

ON THE PERIODIC SOLUTIONS OF SOME SYSTEMS OF HIGHER ORDER DIFFERENCE EQUATIONS

MELIH GOCEN AND ADEM CEBECI

ABSTRACT. In this paper, we obtain the general form of the periodic solutions of some higher order difference equations system

$$x_{n+1} = \frac{\pm x_{n-k} y_{n-(2k+1)}}{y_{n-(2k+1)} \mp y_{n-k}},$$
$$y_{n+1} = \frac{\pm y_{n-k} x_{n-(2k+1)}}{x_{n-(2k+1)} \mp x_{n-k}},$$

$n, k \in \mathbb{N}_0$, where the initial values are arbitrary real numbers such that the denominator is always nonzero. Moreover, some numerical examples are presented to verify our theoretical results.

1. Introduction. Difference equations and systems have been investigated by many researchers in the last few decades, see [1, 4, 5, 6, 10, 11]. The reason is that difference equations have many applications in several mathematical models in biology, economics, genetics, population dynamics, medicine, physiology, and so forth. Moreover, systems of higher order rational difference equations have also been widely studied, see [7, 9, 12, 13], but still have many aspects to be investigated. In particular, studying the qualitative analysis of difference equations and systems is a very rich research field. For example, there are many papers related to the periodicity of the positive solutions of rational difference equation systems, see [2, 3, 8, 10].

Cinar [2] was concerned with the periodicity of positive solutions of the difference equation system:

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1} y_{n-1}}.$$

2010 AMS *Mathematics subject classification.* Primary 39A10.

Keywords and phrases. Periodicity, systems of higher order difference equations, form of the solutions.

The first author is the corresponding author.

Received by the editors on April 24, 2017.

Kurbanli, et al., [8] studied the periodicity of solutions of the following rational difference equation system:

$$x_{n+1} = \frac{x_{n-1} + y_n}{y_n x_{n-1} - 1}, \quad y_{n+1} = \frac{y_{n-1} + x_n}{x_n y_{n-1} - 1}.$$

Touafek and Elsayed [11] investigated the form of solutions of the nonlinear difference equations:

$$x_{n+1} = \frac{x_{n-1}y_n}{y_n \pm y_{n-2}}, \quad y_{n+1} = \frac{x_n y_{n-1}}{x_n \pm x_{n-2}}.$$

El-Metwally [4] dealt with solutions of the difference equation system:

$$x_{n+1} = \frac{x_{n-1}y_n}{\pm x_{n-1} \pm y_{n-2}}, \quad y_{n+1} = \frac{x_n y_{n-1}}{\pm y_{n-1} \pm x_{n-2}}.$$

El-Dessoky and Elsayed [3] obtained the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{x_n y_{n-1}}{y_{n-1} \pm y_n}, \quad y_{n+1} = \frac{y_n x_{n-1}}{x_{n-1} \pm x_n}.$$

Our aim in this paper is to obtain the general form of periodic solutions of some higher order rational difference equations system:

$$(1.1) \quad x_{n+1} = \frac{\pm x_{n-k} y_{n-(2k+1)}}{y_{n-(2k+1)} \mp y_{n-k}}, \quad y_{n+1} = \frac{\pm y_{n-k} x_{n-(2k+1)}}{x_{n-(2k+1)} \mp x_{n-k}},$$

where $n, k \in \mathbb{N}_0$ and the initial conditions $x_{-2k-1}, x_{-2k}, \dots, x_0, y_{-2k-1}, y_{-2k}, \dots, y_0$ are arbitrary real numbers such that the denominator is always nonzero.

Definition 1.1 (Periodicity). A sequence $\{x_n\}_{n=-m}^{+\infty}$ is said to be *periodic* with period p if $x_{n+p} = x_n$ for all $n \geq -m$.

2. The system: $x_{n+1} = (x_{n-k} y_{n-(2k+1)}) / (y_{n-(2k+1)} - y_{n-k}), y_{n+1} = (y_{n-k} x_{n-(2k+1)}) / (x_{n-(2k+1)} - x_{n-k})$. In this section, we investigate the form of solutions and the periodicity of the following rational difference equations system:

$$(2.1) \quad x_{n+1} = \frac{x_{n-k} y_{n-(2k+1)}}{y_{n-(2k+1)} - y_{n-k}}, \quad y_{n+1} = \frac{y_{n-k} x_{n-(2k+1)}}{x_{n-(2k+1)} - x_{n-k}},$$

where $n, k \in \mathbb{N}_0$ and the initial conditions are nonzero real numbers such that

$$(y_{-(2k+1-m)} - y_{-(k-m)})(x_{-(2k+1-m)} - x_{-(k-m)}) \neq 0, \quad m = \overline{0, k}.$$

Theorem 2.1. *Let $\{x_n, y_n\}_{n=-(2k+1)}^{+\infty}$ be solutions of system (2.1). Then, the following results hold:*

(i) $\{x_n\}_{n=-(2k+1)}^{+\infty}$ and $\{y_n\}_{n=-(2k+1)}^{+\infty}$ are periodic with period $6(k+1)$ for $n \geq -(2k+1)$.

(ii) We have

$$(2.2) \quad x_{6(k+1)n+l} = \begin{cases} \frac{x_{(l-k-1)}y_{(l-2k-2)}}{y_{(l-2k-2)} - y_{(l-k-1)}} & 1 \leq l \leq k+1, \\ -\frac{y_{(l-3k-3)}(x_{(l-3k-3)} - x_{(l-2k-2)})}{y_{(l-3k-3)} - y_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ -\frac{y_{(l-3k-3)}(x_{(l-4k-4)} - x_{(l-3k-3)})}{y_{(l-4k-4)} - y_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ -\frac{y_{(l-4k-4)}x_{(l-5k-5)}}{y_{(l-5k-5)} - y_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ x_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6 \end{cases}$$

and

$$(2.3) \quad y_{6(k+1)n+l} = \begin{cases} \frac{y_{(l-k-1)}x_{(l-2k-2)}}{x_{(l-2k-2)} - x_{(l-k-1)}} & 1 \leq l \leq k+1, \\ -\frac{x_{(l-3k-3)}(y_{(l-3k-3)} - y_{(l-2k-2)})}{x_{(l-3k-3)} - x_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ -\frac{x_{(l-3k-3)}(y_{(l-4k-4)} - y_{(l-3k-3)})}{x_{(l-4k-4)} - x_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ -\frac{x_{(l-4k-4)}y_{(l-5k-5)}}{x_{(l-5k-5)} - x_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ y_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6. \end{cases}$$

Proof.

(i) It follows from equation (2.1) that

$$\begin{aligned}
 x_{n+(6k+6)} &= \frac{x_{n+(5k+5)}y_{n+(4k+4)}}{y_{n+(4k+4)}-y_{n+(5k+5)}} \\
 &= \frac{(x_{n+(4k+4)}y_{n+(3k+3)})/(y_{n+(3k+3)}-y_{n+(4k+4)})y_{n+(4k+4)}}{y_{n+(4k+4)}-(y_{n+(4k+4)}x_{n+(3k+3)})/(x_{n+(3k+3)}-x_{n+(4k+4)})} \\
 &= -\frac{y_{n+(3k+3)}(x_{n+(3k+3)}-x_{n+(4k+4)})}{y_{n+(3k+3)}-y_{n+(4k+4)}} \\
 &= -\frac{y_{n+(3k+3)}(x_{n+(3k+3)}-(x_{n+(3k+3)}y_{n+(2k+2)})/(y_{n+(2k+2)}-y_{n+(3k+3)}))}{y_{n+(3k+3)}-(y_{n+(3k+3)}x_{n+(2k+2)})/(x_{n+(2k+2)}-x_{n+(3k+3)})} \\
 &= -\frac{y_{n+(3k+3)}(x_{n+(2k+2)}-x_{n+(3k+3)})}{y_{n+(2k+2)}-y_{n+(3k+3)}} \\
 &= -\frac{(y_{n+(2k+2)}x_{n+(k+1)})/(x_{n+(k+1)}-x_{n+(2k+2)})}{y_{n+(2k+2)}-(y_{n+(2k+2)}x_{n+(k+1)})/(x_{n+(k+1)}-x_{n+(2k+2)})} \\
 &\quad \times (x_{n+(2k+2)}-(x_{n+(2k+2)}y_{n+(k+1)})/(y_{n+(k+1)}-y_{n+(2k+2)})) \\
 &= -\frac{y_{n+(2k+2)}x_{n+(k+1)}}{y_{n+(k+1)}-y_{n+(2k+2)}} \\
 &= -\frac{(y_{n+(k+1)}x_n)/(x_n-x_{n+(k+1)})x_{n+(k+1)}}{y_{n+(k+1)}-(y_{n+(k+1)}x_n)/(x_n-x_{n+(k+1)})} = x_n, \\
 y_{n+(6k+6)} &= \frac{y_{n+(5k+5)}x_{n+(4k+4)}}{x_{n+(4k+4)}-x_{n+(5k+5)}} \\
 &= \frac{(y_{n+(4k+4)}x_{n+(3k+3)})/(x_{n+(3k+3)}-x_{n+(4k+4)})x_{n+(4k+4)}}{x_{n+(4k+4)}-(x_{n+(4k+4)}y_{n+(3k+3)})/(y_{n+(3k+3)}-y_{n+(4k+4)})} \\
 &= -\frac{x_{n+(3k+3)}(y_{n+(3k+3)}-y_{n+(4k+4)})}{x_{n+(3k+3)}-x_{n+(4k+4)}} \\
 &= -\frac{x_{n+(3k+3)}(y_{n+(3k+3)}-(y_{n+(3k+3)}x_{n+(2k+2)})/(x_{n+(2k+2)}-x_{n+(3k+3)}))}{x_{n+(3k+3)}-(x_{n+(3k+3)}y_{n+(2k+2)})/(y_{n+(2k+2)}-y_{n+(3k+3)})} \\
 &= -\frac{x_{n+(3k+3)}(y_{n+(2k+2)}-y_{n+(3k+3)})}{x_{n+(2k+2)}-x_{n+(3k+3)}} \\
 &= -\frac{(x_{n+(2k+2)}y_{n+(k+1)})/(y_{n+(k+1)}-y_{n+(2k+2)})}{x_{n+(2k+2)}-(x_{n+(2k+2)}y_{n+(k+1)})/(y_{n+(k+1)}-y_{n+(2k+2)})} \\
 &\quad \times (y_{n+(2k+2)}-(y_{n+(2k+2)}x_{n+(k+1)})/(x_{n+(k+1)}-x_{n+(2k+2)})) \\
 &= -\frac{x_{n+(2k+2)}y_{n+(k+1)}}{x_{n+(k+1)}-x_{n+(2k+2)}} \\
 &= -\frac{(x_{n+(k+1)}y_n)/(y_n-y_{n+(k+1)})y_{n+(k+1)}}{x_{n+(k+1)}-(x_{n+(k+1)}y_n)/(y_n-y_{n+(k+1)})} = y_n.
 \end{aligned}$$

Thus, $x_{n+(6k+6)} = x_n$, $y_{n+(6k+6)} = y_n$.

(ii) From system (2.1), we obtain that

$$x_1 = \frac{x-ky-(2k+1)}{y-(2k+1)-y-k}, \quad y_1 = \frac{y-kx-(2k+1)}{x-(2k+1)-x-k},$$

$$\begin{aligned}
 x_2 &= \frac{x_{1-k}y_{-2k}}{y_{-2k} - y_{1-k}}, & y_2 &= \frac{y_{1-k}x_{-2k}}{x_{-2k} - x_{1-k}}, \\
 &\vdots & &\vdots \\
 x_{k+1} &= \frac{x_0y_{-(k+1)}}{y_{-(k+1)} - y_0}, & y_{k+1} &= \frac{y_0x_{-(k+1)}}{x_{-(k+1)} - x_0},
 \end{aligned}$$

for $n = 0, 1, \dots, k$. Then, it follows from Theorem (2.1) that

$$\begin{aligned}
 x_1 &= x_{6(k+1)+1} = x_{6(k+1)2+1} = \dots = \frac{x_{-k}y_{-(2k+1)}}{y_{-(2k+1)} - y_{-k}}, \\
 y_1 &= y_{6(k+1)+1} = y_{6(k+1)2+1} = \dots = \frac{y_{-k}x_{-(2k+1)}}{x_{-(2k+1)} - x_{-k}}, \\
 x_2 &= x_{6(k+1)+2} = x_{6(k+1)2+2} = \dots = \frac{x_{1-k}y_{-2k}}{y_{-2k} - y_{1-k}}, \\
 y_2 &= y_{6(k+1)+2} = y_{6(k+1)2+2} = \dots = \frac{y_{1-k}x_{-2k}}{x_{-2k} - x_{1-k}}, \\
 &\vdots \\
 x_{k+1} &= x_{6(k+1)+k+1} = x_{6(k+1)2+k+1} = \dots = \frac{x_0y_{-(k+1)}}{y_{-(k+1)} - y_0}, \\
 y_{k+1} &= y_{6(k+1)+k+1} = y_{6(k+1)2+k+1} = \dots = \frac{y_0x_{-(k+1)}}{x_{-(k+1)} - x_0}.
 \end{aligned}$$

Thus, we have formulas for $1 \leq l \leq k+1$ in (2.2) and (2.3). From (2.1), we get

(2.4)

$$\begin{aligned}
 x_{n+1} &= \frac{x_{(n-k-1)+1}y_{n-(2k+1)}}{y_{n-(2k+1)} - y_{(n-k-1)+1}} \\
 &= \frac{(x_{n-(2k+1)}y_{n-(3k+2)}) / (y_{n-(3k+2)} - y_{n-(2k+1)})y_{n-(2k+1)}}{y_{n-(2k+1)} - (y_{n-(2k+1)}x_{n-(3k+2)}) / (x_{n-(3k+2)} - x_{n-(2k+1)})} \\
 &= -\frac{y_{n-(3k+2)}(x_{n-(3k+2)} - x_{n-(2k+1)})}{y_{n-(3k+2)} - y_{n-(2k+1)}}
 \end{aligned}$$

and

(2.5)

$$y_{n+1} = \frac{y_{(n-k-1)+1}x_{n-(2k+1)}}{x_{n-(2k+1)} - x_{(n-k-1)+1}}$$

$$\begin{aligned} &= \frac{(y_{n-(2k+1)}x_{n-(3k+2)})/(x_{n-(3k+2)} - x_{n-(2k+1)})x_{n-(2k+1)}}{x_{n-(2k+1)} - (x_{n-(2k+1)}y_{n-(3k+2)})/(y_{n-(3k+2)} - y_{n-(2k+1)})} \\ &= -\frac{x_{n-(3k+2)}(y_{n-(3k+2)} - y_{n-(2k+1)})}{x_{n-(3k+2)} - x_{n-(2k+1)}}. \end{aligned}$$

Let $n = k + 1, k + 2, \dots, 2k + 1$. Now, from (2.4) and (2.5), we have

$$\begin{aligned} x_{k+2} &= -\frac{y_{-(2k+1)}(x_{-(2k+1)} - x_{-k})}{y_{-(2k+1)} - y_{-k}}, & y_{k+2} &= -\frac{x_{-(2k+1)}(y_{-(2k+1)} - y_{-k})}{x_{-(2k+1)} - x_{-k}}, \\ x_{k+3} &= -\frac{y_{-2k}(x_{-2k} - x_{1-k})}{y_{-2k} - y_{1-k}}, & y_{k+3} &= -\frac{x_{-2k}(y_{-2k} - y_{1-k})}{x_{-2k} - x_{1-k}}, \\ &\vdots & &\vdots \\ x_{2k+2} &= -\frac{y_{-(k+1)}(x_{-(k+1)} - x_0)}{y_{-(k+1)} - y_0}, & y_{2k+2} &= -\frac{x_{-(k+1)}(y_{-(k+1)} - y_0)}{x_{-(k+1)} - x_0}. \end{aligned}$$

From Theorem 2.1, we obtain

$$\begin{aligned} x_{k+2} &= x_{6(k+1)+k+2} = x_{6(k+1)2+k+2} = \dots \\ &= -\frac{y_{-(2k+1)}(x_{-(2k+1)} - x_{-k})}{y_{-(2k+1)} - y_{-k}}, \\ y_{k+2} &= y_{6(k+1)+k+2} = y_{6(k+1)2+k+2} = \dots \\ &= -\frac{x_{-(2k+1)}(y_{-(2k+1)} - y_{-k})}{x_{-(2k+1)} - x_{-k}}, \\ x_{k+3} &= x_{6(k+1)+k+3} = x_{6(k+1)2+k+3} = \dots \\ &= -\frac{y_{-2k}(x_{-2k} - x_{1-k})}{y_{-2k} - y_{1-k}}, \\ y_{k+3} &= y_{6(k+1)+k+3} = y_{6(k+1)2+k+3} = \dots \\ &= -\frac{x_{-2k}(y_{-2k} - y_{1-k})}{x_{-2k} - x_{1-k}}, \\ &\vdots \\ x_{2k+2} &= x_{6(k+1)+2k+2} = x_{6(k+1)2+2k+2} = \dots \\ &= -\frac{y_{-(k+1)}(x_{-(k+1)} - x_0)}{y_{-(k+1)} - y_0}, \\ y_{2k+2} &= y_{6(k+1)+2k+2} = y_{6(k+1)2+2k+2} = \dots \\ &= -\frac{x_{-(k+1)}(y_{-(k+1)} - y_0)}{x_{-(k+1)} - x_0}. \end{aligned}$$

Hence, we obtain the formulas for $k + 2 \leq l \leq 2k + 2$ in (2.2) and (2.3). From (2.1), (2.4) and (2.5), we can see that

$$\begin{aligned}
 (2.6) \quad x_{n+1} &= -\frac{y_{n-(3k+2)}(x_{n-(3k+2)} - x_{n-(2k+1)})}{y_{n-(3k+2)} - y_{n-(2k+1)}} \\
 &= -\frac{y_{n-(3k+2)}(x_{n-(3k+2)} - (x_{n-(3k+2)}y_{n-(4k+3)})/(y_{n-(4k+3)} - y_{n-(3k+2)}))}{y_{n-(3k+2)} - (x_{n-(3k+2)}x_{n-(4k+3)})/(x_{n-(4k+3)} - x_{n-(3k+2)})} \\
 &= -\frac{y_{n-(3k+2)}(x_{n-(4k+3)} - x_{n-(3k+2)})}{y_{n-(4k+3)} - y_{n-(3k+2)}}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.7) \quad y_{n+1} &= -\frac{x_{n-(3k+2)}(y_{n-(3k+2)} - y_{n-(2k+1)})}{x_{n-(3k+2)} - x_{n-(2k+1)}} \\
 &= -\frac{x_{n-(3k+2)}(y_{n-(3k+2)} - (y_{n-(3k+2)}x_{n-(4k+3)})/(x_{n-(4k+3)} - x_{n-(3k+2)}))}{x_{n-(3k+2)} - (x_{n-(3k+2)}y_{n-(4k+3)})/(y_{n-(4k+3)} - y_{n-(3k+2)})} \\
 &= -\frac{x_{n-(3k+2)}(y_{n-(4k+3)} - y_{n-(3k+2)})}{x_{n-(4k+3)} - x_{n-(3k+2)}}.
 \end{aligned}$$

Let $n = 2k + 2, 2k + 3, \dots, 3k + 2$. Now, from (2.6) and (2.7), we get

$$\begin{aligned}
 x_{2k+3} &= -\frac{y_{-k}(x_{-(2k+1)} - x_{-k})}{y_{-(2k+1)} - y_{-k}}, & y_{2k+3} &= -\frac{x_{-k}(y_{-(2k+1)} - y_{-k})}{x_{-(2k+1)} - x_{-k}}, \\
 x_{2k+4} &= -\frac{y_{1-k}(x_{-2k} - x_{1-k})}{y_{-2k} - y_{1-k}}, & y_{2k+4} &= -\frac{x_{1-k}(y_{-2k} - y_{1-k})}{x_{-2k} - x_{1-k}}, \\
 &\vdots & &\vdots \\
 x_{3k+3} &= -\frac{y_0(x_{-(k+1)} - x_0)}{y_{-(k+1)} - y_0}, & y_{3k+3} &= -\frac{x_0(y_{-(k+1)} - y_0)}{x_{-(k+1)} - x_0}.
 \end{aligned}$$

Thus, we obtain from Theorem (2.1) that

$$\begin{aligned}
 x_{2k+3} &= x_{6(k+1)+2k+3} = x_{6(k+1)2+2k+3} = \dots \\
 &= -\frac{y_{-k}(x_{-(2k+1)} - x_{-k})}{y_{-(2k+1)} - y_{-k}}, \\
 y_{2k+3} &= y_{6(k+1)+2k+3} = y_{6(k+1)2+2k+3} = \dots \\
 &= -\frac{x_{-k}(y_{-(2k+1)} - y_{-k})}{x_{-(2k+1)} - x_{-k}}, \\
 x_{2k+4} &= x_{6(k+1)+2k+4} = x_{6(k+1)2+2k+4} = \dots
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{y_{1-k}(x_{-2k} - x_{1-k})}{y_{-2k} - y_{1-k}}, \\
 y_{2k+4} &= y_{6(k+1)+2k+4} = y_{6(k+1)2+2k+4} = \dots \\
 &= -\frac{x_{1-k}(y_{-2k} - y_{1-k})}{x_{-2k} - x_{1-k}}, \\
 &\quad \vdots \\
 x_{3k+3} &= x_{6(k+1)+3k+3} = x_{6(k+1)2+3k+3} = \dots \\
 &= -\frac{y_0(x_{-(k+1)} - x_0)}{y_{-(k+1)} - y_0}, \\
 y_{3k+3} &= y_{6(k+1)+3k+3} = y_{6(k+1)2+3k+3} = \dots \\
 &= -\frac{x_0(y_{-(k+1)} - y_0)}{x_{-(k+1)} - x_0}.
 \end{aligned}$$

Hence, we have the formulas for $2k + 3 \leq l \leq 3k + 3$ in (2.2) and (2.3). From (2.1), (2.6) and (2.7), we have

(2.8)

$$\begin{aligned}
 x_{n+1} &= \frac{-y_{n-(3k+2)}(x_{n-(4k+3)} - x_{n-(3k+2)})}{y_{n-(4k+3)} - y_{n-(3k+2)}} \\
 &= -\frac{(y_{n-(4k+3)}x_{n-(5k+4)}) / (x_{n-(5k+4)} - x_{n-(4k+3)})}{y_{n-(4k+3)} - (y_{n-(4k+3)}x_{n-(5k+4)}) / (x_{n-(5k+4)} - x_{n-(4k+3)})} \\
 &\quad \times (x_{n-(4k+3)} - (x_{n-(4k+3)}y_{n-(5k+4)}) / (y_{n-(5k+4)} - y_{n-(4k+3)})) \\
 &= -\frac{y_{n-(4k+3)}x_{n-(5k+4)}}{y_{n-(5k+4)} - y_{n-(4k+3)}}
 \end{aligned}$$

and

(2.9)

$$\begin{aligned}
 y_{n+1} &= -\frac{x_{n-(3k+2)}(y_{n-(4k+3)} - y_{n-(3k+2)})}{x_{n-(4k+3)} - x_{n-(3k+2)}} \\
 &= -\frac{(x_{n-(4k+3)}y_{n-(5k+4)}) (y_{n-(5k+4)} - y_{n-(4k+3)})}{x_{n-(4k+3)} - (x_{n-(4k+3)}y_{n-(5k+4)}) (y_{n-(5k+4)} - y_{n-(4k+3)})} \\
 &\quad \times (y_{n-(4k+3)} - (y_{n-(4k+3)}x_{n-(5k+4)}) / (x_{n-(5k+4)} - x_{n-(4k+3)})) \\
 &= -\frac{x_{n-(4k+3)}y_{n-(5k+4)}}{x_{n-(5k+4)} - x_{n-(4k+3)}}.
 \end{aligned}$$

Let $n = 3k + 3, 3k + 4, \dots, 4k + 3$. It follows from (2.8) and (2.9) that

$$x_{3k+4} = -\frac{y_{-k}x_{-(2k+1)}}{y_{-(2k+1)} - y_{-k}}, \quad y_{3k+4} = -\frac{x_{-k}y_{-(2k+1)}}{x_{-(2k+1)} - x_{-k}},$$

$$\begin{aligned}
 x_{3k+5} &= -\frac{y_{1-k}x_{-2k}}{y_{-2k} - y_{1-k}}, & y_{3k+5} &= -\frac{x_{1-k}y_{-2k}}{x_{-2k} - x_{1-k}}, \\
 &\vdots & &\vdots \\
 x_{4k+4} &= -\frac{y_0x_{-(k+1)}}{y_{-(k+1)} - y_0}, & y_{4k+4} &= -\frac{x_0y_{-(k+1)}}{x_{-(k+1)} - x_0}.
 \end{aligned}$$

Then, from Theorem 2.1, we get

$$\begin{aligned}
 x_{3k+4} &= x_{6(k+1)+3k+4} = x_{6(k+1)2+3k+4} = \cdots \\
 &= -\frac{y_{-k}x_{-(2k+1)}}{y_{-(2k+1)} - y_{-k}}, \\
 y_{3k+4} &= y_{6(k+1)+3k+4} = y_{6(k+1)2+3k+4} = \cdots \\
 &= -\frac{x_{-k}y_{-(2k+1)}}{x_{-(2k+1)} - x_{-k}}, \\
 x_{3k+5} &= x_{6(k+1)+3k+5} = x_{6(k+1)2+3k+5} = \cdots \\
 &= -\frac{y_{1-k}x_{-2k}}{y_{-2k} - y_{1-k}}, \\
 y_{3k+5} &= y_{6(k+1)+3k+5} = y_{6(k+1)2+3k+5} = \cdots \\
 &= -\frac{x_{1-k}y_{-2k}}{x_{-2k} - x_{1-k}}, \\
 &\vdots \\
 x_{4k+4} &= x_{6(k+1)+4k+4} = x_{6(k+1)2+4k+4} = \cdots \\
 &= -\frac{y_0x_{-(k+1)}}{y_{-(k+1)} - y_0}, \\
 y_{4k+4} &= y_{6(k+1)+4k+4} = y_{6(k+1)2+4k+4} = \cdots \\
 &= -\frac{x_0y_{-(k+1)}}{x_{-(k+1)} - x_0}.
 \end{aligned}$$

Thus we have the formulas for $3k + 4 \leq l \leq 4k + 4$ in (2.2) and (2.3). From (2.1), (2.8) and (2.9), we have

$$(2.10) \quad x_{n+1} = -\frac{y_{n-(4k+3)}x_{n-(5k+4)}}{y_{n-(5k+4)} - y_{n-(4k+3)}} = x_{n-(6k+5)},$$

and

$$(2.11) \quad y_{n+1} = -\frac{x_{n-(4k+3)}y_{n-(5k+4)}}{x_{n-(5k+4)} - x_{n-(4k+3)}} = y_{n-(6k+5)}.$$

Let $n = 4k + 4, 4k + 5, \dots, 6k + 5$. It follows from (2.10) and (2.11) that

$$\begin{aligned} x_{4k+5} &= x_{-(2k+1)}, & y_{4k+5} &= y_{-(2k+1)}, \\ x_{4k+6} &= x_{-2k}, & y_{4k+6} &= y_{-2k}, \\ & \vdots & & \vdots \\ x_{6k+6} &= x_0, & y_{6k+6} &= y_0. \end{aligned}$$

Then, from Theorem (2.1), we obtain

$$\begin{aligned} x_{4k+5} &= x_{6(k+1)+4k+5} = x_{6(k+1)2+4k+5} = \dots = x_{-(2k+1)}, \\ y_{4k+5} &= y_{6(k+1)+4k+5} = y_{6(k+1)2+4k+5} = \dots = y_{-(2k+1)}, \\ x_{4k+6} &= x_{6(k+1)+4k+6} = x_{6(k+1)2+4k+6} = \dots = x_{-2k}, \\ y_{4k+6} &= y_{6(k+1)+4k+6} = y_{6(k+1)2+4k+6} = \dots = y_{-2k}, \\ & \vdots \\ x_{6k+6} &= x_{6(k+1)+6k+6} = x_{6(k+1)2+6k+6} = \dots = x_0, \\ y_{6k+6} &= y_{6(k+1)+6k+6} = y_{6(k+1)2+6k+6} = \dots = y_0. \end{aligned}$$

Hence, we have the formulas for $4k + 5 \leq l \leq 6k + 6$ in (2.2) and (2.3). The proof is complete. □

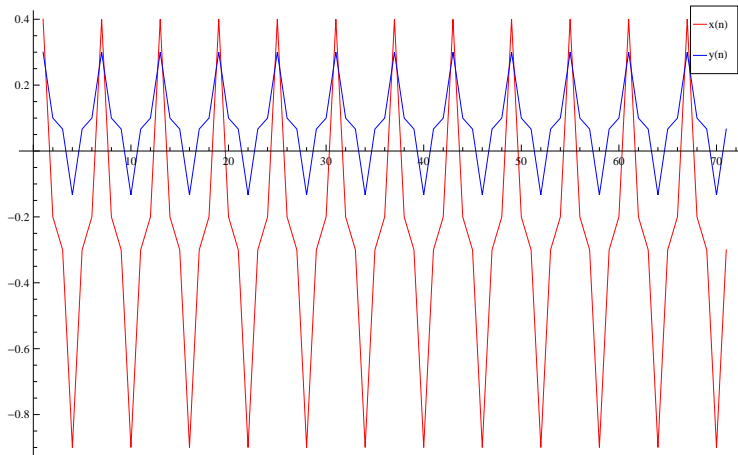


FIGURE 1. Plot of $x_{n+1} = (x_n y_{n-1}) / (y_{n-1} - y_n)$, $y_{n+1} = (y_n x_{n-1}) / (x_{n-1} - x_n)$.

Example 2.2. Consider the system (2.1) with $k = 0$ and the initial conditions $x_{-1} = -0.2, x_0 = 0.4, y_{-1} = 0.1, y_0 = 0.3$ to verify our theoretical results. (See Figure 1.)

3. The system: $x_{n+1} = -(x_{n-k}y_{n-(2k+1)})/(y_{n-(2k+1)} + y_{n-k}), y_{n+1} = -(y_{n-k}x_{n-(2k+1)})/(x_{n-(2k+1)} + x_{n-k})$. In this section, we study the periodic nature of solutions of the following rational difference equations system:

$$(3.1) \quad x_{n+1} = -\frac{x_{n-k}y_{n-(2k+1)}}{y_{n-(2k+1)} + y_{n-k}}, \quad y_{n+1} = -\frac{y_{n-k}x_{n-(2k+1)}}{x_{n-(2k+1)} + x_{n-k}},$$

where $n, k \in \mathbb{N}_0$, and the initial conditions $x_{-2k-1}, x_{-2k}, \dots, x_0, y_{-2k-1}, y_{-2k}, \dots, y_0$ are nonzero real numbers such that

$$(y_{-(2k+1-m)} + y_{-(k-m)})(x_{-(2k+1-m)} + x_{-(k-m)}) \neq 0, \quad m = \overline{0, k}.$$

Theorem 3.1. Let $\{x_n, y_n\}_{n=-(2k+1)}^{+\infty}$ be solutions of system (3.1). Then, the following statements hold:

(i) $\{x_n\}_{n=-(2k+1)}^{+\infty}$ and $\{y_n\}_{n=-(2k+1)}^{+\infty}$ are periodic with period $6(k+1)$ for $n \geq -(2k+1)$.

(ii) We have

$$x_{6(k+1)n+l} = \begin{cases} -\frac{x_{(l-k-1)}y_{(l-2k-2)}}{y_{(l-2k-2)}+y_{(l-k-1)}} & 1 \leq l \leq k+1, \\ \frac{y_{(l-3k-3)}(x_{(l-3k-3)}+x_{(l-2k-2)})}{y_{(l-3k-3)}+y_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ \frac{y_{(l-3k-3)}(x_{(l-4k-4)}+x_{(l-3k-3)})}{y_{(l-4k-4)}+y_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ -\frac{y_{(l-4k-4)}x_{(l-5k-5)}}{y_{(l-5k-5)}+y_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ x_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6 \end{cases}$$

and

$$y_{6(k+1)n+l} = \begin{cases} -\frac{y_{(l-k-1)}x_{(l-2k-2)}}{x_{(l-2k-2)}+x_{(l-k-1)}} & 1 \leq l \leq k+1, \\ \frac{x_{(l-3k-3)}(y_{(l-3k-3)}+y_{(l-2k-2)})}{x_{(l-3k-3)}+x_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ \frac{x_{(l-3k-3)}(y_{(l-4k-4)}+y_{(l-3k-3)})}{x_{(l-4k-4)}+x_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ -\frac{x_{(l-4k-4)}y_{(l-5k-5)}}{x_{(l-5k-5)}+x_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ y_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6. \end{cases}$$

Proof. The proof is similar to Theorem 2.1, and therefore, it is omitted. □

Example 3.2. Consider the system (3.1) with $k = 1$ and the initial conditions $x_{-3} = 0.9, x_{-2} = 0.3, x_{-1} = 0.2, x_0 = 0.5, y_{-3} = 3.9, y_{-2} = 3.3, y_{-1} = 3.2$ and $y_0 = 2.5$ to verify our theoretical results. (See Figure 2.)

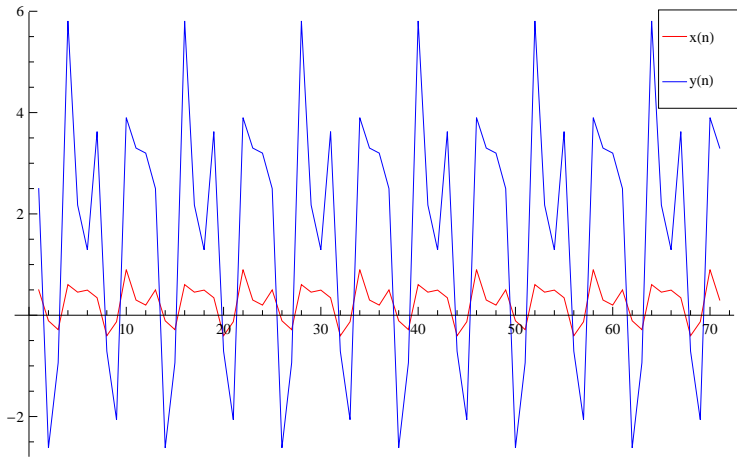


FIGURE 2. Plot of $x_{n+1} = (-x_{n-1}y_{n-3})/(y_{n-3} + y_{n-1}), y_{n+1} = -(y_{n-1}x_{n-3})/(x_{n-3} + x_{n-1})$.

4. The other systems: $x_{n+1} = (x_{n-k}y_{n-(2k+1)})/(\pm y_{n-(2k+1)} + y_{n-k}), y_{n+1} = (y_{n-k}x_{n-(2k+1)})/(\mp x_{n-(2k+1)} + x_{n-k})$. In this section, we give the form of the periodic solutions of the following rational difference equations system:

$$(4.1) \quad x_{n+1} = \frac{x_{n-k}y_{n-(2k+1)}}{\pm y_{n-(2k+1)} + y_{n-k}}, \quad y_{n+1} = \frac{y_{n-k}x_{n-(2k+1)}}{\mp x_{n-(2k+1)} + x_{n-k}},$$

where $n, k \in \mathbb{N}_0$ and the initial conditions $x_{-2k-1}, x_{-2k}, \dots, x_0, y_{-2k-1}, y_{-2k}, \dots, y_0$ are nonzero real numbers such that

$$(\pm y_{-(2k+1-m)} + y_{-(k-m)})(\mp x_{-(2k+1-m)} + x_{-(k-m)}) \neq 0, \quad m = \overline{0, k}.$$

Corollary 4.1. *The solutions $\{x_n, y_n\}_{n=-(2k+1)}^{+\infty}$ of system (4.1) are periodic with period $6(k+1)$ for $n \geq -(2k+1)$. Then, the form of the solutions are as follows:*

$$x_{6(k+1)n+l} = \begin{cases} \frac{x_{(l-k-1)}y_{(l-2k-2)}}{\pm y_{(l-2k-2)} + y_{(l-k-1)}} & 1 \leq l \leq k+1, \\ \frac{\pm y_{(l-3k-3)}(\mp x_{(l-3k-3)} + x_{(l-2k-2)})}{\pm y_{(l-3k-3)} + y_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ \frac{\mp x_{(l-3k-3)}(\mp x_{(l-4k-4)} + x_{(l-3k-3)})}{\pm y_{(l-4k-4)} + y_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ \frac{\pm y_{(l-4k-4)}x_{(l-5k-5)}}{\pm y_{(l-5k-5)} + y_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ x_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6 \end{cases}$$

and

$$y_{6(k+1)n+l} = \begin{cases} \frac{y_{(l-k-1)}x_{(l-2k-2)}}{\mp x_{(l-2k-2)} + x_{(l-k-1)}} & 1 \leq l \leq k+1, \\ \frac{\mp x_{(l-3k-3)}(\pm y_{(l-3k-3)} + y_{(l-2k-2)})}{\mp x_{(l-3k-3)} + x_{(l-2k-2)}} & k+2 \leq l \leq 2k+2, \\ \frac{\pm x_{(l-3k-3)}(\pm y_{(l-4k-4)} + y_{(l-3k-3)})}{\mp x_{(l-4k-4)} + x_{(l-3k-3)}} & 2k+3 \leq l \leq 3k+3, \\ \frac{\mp x_{(l-4k-4)}y_{(l-5k-5)}}{\mp x_{(l-5k-5)} + x_{(l-4k-4)}} & 3k+4 \leq l \leq 4k+4, \\ y_{(l-6k-6)} & 4k+5 \leq l \leq 6k+6. \end{cases}$$

Proof. The proof is similar to Theorem 2.1; thus, the desired results are obtained. □

REFERENCES

1. R.P. Agarwal and E.M. Elsayed, *Periodicity and stability of solutions of higher order rational difference equations*, Adv. Stud. Contemp. Math. **17** (2008), 181–201.
2. C. Cinar, *On the positive solutions of the difference equation system $x_{n+1} = \frac{1}{y_n}$, $y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}$* , Appl. Math. Comp. **158** (2004), 303–305.
3. M.M. El-Dessoky and E.M. Elsayed, *On the solutions and periodic nature of some systems of rational difference equations*, J. Comp. Anal. Appl. **18** (2015), 206–218.
4. H. El-Metwally, *Solutions form for some rational systems of difference equations*, Discr. Dyn. Nat. Soc. **2013**, article ID 903593, 10 pages.

5. H. El-Metwally, E.M. Elsayed and E.M. Elabbasy, *On the solutions of difference equations of order four*, Rocky Mountain J. Math. **43** (2013), 877–894.
6. E.M. Elsayed, *Dynamics and behavior of a higher order rational difference equation*, J. Nonlin. Sci. Appl. **9** (2016), 1463–1474.
7. ———, *On the solutions and periodic nature of some systems of difference equations*, Intern. J. Biomath. **7** (2014), 1450067, 26 pages.
8. A.S. Kurbanli, C. Cinar and D. Simsek, *On the periodicity of solutions of the system of rational difference equations $x_{n+1} = \frac{x_{n-1}+y_n}{y_n x_{n-1}-1}$, $y_{n+1} = \frac{y_{n-1}+x_n}{x_n y_{n-1}-1}$* , Appl. Math. **2** (2011), 410–413.
9. A.Y. Ozban, *On the positive solutions of the system of rational difference equations $x_{n+1} = \frac{1}{y_{n-k}}$, $y_{n+1} = \frac{y_n}{x_{n-m}y_{n-m-k}}$* , J. Math. Anal. Appl. **323** (2006), 26–32.
10. S. Stevic, M.A. Alghamdi, D.A. Maturi and N. Shahzad, *On the periodicity of some classes of systems of nonlinear difference equations*, Abstr. Appl. Anal. (2014), article ID 982378, 6 pages.
11. N. Touafek and E.M. Elsayed, *On a third order rational systems of difference equation*, Anal. Stiint. Univ. Iași Matem. **61** (2015), 367–380.
12. H. Yacine, *Form and periodicity of solutions of some systems of higher-order difference equation*, Math. Sci. Lett. **5** (2016), 79–84.
13. X. Yang, Y. Liu and S. Bai, *On the system of higher order rational difference equation $x_n = \frac{a}{y_{n-p}}$, $y_n = \frac{by_{n-p}}{x_{n-q}y_{n-q}}$* , Appl. Math. Comp. **171** (2005), 853–856.

BULENT ECEVIT UNIVERSITY, DEPARTMENT OF MATHEMATICS, ZONGULDAK 67100, TURKEY

Email address: gocenm@hotmail.com

BULENT ECEVIT UNIVERSITY, DEPARTMENT OF MATHEMATICS, ZONGULDAK 67100, TURKEY

Email address: cebeciadem19@gmail.com