, for each neighborhood V of x, there exists a function $f \in A$ with $\|f\|_\infty = f(x) = 1$ and $M_f \subseteq V$.

Given any f in a Banach function algebra A on a locally compact space X, the *peripheral range* and *peripheral spectrum* of f are defined, respectively, by

$$R_\pi(f) = \{z \in f(X) : |z| = \|f\|_\infty\}$$

and

$$\sigma_\pi(f) = \{z \in \sigma(f) : |z| = r(f)\},$$

where $\sigma(f)$ and $r(f)$ denote the spectrum and the spectral radius of f. It should be noted that, if A is a uniformly closed function algebra, then $R_\pi(f) = \sigma_\pi(f)$ for all $f \in A$ [**22**]. A function $h \in A$ is said to be a *peaking function* provided $\|h\|_\infty = 1$ and $|h(x)| < 1$ whenever $h(x) \neq 1$, i.e., $R_\pi(h) = \{1\}$. The set of all peaking functions in A is denoted by $\mathcal{P}(A)$.

Now, we briefly give preliminaries based on the classes of Banach function algebras which will be considered later in the paper.

Let $1 < p < \infty$ and q be the conjugate to p, i.e., $1/p+1/q = 1$. Given a locally compact group G with the left Haar measure λ, the space $A_p(G)$ consists of all functions $f \in C_0(G)$ which can be represented as

$$f = \sum_{i=1}^{\infty} f_i * \check{g}_i$$

for some $f_i \in L^p(G)$ and $g_i \in L^q(G)$ with

$$\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty,$$

where $\check{g}_i(x) = g_i(x^{-1})$ and

$$(f_i * \check{g}_i)(x) = \int_G f_i(xy) g_i(y)\, d\lambda(y)$$

for all $x \in G$. The norm of $f \in A_p(G)$ is defined by

$$\|f\| = \inf \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q,$$

where the infimum runs over all possible representations of f from the above. A Banach algebra $A_p(G)$ is called the *Figà-Talamanca-Herz algebra*. For $p = 2$, $A_2(G)$ is known as the *Fourier algebra* of G, introduced by Eymard [**4**]. If G is abelian with dual group $\widehat{G}$, $A_2(G)$ is the set of Fourier transforms of all functions in $L^1(\widehat{G})$. From [**10**], $A_p(G)$ is indeed a regular Banach function algebra with the maximal ideal space G.

Another class of Banach function algebras that we shall consider is the class of *abstract Segal algebras* with respect to the Figà-Talamanca-Herz-Lebesgue algebras. Let G be a locally compact group and $1 < p < \infty$. Following [**2**], a subalgebra of $A_p(G)$ is an abstract Segal algebra of $A_p(G)$ if A is a dense ideal in $A_p(G)$, A is a Banach algebra with a norm finer than $A_p(G)$-norm and A is also a Banach $A_p(G)$-

module. We call an abstract Segal algebra of $A_p(G)$ *proper* if it is not equal to $A_p(G)$. The reader may refer to **[2, 3]** for the basic properties of abstract Segal algebras.

The Figà-Talamanca-Herz-Lebesgue algebra is an abstract Segal algebra of $A_p(G)$. We recall from **[5]** that the Figà-Talamanca-Herz-Lebesgue algebra $A_p^r(G)$ of a locally compact group G is $A_p^r(G) = L^r(G) \cap A_p(G)$, where $1 \leq r < \infty$, and the norm $\|f\|$ of $f \in A_p^r(G)$ is the summation $L^r(G)$-norm and $A_p(G)$-norm of f defined above. We note that the first paper dealing with such algebras is **[19]**, where the authors studied $A_2^r(G)$ for the abelian group G. According to **[5,** Theorem 1], $A_p^r(G)$ is an abstract Segal algebra of $A_p(G)$. It is worthwhile pointing out that, for any noncompact locally compact group G, the non-unital Banach function algebras $A_p^r(G)$ and $A_p(G)$ are different **[5]**.

Let G be a locally compact group. For $f \in L^\infty(G)$, the left translation of f by $x \in G$ is denoted by $L_x f(y) = f(xy)$. G is called *amenable* if there exists a continuous linear functional $m \in L^\infty(G)^*$ such that $\|m\| = m(1) = 1$ and $m(L_x f) = m(f)$ for each $x \in G$, $f \in L^\infty(G)$. Examples of amenable groups include abelian and compact groups.

Finally, we recall that a map $f : X \to Y$ between metric spaces is said to be *Lipschitz* if

$$L(f) := \sup_{x \neq x'} \frac{d(f(x), f(x'))}{d(x, x')} < \infty.$$

3. Results. In this paper, we characterize the above-mentioned norm-preserving maps between several non-unital function algebras (with the same proof). For this reason, we first state different contexts for work as follows:

Context 1. Let X be a first countable locally compact space and $A = C_0(X)$.

Context 2. Let X_0 be a pointed compact metric space with the base point e_{X_0}, and let $\mathrm{Lip}_0(X_0)$ be the pointed Lipschitz algebra consisting of all complex-valued Lipschitz functions f on X_0 such that $f(e_{X_0}) = 0$. Note that $\mathrm{Lip}_0(X_0)$ is a Banach algebra with respect to the norm $\|\cdot\| = \|\cdot\|_\infty + L(\cdot)$. In this case, let $X = X_0 \setminus \{e_{X_0}\}$, and let A be the algebra consisting of the restriction of all functions in $\mathrm{Lip}_0(X_0)$ to X, which is a Banach function algebra with maximal ideal space X.

Context 3. Let X be a first countable locally compact group and $A = A_p(X)$, $1 < p < \infty$, the Figà-Talamanca-Herz algebra of X.

Context 4. Let X be a first countable locally compact group and A a proper abstract Segal algebra of $A_p(X)$, $1 < p < \infty$.

From now on, we will assume that we are in one of the above four contexts. This means that, when we refer to function algebras A and B on X and Y, respectively, we suppose that both A and B belong at the same time to one of the above-mentioned contexts. We note that, in all of the above algebras, the spectral-radius of $f \in A$ is the same $\|\cdot\|_\infty$ and $R_\pi(f) = \sigma_\pi(f)$ (see [**2,** Theorem 2.1] for Context 4).

In the sequel, let $\alpha_0 \in \mathbb{C} \setminus \{0\}$, Ω_1, Ω_2 be non-empty sets. Also, let $\rho_1 : \Omega_1 \to A$, $\rho_2 : \Omega_2 \to A$, $\tau_1 : \Omega_1 \to B$ and $\tau_2 : \Omega_2 \to B$ be surjections satisfying

$$\|\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2) + \alpha_0\|_\infty, \quad \omega_1 \in \Omega_1,\ \omega_2 \in \Omega_2.$$

Our aim is to characterize such maps through the next lemmas. Before presenting them, we fix the following notation.

For each $x \in X$, let $\mathcal{P}_x := \{h \in \mathcal{P}(A) : h \geq 0, M_h = \{x\}\}$. We apply the same notation for similar subsets $\mathcal{P}_y$, $y \in Y$.

Now, we state a key result which is essentially a multiplicative version of Bishop's classical lemma for the above algebras (see [**7**] for more details).

Lemma 3.1. *If $f \in A$ and $x \in X$ with $f(x) \neq 0$, then there exists a peaking function $h \in \mathcal{P}_x$ such that $M_h = M_{fh} = \{x\}$. In particular, $R_\pi(fh) = \{f(x)\}$.*

Proof. Let $f \in A$, $x \in X$ and $f(x) \neq 0$. If we show that there exists a peaking function $h \in A$ such that $M_h = M_{fh} = \{x\}$, then the function $h\overline{h}$ satisfies the desired properties, where $\bar{\cdot}$ denotes the complex conjugation. Thus, for Contexts 1 and 2, 3 is obtained by applying [**20,** Corollary 1.1], [**18,** Lemma 2.1] and [**13,** Lemma 5.1], respectively.

In Context 4, according to [**13,** Lemma 5.1], there is a peaking function k in the Figà-Talamanca-Herz algebra $A_p(X)$ such that $M_k = M_{fk} = \{x\}$. On the other hand, from the regularity of $A_p(X)$, we can choose a function g in $A_p(X) \cap C_c(X)$ such that $0 \leq g \leq 1$ and $g(x) = 1$, where $C_c(X)$ denotes the dense subspace of $C_0(X)$ consisting of functions with compact support. Then, $h = gk\overline{k}$ will be a peaking function in $A_p(X) \cap C_c(X)$. Since, by [**26,** Proposition 2.1.14], A contains all functions with compact support in $A_p(X)$, we conclude that $h \in A$. Therefore, apparently, $M_h = M_{fh} = \{x\}$ and $h \in \mathcal{P}_x$, as required. $\square$

We remark that, from Lemma 3.1, it easily follows that each point in X is a strong boundary point for A.

In the following lemma, which is a generalization of [**1,** Lemma 4.1], it is shown that the above maps are jointly norm-multiplicative.

Lemma 3.2. $\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty = \|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty$ *for all* $\omega_1 \in \Omega_1$ *and* $\omega_2 \in \Omega_2$.

Proof. We show the result by applying the method which is used in the proof of [**1,** Lemma 4.1]. First, we prove the following claim.

Claim 3.3. *Given* $t > 0$ *and* $\omega, \omega' \in \Omega_1$ *with* $\rho_1(\omega') = t\rho_1(\omega)$, *we have* $|\tau_1(\omega')(y)| = t|\tau_1(\omega)(y)|$ *for all* $y \in Y$.

Let $y \in Y$. We have two cases to consider. Suppose first that $\tau_1(\omega)(y) \neq 0$. Then, we can choose a function $h \in \mathcal{P}_y$ such that $R_\pi(\tau_1(\omega)h) = \{\tau_1(\omega)(y)\}$, by Lemma 3.1. For each $n \in \mathbb{N}$, let $h_n = nh$. Thus, if $\gamma_n \in \Omega_2$ with $h_n = \tau_2(\gamma_n)$, we have:

$$\begin{aligned} n|\tau_1(\omega')(y)| - |\alpha_0| &= |\tau_1(\omega')(y)h_n(y)| - |\alpha_0| \leq \|\tau_1(\omega')h_n + \alpha_0\|_\infty \\ &= \|\tau_1(\omega')\tau_2(\gamma_n) + \alpha_0\|_\infty = \|\rho_1(\omega')\rho_2(\gamma_n) + \alpha_0\|_\infty \\ &= \|t\rho_1(\omega)\rho_2(\gamma_n) + \alpha_0\|_\infty \\ &\leq t\|\rho_1(\omega)\rho_2(\gamma_n) - \alpha_0 + \alpha_0\|_\infty + |\alpha_0| \\ &\leq t\|\rho_1(\omega)\rho_2(\gamma_n) + \alpha_0\|_\infty + t|\alpha_0| + |\alpha_0| \\ &= t\|\tau_1(\omega)\tau_2(\gamma_n) + \alpha_0\|_\infty + t|\alpha_0| + |\alpha_0| \\ &= t\|\tau_1(\omega)h_n + \alpha_0\|_\infty + t|\alpha_0| + |\alpha_0| \\ &\leq t(n|\tau_1(\omega)(y)| + |\alpha_0|) + t|\alpha_0| + |\alpha_0|. \end{aligned}$$

Hence, $|\tau_1(\omega')(y)| \leq t|\tau_1(\omega)(y)| + (2t+2)|\alpha_0|/n$ and, when $n \to \infty$, it follows that $|\tau_1(\omega')(y)| \leq t|\tau_1(\omega)(y)|$. Let us assume that we have the second case, that is, $\tau_1(\omega)(y) = 0$. Since $\tau_1(\omega)$ is continuous, then, given $n \in \mathbb{N}$, there is a neighborhood V_n of y such that $|\tau_1(\omega)| < 1/n$ on V_n. On the other hand, for the strong boundary point y in V_n, we choose a function $k_n \in B$ with $k_n(y) = 1 = \|k_n\|_\infty$ and

$$|k_n| < \frac{1}{n(\|\tau_1(\omega)\|_\infty + 1)} \quad \text{on } Y \setminus V_n.$$

Now, take $h_n = nk_n$ and $\gamma_n \in \Omega_2$ such that $h_n = \tau_2(\gamma_n)$. Thus, in a similar manner,

$$\begin{aligned} n|\tau_1(\omega')(y)| - |\alpha_0| &= |\tau_1(\omega')(y)h_n(y)| - |\alpha_0| \leq \|\tau_1(\omega')h_n + \alpha_0\|_\infty \\ &= \|\tau_1(\omega')\tau_2(\gamma_n) + \alpha_0\|_\infty = \|\rho_1(\omega')\rho_2(\gamma_n) + \alpha_0\|_\infty \\ &= \|t\rho_1(\omega)\rho_2(\gamma_n) + \alpha_0\|_\infty \\ &\leq t\|\rho_1(\omega)\rho_2(\gamma_n) + \alpha_0\|_\infty + t|\alpha_0| + |\alpha_0| \\ &= t\|\tau_1(\omega)\tau_2(\gamma_n) + \alpha_0\|_\infty + t|\alpha_0| + |\alpha_0| \\ &\leq t\|\tau_1(\omega)h_n\|_\infty + (2t+1)|\alpha_0| \\ &\leq t + (2t+1)|\alpha_0|, \end{aligned}$$

since $\|\tau_1(\omega)h_n\|_\infty \leq 1$. Then, $|\tau_1(\omega')(y)| \leq (t + (2t+2)|\alpha_0|)/n$, and, letting $n \to \infty$, we see that $\tau_1(\omega')(y) = 0 = \tau_1(\omega)(y)$.

In both cases, we have $|\tau_1(\omega')(y)| \leq t|\tau_1(\omega)(y)|$. Moreover, since $\rho_1(\omega) = (1/t)\rho_1(\omega')$, from a similar discussion, it follows that $|\tau_1(\omega)(y)| \leq (1/t)|\tau_1(\omega')(y)|$. Then, we derive that $|\tau_1(\omega')(y)| = t|\tau_1(\omega)(y)|$, as claimed.

We now turn to showing that, for every pair $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$,

$$\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty.$$

Assume that $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. For each $n \in \mathbb{N}$, let δ_n be such that $\rho_1(\delta_n) = n\rho_1(\omega_1)$. From Claim 3.3, we conclude that $|\tau_1(\delta_n)| = n|\tau_1(\omega_1)|$, and thus,

$$\|\tau_1(\delta_n)\tau_2(\omega_2)\|_\infty = n\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty,$$

which implies

$$\begin{aligned} n\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty - |\alpha_0| &\leq \|n\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty \\ &= \|\rho_1(\delta_n)\rho_2(\omega_2) + \alpha_0\|_\infty \\ &= \|\tau_1(\delta_n)\tau_2(\omega_2) + \alpha_0\|_\infty \\ &\leq n\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty + |\alpha_0|. \end{aligned}$$

Thus, for all $n \in \mathbb{N}$, $\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty \leq \|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty + 2|\alpha_0|/n$, and therefore, $\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty \leq \|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty$. Analogously, for all $n \in \mathbb{N}$,

$$\begin{aligned} n\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty - |\alpha_0| &= \|\tau_1(\delta_n)\tau_2(\omega_2)\|_\infty - |\alpha_0| \\ &\leq \|\rho_1(\delta_n)\rho_2(\omega_2) + \alpha_0\|_\infty \\ &= \|n\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty \\ &\leq n\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty + |\alpha_0|, \end{aligned}$$

and hence, $\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty \leq \|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty + 2|\alpha_0|/n$, which verifies that $\|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty \leq \|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty$. Consequently,

$$\|\rho_1(\omega_1)\rho_2(\omega_2)\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2)\|_\infty. \qquad \square$$

According to the next lemma, jointly norm-multiplicative maps T_1 and T_2 induce a homeomorphism between the underlying spaces X and Y (compare with [**1,** Theorem 3.2], [**17,** Section 3] and [**20,** page 114]).

Lemma 3.4. *There exists a homeomorphism $\psi : X \to Y$ such that, for every $\omega_1 \in \Omega_1$, $\omega_2 \in \Omega_2$ and $x \in X$, $|(\rho_1(\omega_1)\rho_2(\omega_2))(x)| = |(\tau_1(\omega_1)\tau_2(\omega_2))(\psi(x))|$.*

Proof. Using Lemma 3.1, we obtain the result by the same arguments as in [**13,** Theorem 3.10]. $\square$

Set $\varphi := \psi^{-1}$. Clearly, φ is a homeomorphism from Y onto X.

Next, we give a straightforward result from [**1**] which will facilitate the reading of the proofs.

Lemma 3.5 ([**1,** Lemma 4.2])**.** *Let $\alpha, \beta \in \mathbb{C}$.*

(i) *If $|\alpha - 1| = |\beta| + 1$ and $|\alpha| = |\beta|$, then $\alpha = -|\beta|$.*

(ii) *If $|\alpha| = 1$ and $0 \leq t \leq 1$, then $|2\alpha t - 1| \leq |2\alpha - 1|$.*

(iii) *If $|\alpha| = |\beta|$, $|\beta - 1| \leq |\alpha - 1|$ and $|\beta + 1| \leq |\alpha + 1|$, then $\alpha = \beta$ or $\alpha = \overline{\beta}$.*

Lemma 3.6. *There are surjective maps $T_1, T_2 : A \to B$ such that $\|T_1 f T_2 g + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in A$.*

Proof. For $j = 1, 2$, define $T_j : A \to B$ by $T_j(f) = \tau_j(\omega_j)$, where ω_j is an element in Ω_j with $\rho_j(\omega_j) = f$. We first show that T_1 (similarly T_2) is well-defined. In order to see this, suppose that $f \in A$ and $\omega, \omega' \in \Omega_1$ with $\rho_1(\omega) = \rho_1(\omega') = f$. According to the assumption, for each $\omega_2 \in \Omega_2$, we have

$$\begin{aligned} \|\tau_1(\omega)\tau_2(\omega_2) + \alpha_0\|_\infty &= \|\rho_1(\omega)\rho_2(\omega_2) + \alpha_0\|_\infty \\ &= \|\rho_1(\omega')\rho_2(\omega_2) + \alpha_0\|_\infty \\ &= \|\tau_1(\omega')\tau_2(\omega_2) + \alpha_0\|_\infty. \end{aligned}$$

Consequently, for each $F \in B$, the following holds

$$\|\tau_1(\omega)F + \alpha_0\|_\infty = \|\tau_1(\omega')F + \alpha_0\|_\infty. \tag{3.1}$$

Note that, from the claim proved in the proof of Lemma 3.2 (or using Lemmas 3.1 and 3.4), it is easy to see that $|\tau_1(\omega)| = |\tau_1(\omega')|$. Then, given $y \in Y$, $\tau_1(\omega)(y) = 0$ if and only if $\tau_1(\omega')(y) = 0$. Now, assume that $y \in Y$ and $\tau_1(\omega)(y) \neq 0$ (and thus, $\tau_1(\omega')(y) \neq 0$). Choose peaking functions $h_1, h_2 \in \mathcal{P}_y$ such that $M_{h_1} = M_{h_1\tau_1(\omega)} = \{y\}$ and $M_{h_2} = M_{h_2\tau_1(\omega')} = \{y\}$, by Lemma 3.1. Now, taking $h = h_1h_2$, we conclude that $h \in \mathcal{P}_y$ and $M_h = M_{h\tau_1(\omega)} = M_{h\tau_1(\omega')} = \{y\}$. Then, from (3.1), it follows that

$$2|\alpha_0| = \left\|\alpha_0\tau_1(\omega)\frac{h}{\tau_1(\omega)(y)} + \alpha_0\right\|_\infty = \left\|\alpha_0\tau_1(\omega')\frac{h}{\tau_1(\omega)(y)} + \alpha_0\right\|_\infty.$$

Thus, $\|\alpha_0\tau_1(\omega')h/(\tau_1(\omega)(y)) + \alpha_0\|_\infty = 2|\alpha_0|$. On the other hand, as $M_{h\tau_1(\omega')} = \{y\}$, for each $y' \neq y$, we have

$$\begin{aligned} |\alpha_0|\left|\tau_1(\omega')(y')\frac{h(y')}{\tau_1(\omega)(y)} + 1\right| &< |\alpha_0|\left(\frac{\|\tau_1(\omega')h\|_\infty}{|\tau_1(\omega)(y)|} + 1\right) \\ &= |\alpha_0|\left(\frac{|\tau_1(\omega')(y)|}{|\tau_1(\omega)(y)|} + 1\right) = 2|\alpha_0|. \end{aligned}$$

Therefore, we conclude that

$$\left|\frac{\tau_1(\omega')(y)}{\tau_1(\omega)(y)} + 1\right| = 2,$$

and, since

$$\frac{|\tau_1(\omega')(y)|}{|\tau_1(\omega)(y)|} = 1,$$

then, from Lemma 3.5 (i), it is deduced that $\tau_1(\omega)(y) = \tau_1(\omega')(y)$. Therefore, $\tau_1(\omega) = \tau_1(\omega')$, which demonstrates that T_1 is well defined.

Furthermore, it is easily seen that $T_1, T_2 : A \to B$ are surjections such that $\|T_1 f T_2 g + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in A$. □

Remark 3.7. Let $\beta_0 \in \mathbb{C}$ with $\beta_0^2 = -\alpha_0$, and define $T_j' f = \beta_0^{-1} T_j(\beta_0 f)$ for all $f \in A$ and $j = 1, 2$. Clearly, $T_1', T_2' : A \to B$ are surjections satisfying $\|T_1'(f)T_1'(g) - 1\|_\infty = \|fg - 1\|_\infty$ for all $f, g \in A$. Hence, it is sufficient to consider the case where $\alpha_0 = -1$, and, in the next lemmas, we suppose that T_1 and T_2 are as in Lemma 3.6 with $\alpha_0 = -1$.

The following lemmas give some properties of T_1 and T_2.

Lemma 3.8. *If $x \in X$, $\alpha \in \mathbb{T}$ and $h, h' \in \mathcal{P}_x$, then $T_1(\alpha h)(\psi(x))T_2(-\overline{\alpha}h')(\psi(x)) = -1$. In particular, $T_j(\alpha h)(\psi(x)) = T_j(\alpha h')(\psi(x))$ for $j = 1, 2$.*

Proof. Let $x \in X$, $\alpha \in \mathbb{T}$ and $h, h' \in \mathcal{P}_x$. By assumption, it is obvious that

$$\|T_1(\alpha h)T_2(-\overline{\alpha}h') - 1\|_\infty = \| - hh' - 1\|_\infty = 2.$$

Then, there exists an $x' \in X$ such that $|(T_1(\alpha h)T_2(-\overline{\alpha}h'))(\psi(x')) - 1| = 2$. Hence, by Lemma 3.4,

$$2 \le |(T_1(\alpha h)T_2(-\overline{\alpha}h'))(\psi(x'))| + 1 = |(hh')(x')| + 1,$$

which implies that $x = x'$ since $h, h' \in \mathcal{P}_x$. Thus, $|(T_1(\alpha h)T_2(-\overline{\alpha}h'))(\psi(x)) - 1| = 2$.

Next, taking into account $|(T_1(\alpha h)T_2(-\overline{\alpha}h'))(\psi(x))| = 1$, from Lemma 3.5 (i), we get $(T_1(\alpha h)T_2(-\overline{\alpha}h'))(\psi(x)) = -1$, as asserted. Now, the specific case immediately follows. □

Lemma 3.9. *Given $x \in X$ and $f = \alpha h$ for some $\alpha \in \{\pm 1, \pm i\}$ and $h \in \mathcal{P}_x$, we have*

$$T_j(2f)(\psi(x))T_{j'}(\overline{f})(\psi(x)) = 2, \quad \{j, j'\} = \{1, 2\}.$$

Proof. From Lemma 3.4, $|T_j(2f)(\psi(x))T_{j'}(\overline{f})(\psi(x))| = 2$, and, consequently, by assumption, we get

$$\begin{aligned} 1 &\leq |T_j(2f)(\psi(x))T_{j'}(\overline{f})(\psi(x)) - 1| \\ &\leq \|T_j(2f)T_{j'}(\overline{f}) - 1\|_\infty \\ &= \|2|f|^2 - 1\|_\infty = 1. \end{aligned}$$

Now, Lemma 3.5 (i) implies that $T_j(2f)(\psi(x))T_{j'}(\overline{f})(\psi(x)) = 2$. □

Lemma 3.10. *Given $x \in X$ and $h \in \mathcal{P}_x$, the following cases hold*:

(a) $T_j(2h)(\psi(x)) = -2T_j(-h)(\psi(x))$;

(b) $T_j(-2h)(\psi(x)) = -T_j(2h)(\psi(x))$;

(c) $T_j(2h)(\psi(x)) = 2T_j(h)(\psi(x))$;

(d) $T_j(-h)(\psi(x)) = -T_j(h)(\psi(x))$;

(e) $T_j(2ih)(\psi(x)) = 2T_j(ih)(\psi(x))$,

for $j = 1, 2$.

Proof. Since the conditions are the same for T_1 and T_2, it is enough to prove only the above facts for $j = 1$.

(a) From Lemmas 3.8 and 3.9, we conclude that

$$T_1(2h)(\psi(x)) = \frac{2}{T_2(h)(\psi(x))} = -2T_1(-h)(\psi(x)).$$

Therefore, $T_1(2h)(\psi(x)) = -2T_1(-h)(\psi(x))$.

(b) By assumption,

$$\|T_1(-2h)T_2(h) - 1\|_\infty = \| - 2h^2 - 1\|_\infty = 3,$$

and also, by Lemma 3.4,

$$|T_1(-2h)(\psi(x'))T_2(h)(\psi(x')) - 1| < 3, \quad x' \neq x,$$

which yield $|T_1(-2h)(\psi(x))T_2(h)(\psi(x)) - 1| = 3$. Then, since $|T_1(-2h)(\psi(x))T_2(h)(\psi(x))| = 2$, by Lemma 3.5 (i), we infer that $T_1(-2h)(\psi(x))T_2(h)(\psi(x)) = -2$. Next, from Lemma 3.9, we have

$$-2 = T_1(-2h)(\psi(x))T_2(h)(\psi(x)) = T_1(-2h)(\psi(x)) \frac{2}{T_1(2h)(\psi(x))},$$

which verifies $T_1(-2h)(\psi(x)) = -T_1(2h)(\psi(x))$.

(c) Similarly, the following relations

$$\|T_1(2h)T_2(-h) - 1\|_\infty = 3,$$

and

$$|T_1(2h)(\psi(x'))T_2(-h)(\psi(x'))| < 2 \quad \text{for all } x' \neq x,$$

lead to the fact that $T_1(2h)(\psi(x))T_2(-h)(\psi(x)) = -2$. Now, using Lemma 3.8, we conclude that

$$T_1(2h)(\psi(x))\,\frac{-1}{T_1(h)(\psi(x))} = -2,$$

and then, $T_1(2h)(\psi(x)) = 2T_1(h)(\psi(x))$.

(d) This is an immediate conclusion from the combination of parts (a) and (c).

(e) As in the above reasoning, since $|T_1(ih)(\psi(x'))T_2(2ih)(\psi(x'))| = 2$ if and only if $x' = x$ and $\|T_1(ih)T_2(2ih) - 1\|_\infty = 3$, it follows that $T_1(ih)(\psi(x))T_2(2ih)(\psi(x)) = -2$; hence,

$$\frac{-2}{T_2(2ih)(\psi(x))} = T_1(ih)(\psi(x)).$$

On the other hand, again by Lemmas 3.8 and 3.9,

$$T_1(2ih)(\psi(x)) = \frac{2}{T_2(-ih)(\psi(x))} = -2T_1(-ih)(\psi(x)) = \frac{-4}{T_2(2ih)(\psi(x))}.$$

Now, by comparison of the above two relations,

$$T_1(2ih)(\psi(x)) = 2T_1(ih)(\psi(x))$$

is easily obtained. □

Lemma 3.11. *Let $x \in X$. Then, we either have $T_j(\alpha h)(\psi(x)) = \alpha T_j(h)(\psi(x))$ for all $\alpha \in \mathbb{T}$, $h \in \mathcal{P}_x$ and $j \in \{1,2\}$, or $T_j(\alpha h)(\psi(x)) = \overline{\alpha} T_j(h)(\psi(x))$ for all $\alpha \in \mathbb{T}$, $h \in \mathcal{P}_x$ and $j \in \{1,2\}$.*

Proof. Let $h \in \mathcal{P}_x$. Given $\alpha \in \mathbb{T}$, from Lemmas 3.9 and 3.10 (a), (b), we conclude that

(3.2)
$$T_1(2h)(\psi(x)) = -2T_1(-h)(\psi(x)) = \frac{-4}{T_2(-2h)(\psi(x))} = \frac{4}{T_2(2h)(\psi(x))}.$$

According to (3.2) and Lemma 3.5 (ii),

$$\begin{aligned}\left|T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))}+1\right| &= \left|-T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))}-1\right| \\ &= |T_1(\alpha h)(\psi(x))T_2(-2h)(\psi(x))-1| \\ &\leq \|T_1(\alpha h)T_2(-2h)-1\|_\infty \\ &= \|-2\alpha h^2-1\|_\infty \\ &= |2\alpha+1|.\end{aligned}$$

On the other hand, again applying (3.2) and Lemma 3.5 (ii), we have

$$\begin{aligned}\left|T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))}-1\right| &= |T_1(\alpha h)(\psi(x))T_2(2h)(\psi(x))-1| \\ &\leq \|T_1(\alpha h)T_2(2h)-1\|_\infty \\ &= \|2\alpha h^2-1\|_\infty \\ &= |2\alpha-1|.\end{aligned}$$

Moreover,

$$\left|T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))}\right| = |T_1(\alpha h)(\psi(x))T_2(2h)(\psi(x))| = |2\alpha|,$$

by Lemma 3.4. Then, from Lemma 3.5 (iii), it is deduced that either

$$T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))} = 2\alpha \tag{3.3}$$

or

$$T_1(\alpha h)(\psi(x))\frac{4}{T_1(2h)(\psi(x))} = 2\overline{\alpha}.$$

In other words, using Lemma 3.10 (c), we either have

$$T_1(\alpha h)(\psi(x)) = \alpha T_1(h)(\psi(x)),$$

or

$$T_1(\alpha h)(\psi(x)) = \overline{\alpha}T_1(h)(\psi(x)).$$

In particular, from the above, we conclude that $T_1(ih)(\psi(x)) = iT_1(h)(\psi(x))$ or $T_1(ih)(\psi(x)) = -iT_1(h)(\psi(x))$.

Now, we show that, if $T_1(ih)(\psi(x)) = iT_1(h)(\psi(x))$, then $T_1(\alpha h)(\psi(x)) = \alpha T_1(h)(\psi(x))$ for given $\alpha \in \mathbb{T}$. First, note that $T_2(ih)(\psi(x)) = iT_2(h)(\psi(x))$ since, applying Lemmas 3.8 and 3.10 (d), we have

$$\begin{aligned} T_2(ih)(\psi(x)) &= \frac{-1}{T_1(ih)(\psi(x))} = \frac{-1}{iT_1(h)(\psi(x))} \\ &= -iT_2(-h)(\psi(x)) = iT_2(h)(\psi(x)). \end{aligned} \tag{3.4}$$

We next claim that $T_2(2ih)(\psi(x)) = iT_2(2h)(\psi(x))$. Note that, according to Lemma 3.10 (c), (e), $T_2(2ih)(\psi(x)) = 2T_2(ih)(\psi(x)) = 2iT_2(h)(\psi(x)) = iT_2(2h)(\psi(x))$, and hence, $T_2(2ih)(\psi(x)) = iT_2(2h)(\psi(x))$, as claimed. Now, using (3.2), Lemmas 3.5 (ii), 3.10 (b) and (3.4), we have

$$\begin{aligned} \left|T_1(\alpha h)(\psi(x))\,\frac{4i}{T_1(2h)(\psi(x))} - 1\right| &= \left|T_1(\alpha h)(\psi(x))\,\frac{4}{-iT_1(2h)(\psi(x))} - 1\right| \\ &= \left|T_1(\alpha h)(\psi(x))\,\frac{T_2(-2h)(\psi(x))}{i} - 1\right| \\ &= |iT_1(\alpha h)(\psi(x))T_2(2h)(\psi(x)) - 1| \\ &= |T_1(\alpha h)(\psi(x))T_2(2ih)(\psi(x)) - 1| \\ &\le \|T_1(\alpha h)T_2(2ih) - 1\|_\infty \\ &= \|2i\alpha h^2 - 1\|_\infty = |2i\alpha - 1|. \end{aligned}$$

On the other hand, Lemmas 3.9 and 3.10 (c) give

$$iT_1(2h)(\psi(x)) = 2iT_1(h)(\psi(x)) = 2T_1(ih)(\psi(x)) = \frac{4}{T_2(-2ih)(\psi(x))}.$$

Consequently, using the above relation and Lemma 3.5 (ii), it is seen that

$$\begin{aligned} \left|T_1(\alpha h)(\psi(x))\,\frac{4i}{T_1(2h)(\psi(x))} + 1\right| &= \left|T_1(\alpha h)(\psi(x))\,\frac{-4i}{T_1(2h)(\psi(x))} - 1\right| \\ &= |T_1(\alpha h)(\psi(x))T_2(-2ih)(\psi(x)) - 1| \\ &\le \|T_1(\alpha h)T_2(-2ih) - 1\|_\infty \\ &= \|-2i\alpha h^2 - 1\|_\infty = |2i\alpha + 1|. \end{aligned}$$

Now, since

$$\left|T_1(\alpha h)(\psi(x))\,\frac{4i}{T_1(2h)(\psi(x))}\right| = |2i\alpha|,$$

from the above inequalities and Lemma 3.5 (iii), it follows that

$$\mathrm{Re}\left(T_1(\alpha h)(\psi(x))\,\frac{4i}{T_1(2h)(\psi(x))}\right) = 2\ \mathrm{Re}\,(i\alpha).$$

Equivalently, we have

$$\mathrm{Im}\left(T_1\,(\alpha h)(\psi(x))\,\frac{4}{T_1(2h)(\psi(x))}\right) = 2\ \mathrm{Im}\,(\alpha).$$

Then, from (3.3) and Lemma 3.10 (c), it is derived that

$$T_1(\alpha h)(\psi(x)) = \alpha T_1(h)(\psi(x)).$$

Moreover, taking into account that $T_2(ih)(\psi(x)) = iT_2(h)(\psi(x))$, similar arguments show that $T_2(\alpha h)(\psi(x)) = \alpha T_2(h)(\psi(x))$. Finally, we conclude that

$$T_j(\alpha h')(\psi(x)) = \alpha T_j(h')(\psi(x)), \quad h' \in \mathcal{P}_x,\ j \in \{1,2\},$$

from Lemma 3.8.

Analogously, the same reasoning can be applied to the equation $T_1(ih)(\psi(x)) = -iT_1(h)(\psi(x))$, and we deduce that

$$T_j(\alpha h')(\psi(x)) = \overline{\alpha}\,T_j(h')(\psi(x)),$$

for all $h' \in \mathcal{P}_x$, $\alpha \in \mathbb{T}$ and $j \in \{1,2\}$. □

Set

$$K := \{y \in Y : T_1(ih)(y) = iT_1(h)(y) \text{ for all } h \in \mathcal{P}_x\}.$$

From the preceding lemma, it is apparent that

$$Y \setminus K = \{y \in Y : T_1(ih)(y) = -iT_1(h)(y) \text{ for all } h \in \mathcal{P}_x\},$$

and then simple calculation shows that K is a clopen subset of Y. Moreover, note that, given $h \in \mathcal{P}_x$ and $\alpha \in \mathbb{T}$,

$$T_j(\alpha h)(y) = \begin{cases} \alpha T_j(h)(y) & y \in K, \\ \overline{\alpha}\,T_j(h)(y) & y \in Y \setminus K, \end{cases}$$

for $j = 1, 2$.

Next we give a complete description of the maps T_1 and T_2.

Lemma 3.12. *There are continuous functions $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ such that $\Gamma_1\Gamma_2 = 1$. Furthermore, for each $f \in A$ and $j = 1, 2$, $T_j(f) = \Gamma_j f \circ \varphi$ on K and $T_j(f) = \Gamma_j \overline{f \circ \varphi}$ on $Y \setminus K$.*

Proof. We prove this lemma by a method similar to [**1,** Theorem 4.8]. Let $f \in A$ and $y \in Y$. If $f(\varphi(y)) = 0$, then, taking into account Lemma 3.4, it is not difficult to see that $T_1(f)(y) = 0$. Now, suppose that $f(\varphi(y)) \neq 0$. From Lemma 3.1, there exists an $h \in \mathcal{P}_{\varphi(y)}$ such that $M_{fh} = \{\varphi(y)\}$, and, in particular, $R_\pi(fh) = \{f(\varphi(y))\}$. Put

$$\alpha := \frac{-\overline{T_2(h)(y)}\,\overline{T_1(f)(y)}}{|f(\varphi(y))|}, \qquad \beta := \alpha$$

if $y \in K$ and $\beta := \overline{\alpha}$ if $y \in Y \setminus K$. Clearly, $\beta \in \mathbb{T}$. Then, from the explanations before the lemma, we have

$$\begin{aligned} |T_1(f)(y)T_2(\beta h)(y) - 1| &= |T_1(f)(y)\alpha T_2(h)(y) - 1| \\ &= |f(\varphi(y))| + 1. \end{aligned}$$

Moreover, by Lemma 3.4, for every $y' \neq y$,

$$\begin{aligned} |T_1(f)(y')T_2(\beta h)(y') - 1| &\leq |T_1(f)(y')T_2(\beta h)(y')| + 1 \\ &= |(fh)(\varphi(y'))| + 1 < |f(\varphi(y))| + 1. \end{aligned}$$

Then, $|f(\varphi(y))| + 1 = \|T_1(f)T_2(\beta h) - 1\|_\infty = \|f\beta h - 1\|_\infty$. On the other hand, for every $x' \neq \varphi(y)$, $|\beta(fh)(x') - 1| \leq |\beta(fh)(x')| + 1 < |f(\varphi(y))| + 1$. Thus,

$$|f(\varphi(y))| + 1 = |\beta(fh)(\varphi(y)) - 1| = |\beta f(\varphi(y)) - 1|.$$

Now, by Lemma 3.5 (i), $\beta f(\varphi(y)) = -|f(\varphi(y))|$, which implies that

$$T_1(f)(y) = \begin{cases} \dfrac{f(\varphi(y))}{T_2(h)(y)} = -T_1(-h)(y)f(\varphi(y)) = T_1(h)(y)f(\varphi(y)) \\ \hfill y \in K, \\ \dfrac{\overline{f(\varphi(y))}}{T_2(h)(y)} = -T_1(-h)(y)\overline{f(\varphi(y))} = T_1(h)(y)\overline{f(\varphi(y))} \\ \hfill y \in Y \setminus K, \end{cases}$$

by Lemmas 3.8 and 3.10 (d). Thus,

$$T_1(f)(y) = \begin{cases} T_1(h)(y)f(\varphi(y)) & y \in K, \\ T_1(h)(y)\overline{f(\varphi(y))} & y \in Y \setminus K. \end{cases}$$

Similarly, the following representation holds for $T_2(f)$:

$$T_2(f)(y) = \begin{cases} T_2(h)(y)f(\varphi(y)) & y \in K, \\ T_2(h)(y)\overline{f(\varphi(y))} & y \in Y \setminus K. \end{cases}$$

Furthermore, combining Lemmas 3.8 and 3.10 (d), we have

$$T_1(h)(y)T_2(h)(y) = T_1(h)(y)\,\frac{1}{-T_1(-h)(y)} = T_1(h)(y)\,\frac{1}{T_1(h)(y)} = 1,$$

and thus, $T_1(h)(y)T_2(h)(y) = 1$. Now, define the functions $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ by $\Gamma_j(y) = T_j(h_0)(y)$ for some $h_0 \in \mathcal{P}_{\varphi(y)}$, $j = 1, 2$. Note that these functions are well defined since, according to Lemma 3.8, this assignment is independent of the choice of h_0. Therefore, for each $f \in A$, $y \in Y$ and $j = 1, 2$, we have

$$T_j(f)(y) = \begin{cases} \Gamma_j(y)f(\varphi(y)) & y \in K, \\ \Gamma_j(y)\overline{f(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

and $\Gamma_1(y)\Gamma_2(y) = 1$.

Finally, we check the continuity of Γ_j for $j = 1, 2$. Let $y_0 \in Y$ and $j \in \{1, 2\}$. We may choose a function $f \in A$ such that $f(\varphi(y_0)) \neq 0$. Take $U = \varphi^{-1}(\{x \in X : f(x) \neq 0\})$, which is a neighborhood of y_0. Hence,

$$\Gamma_j(y) = \frac{T_j(f)(y)}{(f \circ \varphi)(y)^\star}, \quad y \in U,$$

where $(f \circ \varphi)(y)^\star = (f \circ \varphi)(y)$ if $y \in K$ and $(f \circ \varphi)(y)^\star = \overline{(f \circ \varphi)(y)}$ if $y \in Y \setminus K$. Now, since K is clopen, from the continuity of $T_j(f)$ and $f \circ \varphi$ we easily conclude that Γ_j is continuous at y_0. $\square$

Now, we establish the following result which has been proved through the above lemmas and gives a complete description of the maps τ_1 and τ_2.

Theorem 3.13. *For surjections $\rho_1 : \Omega_1 \to A$, $\rho_2 : \Omega_2 \to A$, $\tau_1 : \Omega_1 \to B$ and $\tau_2 : \Omega_2 \to B$ satisfying*

$$\|\rho_1(\omega_1)\rho_2(\omega_2) + \alpha_0\|_\infty = \|\tau_1(\omega_1)\tau_2(\omega_2) + \alpha_0\|_\infty$$

for all $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, there exist a homeomorphism $\varphi : Y \to X$, a clopen subset K of Y and continuous functions $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ such that $\Gamma_1\Gamma_2 = 1$ and

$$\tau_j(\omega_j)(y) = \Gamma_j(y) \begin{cases} \rho_j(\omega_j)(\varphi(y)) & y \in K, \\ -\dfrac{\alpha_0}{|\alpha_0|}\overline{\rho_j(\omega_j)(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

for all $\omega_j \in \Omega_j$, $y \in Y$ and $j = 1, 2$.

It is worthwhile mentioning that the converse of Theorem 3.14 is also valid. This means that maps of the forms that appear in the conclusion of the theorem preserve the above-considered norm condition. Meanwhile, as observed, according to Lemma 3.6, we can always ignore the index sets Ω_1, Ω_2 and define induced maps T_1, T_2 on the algebras without considering any structure of the sets.

The following result is immediately obtained.

Theorem 3.14. *For surjective maps $T_1, T_2 : A \to B$ such that $\|T_1 f T_2 g + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in A$, there exist a homeomorphism $\varphi : Y \to X$, a clopen subset K of Y and continuous functions $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ such that $\Gamma_1\Gamma_2 = 1$ and*

$$T_j(f)(y) = \begin{cases} \Gamma_j(y) f(\varphi(y)) & y \in K, \\ -\dfrac{\alpha_0}{|\alpha_0|}\Gamma_j(y)\overline{f(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

for all $f \in A$, $y \in Y$ and $j = 1, 2$. If, furthermore, $T_j(ih_0) = iT_j(h_0)$ for some $j \in \{1, 2\}$ and $h_0 \in A$ with $h_0 \neq 0$ on X, then $K = Y$ and T_1, T_2 are weighted composition operators.

Remark 3.15.

(1) We remark that the previous theorems are valid, with the same proofs for all Banach function algebras satisfying Lemma 3.1. In par-

ticular, this lemma holds for any Banach function algebra A with the following property:

> *there exists a constant $c > 0$ such that for each point x in the first countable underlying space X, each neighborhood U of x and each scalar $\epsilon > 0$, there is a non-negative function $u \in A$ with $\|u\| \leq c$, $u(x) = 1 = \|u\|_\infty$, and $|u| < \epsilon$ on $X \setminus U$.*

In order to show this, we apply a modification of the proofs of Lemmas 3.1 and 5.1 in [**13**]. Let $x \in X$ and $f \in A$ with $f(x) \neq 0$. Without loss of generality, we may assume that $f(x) = 1$. Since X is first countable, there is a sequence $\{V_n\}$ of neighborhoods of x such that

$$\bigcap_{n=1}^{\infty} V_n = \{x\}.$$

Consider the following open sets

$$U_n = \left\{ z \in V_n : |f(z) - 1| < \frac{1}{2^{n+1}} \right\}, \quad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$, we choose a non-negative function u_n in A with $\|u_n\| \leq c$, $u_n(x) = 1 = \|u_n\|_\infty$ and

$$|u_n| < \frac{1}{(2^n + 1)\|f\|_\infty} \quad \text{on } X \setminus U_n.$$

Set $h = \sum_{n=1}^{\infty} u_n/2^n$. Since $\|u_n\| \leq c$, the series converges and $h \in A$. It is apparent that

$$M_h \subseteq \bigcap_{n=1}^{\infty} M_{u_n} \subseteq \bigcap_{n=1}^{\infty} V_n = \{x\},$$

and, consequently, $h \in \mathcal{P}_x$ with $M_h = \{x\}$.

Finally, we show that $M_{fh} = \{x\}$. Clearly, $(fh)(x) = 1$. Now, let $z \in X \setminus \{x\}$. If

$$z \notin \bigcup_{n=1}^{\infty} U_n,$$

then

$$|(fh)(z)| < |f(z)| \sum_{i=1}^{\infty} \frac{1}{2^i \|f\|_\infty} \leq 1.$$

Otherwise, z belongs to $U_1, \dots, U_{n-1}$ but not to U_n. Then,

$$\begin{aligned} |(fh)(z)| &\le (1 + |f(z) - 1|)|h(z)| \\ &\le \left(1 + \frac{1}{2^n}\right)\left(\sum_{i \neq n} \frac{1}{2^i} + \frac{u_n(z)}{2^n}\right) \\ &< \left(1 + \frac{1}{2^n}\right)\left(1 - \frac{1}{2^n} + \frac{1}{2^n(2^n+1)}\right) = 1. \end{aligned}$$

Therefore, $M_{fh} = \{x\}$, as claimed.

(2) The representations given in Theorem 3.14 show that T_1 and T_2 are real-linear. Indeed, T_1 and T_2 are not necessarily complex-linear. For instance, consider the case where $T_1(f) = T_2(f) = \overline{f}$ for all $f \in A$. We shall show that T_1, T_2 are complex-linear if and only if they satisfy a stronger condition (see Theorem 3.19).

(3) Since $\|\cdot\|_\infty \le \|\cdot\|$ on A, from the representation of T_j, $j = 1, 2$, appearing above, it is easily inferred that the graph of T_j is closed. We also point out that it can be verified that the closed graph theorem holds for real-linear maps. Therefore, we conclude that T_j is continuous with respect to the complete norm $\|\cdot\|$.

Corollary 3.16. *Assume that, in Contexts* 1–3, $\alpha_0 \in \mathbb{C} \setminus s\{0\}$, *and* $T : A \to B$ *is a surjection such that* $\|Tf\overline{Tg} + \alpha_0\|_\infty = \|f\overline{g} + \alpha_0\|_\infty$ *for all* $f, g \in A$. *Then, there exist a homeomorphism* $\varphi : Y \to X$, *a clopen subset* K *of* Y *and a unimodular continuous function* $\Gamma : Y \to \mathbb{T}$ *such that, for each* $f \in A$,

$$T(f) = \Gamma \begin{cases} f \circ \varphi & \text{on } K, \\ -\dfrac{\alpha_0}{|\alpha_0|}\,\overline{f \circ \varphi} & \text{on } Y \setminus K. \end{cases}$$

Proof. Define surjections $T_1, T_2 : A \to B$ by $T_1(f) := T(f)$ and $T_2(f) := \overline{T(\overline{f})}$ for all $f \in A$. Then, the result follows from Theorem 3.14. □

Below, we give a corollary of our main result which may be considered an extension of [**1,** Theorem 4.8].

Corollary 3.17. *Let $\alpha_0 \in \mathbb{C} \setminus \{0\}$, X and Y be pointed compact metric spaces, and let $T_1, T_2 : \mathrm{Lip}_0(X) \to \mathrm{Lip}_0(Y)$ be surjective maps satisfying $\|T_1 f T_2 g + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in \mathrm{Lip}_0(X)$. Then there exist a base point preserving bi-Lipschitz homeomorphism $\varphi : Y \to X$, functions $\eta : Y \to \{0,1\}$, $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ such that $\Gamma_1 \Gamma_2 = 1$ and*

$$T_j(f) = \Gamma_j \left(\eta \cdot f \circ \varphi + (1-\eta) \cdot \frac{-\alpha_0}{|\alpha_0|} \overline{f \circ \varphi} \right),$$

for each $f \in \mathrm{Lip}_0(X)$ and $j = 1, 2$.

Proof. Let e_X and e_Y be the base points of metric spaces X and Y, respectively. From Theorem 3.14, there exist a homeomorphism $\varphi : Y \setminus \{e_Y\} \to X \setminus \{e_X\}$, and a clopen subset K of $Y \setminus \{e_Y\}$, continuous functions $\Gamma_1, \Gamma_2 : Y \setminus \{e_Y\} \to \mathbb{C}$ such that $\Gamma_1 \Gamma_2 = 1$ and

$$T_j(f) = \Gamma_j \begin{cases} f \circ \varphi & \text{on } K, \\ \dfrac{-\alpha_0}{|\alpha_0|} \overline{f \circ \varphi} & \text{on } (Y \setminus \{e_Y\}) \setminus K, \end{cases}$$

for all $f \in \mathrm{Lip}_0(X)$ and $j = 1, 2$. Next, extend φ, Γ_1 and Γ_2 to the functions on Y by letting

$$\varphi(e_Y) = e_X, \qquad \Gamma_1(e_Y) = \Gamma_2(e_Y) = 1.$$

Moreover, define the function η as follows: $\eta(y) = 1$ if $y \in K \cup \{e_Y\}$ and $\eta(y) = 0$ if $y \in Y \setminus (\{e_Y\} \cup K)$. Therefore, for each $f \in \mathrm{Lip}_0(X)$ and $j = 1, 2$, we have

$$T_j(f) = \Gamma_j \left(\eta \cdot f \circ \varphi + (1-\eta) \cdot \frac{-\alpha_0}{|\alpha_0|} \overline{f \circ \varphi} \right).$$

Now, it suffices to prove that φ is a bi-Lipschitz function, which follows from a standard argument as in [**1,** Page 10]. We include the proof here for completeness. From the previous remark, we get T_1 is continuous. Then, there exists a $t > 0$ such that $\|T_1(f)\| \leq t\|f\|$ for all $f \in \mathrm{Lip}_0(X)$. Let y and y' be two distinct points in Y, and suppose, without loss of generality, that $d(\varphi(y), e_X) \leq d(\varphi(y'), e_X)$. Let $f_0 \in \mathrm{Lip}_0(X)$ be defined by

$$f_0(x) = d(\varphi(y), \varphi(y')) \max\left(0, 1 - \frac{d(\varphi(y'), x)}{\delta}\right),$$

where $\delta = \min\{d(\varphi(y'), e_X), d(\varphi(y), \varphi(y'))\}$. Taking into account that $d(\varphi(y), \varphi(y')) \leq 2\delta$, it is easy to see that $L(f_0) \leq 2$, and, consequently, $\|f_0\| \leq 2 + \mathrm{diam}(X)$. Thus, $\|T(f_0)\| \leq t(2 + \mathrm{diam}(X))$. Now, letting $s = \min\{|\Gamma_1(y)| : y \in Y\}$, we have

$$\begin{aligned}\frac{d(\varphi(y), \varphi(y'))}{d(y, y')} &= \frac{|f_0(\varphi(y)) - f_0(\varphi(y'))|}{d(y, y')} \leq \frac{|\Gamma_1(y) f_0(\varphi(y)) - \Gamma_1(y) f_0(\varphi(y'))|}{s d(y, y')} \\ &= \frac{|T(f_0)(y) - T(f_0)(y')|}{s d(y, y')} \leq \frac{L(T(f_0))}{s} \\ &\leq \frac{\|T(f_0)\|}{s} \leq \frac{t(2 + \mathrm{diam}(X))}{s},\end{aligned}$$

which shows that φ is a Lipschitz function. Analogously, it can be seen that φ^{-1} is a Lipschitz function. $\square$

The next result states that each non-symmetric multiplicatively norm-preserving map between the Figà-Talamanca-Herz algebras on amenable groups induces a real-algebra isomorphism.

Corollary 3.18. *Assume that, in Context* 3*, the groups X and Y are amenable. If $\alpha_0 \in \mathbb{C} \setminus \{0\}$ and $T : A \to B$ is a surjection such that $\|TfTg + \alpha_0\|_\infty = \|fg + \alpha_0\|_\infty$ for all $f, g \in A$, then A and B are real-algebra isomorphic.*

Proof. Taking into account Remark 3.7, we may assume, without loss of generality, that $\alpha_0 = -1$. From Theorem 3.14, we infer that there exist a homeomorphism $\varphi : Y \to X$, a clopen subset K of Y and a continuous function $\Gamma : Y \to \{\pm 1\}$ such that

$$T(f) = \Gamma \begin{cases} f \circ \varphi & \text{on } K, \\ \overline{f \circ \varphi} & \text{on } Y \setminus K, \end{cases}$$

and

$$T^{-1}(F) = \Gamma \circ \psi \begin{cases} F \circ \psi & \text{on } \varphi(K), \\ \overline{F \circ \psi} & \text{on } X \setminus \varphi(K), \end{cases}$$

for all $f \in A$ and $F \in B$, where $\psi = \varphi^{-1}$. Although T is real-linear (not necessarily complex-linear), nevertheless, the same arguments as in the proof of [**24,** Theorem 4.22] with slight modifications show that the functions Γ and $\Gamma \circ \psi$ belong to the multiplier algebra of $A_p(Y)$

and $A_p(X)$, respectively. Now, it is easily seen that, for each $f \in A$, $\Gamma\circ\psi\cdot f \in A$, and thus, the function g defined as $g(y) = f(\varphi(y))$ if $y \in K$, and $g(y) = \overline{f(\varphi(y))}$ if $y \in Y \setminus K$, belongs to B; indeed, $g = T(\Gamma\circ\psi\cdot f)$. Therefore, the operator $\mathcal{T}$ defined by

$$\mathcal{T}(f) = \begin{cases} f\circ\varphi & \text{on } K, \\ \overline{f\circ\varphi} & \text{on } Y \setminus K, \end{cases}$$

is a real-algebra isomorphism from A onto B. □

Finally, we present a sufficient (clearly, necessary) condition for the maps T_1 and T_2 to be weighted composition operators.

Theorem 3.19. *Let $\alpha_0 \in \mathbb{C} \setminus \{0\}$ and $T_1, T_2 : A \to B$ be surjective maps. If $R_\pi(fg+\alpha_0) \cap R_\pi(T_1fT_2g+\alpha_0) \neq \varnothing$ for all $f, g \in A$, then $T_jf(y) = \Gamma_j(y)\, f(\varphi(y))$, where Γ_j and φ are given by Theorem* 3.14.

Proof. According to Remark 3.7, we can assume, without loss of generality, that $\alpha_0 = -1$. Since, for each $f, g \in A$, $R_\pi(fg-1) \cap R_\pi(T_1fT_2g-1) \neq \varnothing$, then $\|T_1fT_2g-1\|_\infty = \|fg-1\|_\infty$. Thus, by Theorem 3.14, there exist a homeomorphism $\varphi : Y \to X$, a clopen subset K of Y, and continuous functions $\Gamma_1, \Gamma_2 : Y \to \mathbb{C}$ such that $\Gamma_1\Gamma_2 = 1$ and

$$T_j(f)(y) = \Gamma_j(y) \begin{cases} f(\varphi(y)) & y \in K, \\ \overline{f(\varphi(y))} & y \in Y \setminus K, \end{cases}$$

for all $f \in A$ and $j = 1, 2$. We claim that $K = Y$. Suppose, on the contrary, that $Y \setminus K \neq \emptyset$. Take $y_0 \in Y \setminus K$. Since B is a regular Banach function algebra and the strong boundary point y_0 does not belong to the closed set K, we may select $F \in B$ such that $F(y_0) = 1$, $0 \le F \le 1$ and $F = 0$ on K, by Lemma 3.1. Let $f, g \in A$ be such that $T_1f = iF$ and $T_2g = F$. It is obvious that $R_\pi(T_1fT_2g-1) = \{i-1\}$, while $R_\pi(fg-1) = \{-i-1\}$ since, according to the previously obtained representations for T_1 and T_2, we see that

$$fg = \begin{cases} 0 & \text{on } K, \\ \dfrac{-iF^2}{\overline{\Gamma_1\Gamma_2}} = -iF^2 & \text{on } Y \setminus K. \end{cases}$$

This contradiction shows that $K = Y$, and hence, $T_j f(y) = \Gamma_j(y)\, f(\varphi(y))$ for all $f \in A$, $y \in Y$, and $j = 1, 2$. $\square$

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REFERENCES

1. M. Burgos, A. Jiménez-Vargas and M. Villegas-Vallecillos, *Nonlinear conditions for weighted composition operators between Lipschitz algebras*, J. Math. Anal. Appl. **359** (2009), 1–14.

2. J.T. Burnham, *Closed ideals in subalgebras of Banach algebras*, I, Proc. Amer. Math. Soc. **32** (1972), 551–555.

3. J.T. Burnham and R.R. Goldberg, *Basic properties of Segal algebras*, J. Math. Anal. Appl. **42** (1973), 323–329.

4. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.

5. E.E. Granirer, *The Figà-Talamanca-Herz-Lebesgue Banach algebras* $A_p^r(G) = A_p(G) \cap L^r(G)$, Math. Proc. Cambridge Philos. Soc. **140** (2006), 401–416.

6. O. Hatori, A. Jiménez-Vargas and M. Villegas-Vallecillos, *Maps which preserve norms of non-symmetrical quotients between groups of exponentials of Lipschitz functions*, J. Math. Anal. Appl. **415** (2014), 825–845.

7. O. Hatori, S. Lambert, A. Lutman, T. Miura, T. Tonev and R. Yates, *Spectral preservers in commutative Banach algebras*, Contemp. Math. **547** (2011), 103–124.

8. O. Hatori, T. Miura, R. Shindo and H. Takagi, *Generalizations of spectrally multiplicative surjections between uniform algebras*, Rend. Circ. Mat. Palermo **59** (2010), 161–183.

9. O. Hatori, T. Miura and H. Takagi, *Multiplicatively spectrum-preserving and norm-preserving maps between invertible groups of commutative Banach algebras*, arxiv:0904.1939, 2006.

10. C.S. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier (Grenoble) **23** (1973), 91–123.

11. G. Hirasawa, T. Miura and H. Takagi, *Spectral radii conditions for isomorphisms between unital semisimple commutative Banach algebras*, Contemp. Math. **547** (2011), 125–134.

12. D. Honma, *Norm-preserving surjections on algebras of continuous functions*, Rocky Mountain J. Math. **39** (2009), 1517–1531.

13. M. Hosseini and M. Amini, *Peripherally multiplicative maps between Figà-Talamanca-Herz algebras*, Publ. Math. Debrecen **86** (2015), 71–88.

14. M. Hosseini and F. Sady, *Banach function algebras and certain polynomially norm-preserving maps*, Banach J. Math. Anal. **6** (2012), 1–18.

15. ———, *Maps between Banach function algebras satisfying certain norm conditions*, Central Europ. J. Math. **11** (2013), 1020–1033.

16. M. Hosseini and F. Sady, *Multiplicatively and non-symmetric multiplicatively norm-preserving maps*, Central Europ. J. Math. **8** (2010), 878–889.

17. A. Jiménez-Vargas, K. Lee, A. Luttman and M. Villegas-Vallecillos, *Generalized weak peripheral multiplicativity in algebras of Lipschitz functions*, Central Europ. J. Math. **11** (2013), 1197–1211.

18. A. Jiménez-Vargas, A. Luttman and M. Villegas-Vallecillos, *Weakly peripherally multiplicative surjections of pointed Lipschitz algebras*, Rocky Mountain J. Math. **40** (2010), 1903–1922.

19. R. Larsen, T.-S. Liu and J.-K. Wang, *On functions with Fourier transforms in L_p*, Michigan Math. J. **11** (1964), 369–378.

20. K. Lee and A. Luttman, *Generalizations of weakly peripherally multiplicative maps between uniform algebras*, J. Math. Anal. Appl. **375** (2011), 108–117.

21. A. Luttman and S. Lambert, *Norm conditions for uniform algebra isomorphisms*, Central Europ. J. Math. **6** (2008), 272–280.

22. A. Luttman and T. Tonev, *Uniform algebra isomorphisms and peripheral multiplicativity*, Proc. Amer. Math. Soc. **135** (2007), 3589–3598.

23. T. Miura, D. Honma and R. Shindo, *Divisibly norm-preserving maps between commutative Banach algebras*, Rocky Mountain J. Math. **41** (2011), 1675–1699.

24. M.S. Monfared, *Extensions and isomorphisms for the generalized Fourier algebras of a locally compact group*, J. Funct. Anal. **198** (2003), 413–444.

25. N.V. Rao, T. Tonev and E. Toneva, *Uniform algebra isomorphisms and peripheral spectra*, Contemp. Math. **427** (2007), 401–416.

26. H. Reiter and J.D. Stegeman, *Classical harmonic analysis and locally compact groups*, Oxford University Press, Oxford, 2000.

27. R. Shindo, *Maps between uniform algebras preserving norms of rational functions*, Mediterranean J. Math. **8** (2010), 81–95.

28. ______, *Norm conditions for real-algebra isomorphisms between uniform algebras*, Central Europ. J. Math. **8** (2010), 135–147.

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