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A NOTE ON SUMS OF ROOTS

TIMOTHY FERDINANDS AND ANNETTE PILKINGTON

ABSTRACT. In this paper, we look at properties of roots which can be written as sums of roots in crystallographic root systems. We derive properties of the poset associated to such a sum consisting of the subsums which are themselves roots.

1. Introduction. Bourbaki [**1,** Chapter VI] discussed basic properties of crystallographic root systems of finite Weyl groups which are of fundamental importance in Lie theory. In particular, it was shown that, given a set of positive roots $\alpha_1, \alpha_2, \dots, \alpha_n$, in a crystallographic root system Φ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is a root, we can find a permutation π of the indices $1, \dots, n$, such that $\alpha_{\pi(1)} + \alpha_{\pi(2)} + \cdots + \alpha_{\pi(i)}$ is a root for $1 \leq i \leq n$. In this paper, we include a generalization of this result due to Dyer, and we examine the structure of a poset associated to such a set of roots.

If Φ is a crystallographic root system and $\alpha_1, \alpha_2, \dots, \alpha_n \in \Phi$ has the property that $\alpha_1 + \alpha_2 + \cdots + \alpha_n \in \Phi$, we can define an associated poset. Let $\alpha_I = \sum_{i \in I} \alpha_i$ for $I \subseteq [n] = \{1, 2, \dots, n\}$. The set

$$C = \{I \subseteq [n] \mid \alpha_I \in \Phi\}$$

forms a poset under the containment partial order. In this paper, we show that, for a given i with $1 \leq i \leq n$, the cardinality of the set $\{I \in C \mid |I| = i\}$ is always greater than or equal to $n - i + 1$, and C is a graded poset. Furthermore, given any $k \in [n]$ and any $1 \leq i \leq n$, there is at least one $I \in C$ with $|I| = i$ and $k \in I$. We show that, when the root system Φ is of type A_n, the poset C is a lattice, but give a counterexample to show that, this is not the case in general.

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2. Notation and definitions.

Definition 2.1. Let V be a finite-dimensional vector space over $\mathbb{R}$ with a positive definite, symmetric bilinear form $(\_|\_) : V \times V \to \mathbb{R}$. Let Φ be a subset of V. Then, Φ is said to be *a root system* in V if the following conditions are satisfied:

(1) Φ is finite, does not contain 0, and spans V.
(2) For all $\alpha \in \Phi$, the reflection $s_\alpha : V \to V$, defined by $s_\alpha(v) = v - (\alpha \mid \alpha^\vee)\alpha$, where
$$\alpha^\vee = \frac{2\alpha}{(\alpha \mid \alpha)},$$
leaves Φ stable.
(3) For $\alpha, \phi \in \Phi$, $(\phi \mid \alpha^\vee) \in \mathbb{Z}$.

This type of root system is frequently referred to as a *crystallographic root system.*

We can choose a system Φ^+ of positive roots for Φ as in Bourbaki [**1,** Theorem 3]. Then, $\Phi = \Phi^+ \cup \Phi^-$ is a disjoint union where $-\Phi^+ = \Phi^-$. If $\alpha, \beta \in \Phi$ such that $\beta = c\alpha$, where $c \in \mathbb{R}$, then $c \in \{\pm 1, \pm 1/2, \pm 2\}$ [**1,** Proposition 8]. If a root $\alpha \in \Phi$ is such that $1/2\alpha \notin \Phi$, then α is called *an indivisible root.*

Definition 2.2. A root system Φ is *reduced* if every root of the system is indivisible.

We let $\Phi' = \Phi \cup \{0\}$. If Φ is reduced, Φ' is the set of weights of the adjoint representation of the corresponding semisimple complex Lie algebra. The set $\{i \in \mathbb{Z} \mid 1 \le i \le n\}$ will be denoted by $[n]$. For $\pi \in S_n$, the permutations of $[n]$, and $I \subseteq [n]$, let $\pi(I)$ denote $\{\pi(j) \mid j \in I\}$. If $\alpha_i \in \Phi'$ for $i \in [n]$ and $I \subseteq [n]$, we let $\alpha_I = \sum_{i \in I} \alpha_i$ (here, $\alpha_\emptyset = 0$).

3. Sums of roots. If $\alpha \in \Phi$ and $\beta \in \Phi'$, we will refer to the set $\{\beta + k\alpha \mid k \in \mathbb{Z}\} \cap \Phi'$ as a root string. We have the following results on root strings from Bourbaki [**1**] and from Dyer (unpublished).

Lemma 3.1 ([**1,** Chapter VI, Propositions 8, 9 and Theorem 1])**.** *Let $\alpha \in \Phi$ and $\beta \in \Phi'$.*

(i) $\{k \in \mathbb{Z} \mid \beta + k\alpha \in \Phi'\} = [-q, p]$ *for some* $p, q \geq 0$, $p, q \in \mathbb{Z}$, *with* $p - q = -\langle \beta, \alpha^\vee \rangle$.

(ii) *If* $\langle \beta, \alpha^\vee \rangle > 0$, *then* $\beta - \alpha \in \Phi'$ *and, if* $\langle \beta, \alpha^\vee \rangle < 0$, *then* $\beta + \alpha \in \Phi'$.

Proposition 3.2 ([**1,** Chapter VI, Proposition 19])**.** *Let* $\{\beta_i\}_{1 \leq i \leq n}$ *be a sequence of positive roots such that* $\beta_{[n]}$ *is a root. Then, there exists a permutation* $\pi \in S_n$ *such that* $\beta_{\pi([i])}$ *is a root for* $1 \leq i \leq n$.

A version of the following lemma involving more restrictive hypotheses is applied in [**3**] to give an elementary proof, independent of the theory of semisimple Lie algebras, of a lemma of Oshima on parabolic subgroup orbits on finite root systems. The main result of this paper, Theorem 3.7, extends part (ii) of Lemma 3.3 from a set of three roots to a set of n roots. It would be interesting to have a proof of Theorem 3.7 in the case where Φ is reduced using semisimple complex Lie algebras, similar to that stated in [**3,** Remark 4.2(2)] for the case $n = 3$.

Lemma 3.3 ([**3**])**.** *Let* $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Phi'$ *be such that* $\alpha_{[n]} \in \Phi'$. *Then*:

(i) *there exists a permutation* $\pi \in S_n$ *such that* $\alpha_{\pi([i])} \in \Phi'$ *for* $1 \leq i \leq n$.

(ii) *Suppose that* $n = 3$ *and* $\alpha_1 + \alpha_2 \in \Phi$, *but* $\alpha_2 + \alpha_3 \notin \Phi'$. *Then,* $\alpha_1 + \alpha_3 \in \Phi'$.

(iii) *Assume that* $\alpha_I \neq 0$ *if* $\emptyset \subsetneq I \subseteq [n]$ *and* $\alpha_i + \alpha_j \notin \Phi$ *for any* i, j *with* $2 \leq i < j \leq n$. *Then,* $\alpha_I \in \Phi$ *for all* $I \subseteq [n]$ *with* $1 \in I$.

Note 3.4. If $\alpha_i \in \Phi^+$ for all $i \in [n]$, then, for $\emptyset \neq I \subseteq [n]$, $\alpha_I \in \Phi' \Leftrightarrow \alpha_I \in \Phi^+$.

Proof. The proof is trivial if $n \leq 2$. If $n \geq 2$, we use induction on n.

(i)

Case 1. $\alpha = \alpha_{[n]} \neq 0$. Here, $(\alpha, \alpha) = \sum_{i \in [n]} (\alpha, \alpha_i) > 0$; thus, there exists an $i \in [n]$ with $(\alpha, \alpha_i) > 0$. Without loss of generality, we may assume that $i = n$. Then, $(\alpha, \alpha_n) > 0$, which implies that $\langle \alpha, \alpha_n^\vee \rangle > 0$, which, in turn, implies that $\alpha_{[n-1]} = \alpha - \alpha_n \in \Phi'$. By induction, there exists a $\widehat{\sigma} \in S_{n-1}$ such that $\alpha_{\widehat{\sigma}([i])} \in \Phi'$ for $i \in [n-1]$. We can extend $\widehat{\sigma}$ to $\sigma \in S_n$ with $\sigma(n) = n$. Then, $\alpha_{\sigma([i])} \in \Phi'$ for $i \in [n]$.

Case 2. $\alpha_{[n]} = 0$. In this case, $\alpha_{[n-1]} = -\alpha_n \in \Phi'$. By induction, there exists a $\widehat{\sigma} \in S_{n-1}$ with $\alpha_{\hat{\sigma}([i])} \in \Phi'$ for $i \in [n-1]$. We can extend $\widehat{\sigma}$ to $\sigma \in S_n$ with $\sigma(n) = n$. Then, $\alpha_{\sigma([i])} \in \Phi'$ for $i \in [n]$.

(ii) Suppose that $\alpha_1 + \alpha_2 \in \Phi$, $\alpha_2 + \alpha_3 \notin \Phi'$ and $\alpha_1 + \alpha_3 \notin \Phi'$. Since $\alpha_2 \in \Phi'$ and $\alpha_2 + \alpha_3 \notin \Phi'$, $\alpha_3 \neq 0$; thus, $\alpha_3 \in \Phi$. Hence, $\alpha_3 + \alpha_2 \notin \Phi'$, $\alpha_2 \neq 0$ and $\alpha_2 \in \Phi$. By symmetry, $\alpha_1 \in \Phi$ also. Since $\alpha_2 + \alpha_3 \notin \Phi'$, we must have $\langle \alpha_2, \alpha_3^\vee \rangle \geq 0$, and similarly, we must have $\langle \alpha_1, \alpha_3^\vee \rangle \geq 0$. Since $\alpha_1 + \alpha_2 + \alpha_3 \in \Phi'$, $-\alpha_2 \in \Phi$ and $(\alpha_1 + \alpha_2 + \alpha_3) + (-\alpha_2) \notin \Phi'$, we must have $\langle \alpha_1 + \alpha_2 + \alpha_3, -\alpha_2^\vee \rangle \geq 0$. Similarly, $\langle \alpha_1 + \alpha_2 + \alpha_3, -\alpha_1^\vee \rangle \geq 0$. Thus, we have

$$(\alpha_2, \alpha_3) \geq 0, \qquad (\alpha_1, \alpha_3) \geq 0,$$

$$(\alpha_1 + \alpha_2 + \alpha_3, \alpha_2) \leq 0, \qquad (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1) \leq 0.$$

Hence,

$$(\alpha_1 + \alpha_2, \alpha_2) \leq -(\alpha_3, \alpha_2) \leq 0$$

and

$$(\alpha_1 + \alpha_2, \alpha_1) \leq -(\alpha_3, \alpha_1) \leq 0.$$

Therefore, we have

$$(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) = (\alpha_1 + \alpha_2, \alpha_1) + (\alpha_1 + \alpha_2, \alpha_2) \leq 0,$$

which implies that $\alpha_1 + \alpha_2 = 0$ and contradicts the assumption that $\alpha_1 + \alpha_2 \neq 0$, thus proving (ii).

Note 3.5. In a root system of type $A_1 \times A_1$, say $\Phi = \{\pm\alpha, \pm\beta\}$, $\alpha + (-\alpha) + \beta \in \Phi'$, $\alpha + (-\alpha) = 0 \in \Phi'$; however, $\alpha + \beta \notin \Phi'$ and $-\alpha + \beta \notin \Phi'$. Thus, result (ii) fails if we assume that $\alpha_1 + \alpha_2 \in \Phi'$.

(iii) We will use induction on n. If $n \leq 2$, the result is trivial. Suppose that $n \geq 3$. By (i), there is a $\sigma \in S_n$ such that $\alpha_{\sigma([i])} \in \Phi'$ for all $i \in [n]$. By the assumption that $\alpha_I \neq 0$ for $I \neq \emptyset$, we have $\alpha_I \in \Phi' \Leftrightarrow \alpha_I \in \Phi$ for $I \neq \emptyset$. Thus, $\alpha_{\sigma(1)}$, $\alpha_{\sigma(2)}$ and $\alpha_{\sigma(1)} + \alpha_{\sigma(2)}$ are in Φ. By the assumption of (iii), either $\sigma(1) = 1$ or $\sigma(2) = 1$. Let $\widetilde{\sigma} \in S_n$ denote the permutation $\sigma(1, 2)$ (first, apply the transposition (1,2), followed by σ). Then, $\alpha_{\tilde{\sigma}([i])} = \alpha_{\sigma([i])}$ for $i \geq 3$. Thus, we may assume that $\sigma(1) = 1$. By reindexing $\alpha_2, \alpha_3, \ldots, \alpha_n$, we may assume that σ is the identity permutation.

We claim that $\alpha_1 + \alpha_i \in \Phi$ for $i \in [n] \setminus [1]$. This is true for $i = 2$. Set $\beta_1 = \alpha_1 + \alpha_2$, $\beta_i = \alpha_{i+1}$ for $i \in [n-1] \setminus [1]$. Since $\beta_{[n-1]} \in \Phi$ and $\beta_i + \beta_j \notin \Phi$ for $2 \le i < j \le n-1$, induction gives $\beta_{I'} \in \Phi$ for all $I' \subseteq [n-1]$ with $1 \in I'$. Equivalently, $\alpha_I \in \Phi$ for all $I \subseteq [n]$ with $[2] \subseteq I$. In particular, for any $j \in [n] \setminus [2], \alpha_1 + \alpha_2 + \alpha_j \in \Phi$. Since $\alpha_1 + \alpha_2 \in \Phi$ but $\alpha_2 + \alpha_j \notin \Phi$, by (ii), this implies $\alpha_1 + \alpha_j \in \Phi$ for $j \in [n] \setminus [2]$, proving our claim.

Now, we fix a $j \in [n] \setminus [2]$ such that $\alpha_1 + \alpha_j \in \Phi$. Let $\gamma_1 = \alpha_1 + \alpha_j$, $\gamma_i = \alpha_i$ for $i \in [j-1] \setminus [1]$ and $\gamma_i = \alpha_{i+1}$ for $i \in [n-1] \setminus [j-1]$. Since $\gamma_{[n-1]} \in \Phi$ and $\gamma_p + \gamma_q \notin \Phi$ if $2 \le p < q \le n$, induction gives $\gamma_{I'} \in \Phi$ if $I' \subseteq [n-1]$ with $1 \in I'$, that is, $\alpha_I \in \Phi$ if $\{1, j\} \subseteq I$. We have now shown that $\alpha_I \in \Phi$ for all $I \subseteq [n]$ such that either $[2] \subseteq I$ or $\{1, j\} \subseteq I$, $j \in [n] \setminus [2]$. Since $\alpha_1 \in \Phi$, this gives $\alpha_I \in \Phi$ for all $I \subseteq [n]$ with $1 \in I$. □

We now turn our attention to the poset associated to a root, which can be expressed as a sum of roots in a crystallographic root system.

Definition 3.6. Let Φ denote a crystallographic root system, and let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \Phi$ with $\alpha_{[n]} \in \Phi$. The *poset associated to the sum* $\alpha_{[n]}$ is the set

$$C = \{I \subseteq [n] \mid \alpha_I \in \Phi'\}$$

ordered by inclusion. For $i \ge 1$, C_i denotes the subset of elements of C with cardinality i, $C_i = \{I \in C \mid |I| = i\})$ and c_i denotes its cardinality, $c_i = |C_i|$.

Theorem 3.7. *Let Φ denote a crystallographic root system. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Phi$ with $\alpha_{[n]} \in \Phi$ be such that $\alpha_I \neq 0$ if $I \subseteq [n]$ and $I \neq \emptyset$. Let c_i be defined as above. Then, for $i \ge 1$, $c_i \ge n - i + 1$. Furthermore, given any $k \in [n]$, there exists at least one $I \in C_i$ such that $k \in I$.*

Note 3.8. The assumption that $\alpha_I \neq 0$ if $I \subseteq [n]$ and $I \neq \emptyset$ holds for $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Phi^+$.

Proof. We use induction on n. Let us assume that, if $0 < j < n$ and $\alpha_1, \alpha_2, \ldots, \alpha_j \in \Phi$ with $\alpha_{[j]} \in \Phi$, then, for $i \in [j]$, the cardinality of the set $\{I \subseteq [j] \mid |I| = i \text{ and } \alpha_I \in \Phi\}$ is $\ge j - i + 1$ and, given any $k \in [j]$, there exists at least one $I \subseteq [j]$ with $|I| = i$, $\alpha_I \in \Phi$ and $k \in I$.

Clearly, this is true for $j = 1$ and 2. It is also true for $j = 3$ by Lemma 3.3 (ii). We may assume that the roots $\alpha_1, \alpha_2, \dots, \alpha_n$ are ordered in a manner such that $\alpha_{[i]} \in \Phi$ for $1 \le i \le n$.

Assume that $n > 3$. For $i = n$, the set $I = [n]$ has the property that $\alpha_{[n]} \in \Phi$; hence, $c_i \ge 1$ as desired, and, trivially, $k \in I$ for all $k \in [n]$. Thus, we can limit our attention to the case $0 < i \le n-1$. We fix such an i. Since $\alpha_{[n-1]} \in \Phi$, we have by induction that

$$\#(\{I \subseteq [n-1] \mid |I| = i \text{ and } \alpha_I \in \Phi\}) \ge (n-1) - i + 1 = n - i$$

and that, for each k with $1 \le k \le n-1$, there exists at least one $I \subseteq [n-1]$ with $|I| = i$, $\alpha_I \in \Phi$ and $k \in I$. Therefore, it suffices to show that there is at least one $I \subseteq [n]$ with $|I| = i$, $n \in I$ and $\alpha_I \in \Phi$.

We have $\alpha_{[n]} = \alpha_{[n-2]} + \alpha_{n-1} + \alpha_n \in \Phi$. If $\alpha_{[n-2]} + \alpha_n \in \Phi$, then, by induction, there exists an $I \subseteq \{1, 2, \dots, n-2, n\}$ such that $n \in I$, $|I| = i$ and $\alpha_I \in \Phi$. If $\alpha_{[n-2]} + \alpha_n \notin \Phi$, then, by Lemma 3.3 (ii) above, we have $\alpha_{n-1} + \alpha_n \in \Phi$. Therefore, $\alpha_{[n]} = \alpha_{[n-3]} + \alpha_{n-2} + (\alpha_{n-1} + \alpha_n) \in \Phi$. If $\alpha_{[n-3]} + (\alpha_{n-1} + \alpha_n) \in \Phi$, then we can use induction to find $I \subseteq [n]$ such that $\alpha_I \in \Phi$, $|I| = i$ and $n \in I$. Otherwise, according to Lemma 3.3 (ii), we have that $\alpha_{n-2} + (\alpha_{n-1} + \alpha_n) \in \Phi$. This process can continue for at most $i - 1$ steps prior to finding $I \subseteq [n]$ with $n \in I$ such that $|I| = i$ and $\alpha_I \in \Phi$. □

4. Graded posets from sums of roots.

Definition 4.1 (**[2, 5]**). A *graded poset* P is a finite poset with a minimum element m and a maximum element M such that every maximal chain $m = p_0 < p_1 < \cdots < p_r = M$ has the same length r, called the *rank* P. If P is a graded poset, then, for any $x \in P$, the closed interval $[m, x]$ is graded. The *rank* of x is the rank of the interval $[m, x]$.

Theorem 4.2. *Let Φ denote a crystallographic root system. Let $\alpha_1, \alpha_2, \dots, \alpha_n \in \Phi$ with $\alpha_{[n]} \in \Phi$ be such that $\alpha_I \ne 0$ if $I \subseteq [n]$ and $I \ne \emptyset$. Let C be the poset associated to the sum $\alpha_{[n]}$. If $I, J \in C$ with $I \subseteq J$ and $k = |J \setminus I| \ge 2$, then there exists a $K \in C$ with $I \subseteq K \subseteq J$ and $|K| = |I| + 1$. Consequently, we can find $I_1, I_2, \dots, I_k \in C$ such that $|I_l \setminus I_{l-1}| = 1$ for $l \in [k]$, and*

$$I = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k = J.$$

Proof. Let $J \setminus I = \{\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_k}\}$. We use induction on k. The result is obviously true if $k = 1$. We have $\alpha_J = \alpha_I + \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_k} \in \Phi$. Letting $\alpha_{i_1} = \alpha_I, \alpha_{i_2} = \alpha_{j_1}, \alpha_{i_3} = \alpha_{j_2}, \dots, \alpha_{i_{k+1}} = \alpha_{j_k}$, by Theorem 3.7, we have at least $k+1-2+1 = k > 1$ subsets L of the set of indices $\{i_1, i_2, \dots, i_{k+1}\}$ for which $|L| = 2$ and $\alpha_L \in \Phi$. Furthermore, at least one of those subsets contains i_1. Therefore there exists a $j_l \in J \setminus I$ such that $\alpha_I + \alpha_{j_l} \in \Phi$. Letting $K = I \cup \{j_l\}$, we have $|K \setminus I| = 1$, $K \subseteq J$ and $\alpha_K \in \Phi$. Therefore, $K \in C$ and, since $|J \setminus K| = k - 1$, we can use induction to find $I_2, I_3, \dots, I_k \in C$, such that $|I_l \setminus I| = l$ and

$$I = I_0 \subseteq K = I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k = J. \qquad \square$$

Theorems 3.7 and 4.2 yield the following:

Theorem 4.3. *Let $\alpha_i \in \Phi$ for $i \in [n]$ be such that $\alpha_{[n]} \in \Phi$ and $\alpha_I \neq 0$ if $I \subseteq [n]$, $I \neq \emptyset$. Let C be the poset associated to the sum $\alpha_{[n]}$. Then:*

(i) *C is a graded poset with minimum element $\emptyset$, maximum element $[n]$ and rank function given by $I \mapsto |I|$ for $I \in C$.*
(ii) *For $0 \le i \le n$, let $C_i = \{I \in C \mid |I| = i\}$, and let $c_i = |C_i|$. Then, for $i \in [n]$, $c_i \ge n - i + 1$.*
(iii) *If $i, k \in [n]$, then there exists an $I \in C_i$ with $k \in I$.*

Definition 4.4. A poset P with partial order $\le$ is a *lattice* if, for every pair $x, y \in P$, there exist elements $x \vee y$ and $x \wedge y$ in P such that

- $x \le x \vee y$, $y \le x \vee y$ and, if $u \in P$ with $x \le u$ and $y \le u$, then $x \vee y \le u$,
- $x \ge x \wedge y$, $y \ge x \wedge y$ and, if $l \in P$ with $x \ge l$ and $y \ge l$, then $x \wedge y \ge l$.

Below, we present some examples with root systems of type A_n and B_n. We see that, in root systems of type A_n, the posets defined above associated to sums of roots, are lattices. We see with the example from B_4 presented below that this is not always the case.

4.1. Root system of type A_n ([1]). Let $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n+1}\}$ denote the canonical basis of $V = \mathbb{R}^{n+1}$. The set of vectors $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j, 1 \le i \le n+1, 1 \le j \le n+1\}$ is a root system of type A_n, with positive roots given by $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i < j, 1 \le i \le n, 1 \le j \le n+1\}$. Let $s_\alpha : V \to V$ denote the reflection associated to $\alpha \in \Phi$. The associated Weyl group W is the group generated by the reflections $\{s_{\varepsilon_i-\varepsilon_j} \mid 1 \leq i < j \leq n+1\}$. Since $s_{\varepsilon_i-\varepsilon_j}$ switches ε_i and ε_j for $1 \leq i < j \leq n+1$ and leaves ε_k fixed for $k \notin \{i,j\}$, we can identify W with the symmetric group on $n+1$ letters S_{n+1}.

Example 4.5. A sum of roots form a root system of type A_3.

Suppose that Φ is of type A_3 with positive roots

$$\Phi^+ = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \\ \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}.$$

Clearly, $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ has the property that $\alpha_I \in \Phi$ for $\emptyset \subsetneq I \subseteq [3]$ and $I \neq \{1,3\}$. The poset corresponding to this sum of roots is the lattice consisting of all subsets of $[3]$ except $\{1,3\}$.

Example 4.6. Posets associated to sums of roots in a root system of type A_{n-1} are lattices.

Suppose that Φ is of type A_{n-1} with $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$. Let $\alpha_1, \ldots, \alpha_m \in \Phi$ be such that $\alpha_{[m]} \in \Phi$ and $\alpha_I \neq 0$ for all I with $\emptyset \subsetneq I \subseteq [m]$.

Claim 4.7. *We claim that there is a $\sigma \in S_m$ and a $w \in W$ such that $w(\alpha_{\sigma(i)}) = \varepsilon_i - \varepsilon_{i+1}$.*

Proof. This may be proven by induction on m. The result is trivial if $m = 1$. Suppose that $m \geq 2$. By Lemma 3.3, we may assume that, by permuting the α_i, $\alpha_{[m-1]} \in \Phi$. By induction, we may assume that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \ldots, m-1$. Then, since $\alpha_{[m-1]} \in \Phi$, we must have $m-1 \leq n$ and $\alpha_{[m-1]} = \varepsilon_1 - \varepsilon_m$. Since $\alpha_{[m]} = \alpha_{[m-1]} + \alpha_m \in \Phi$ and $\alpha_I \neq 0$ for all I with $\emptyset \subsetneq I \subseteq [m]$, we must have either $\alpha_m = \varepsilon_j - \varepsilon_1$ for some $j \geq m+1$ or $\alpha_m = \varepsilon_m - \varepsilon_j$ for some $m+1 \leq j \leq n+1$.

In the first case, we can choose $w \in W$ with $w(\varepsilon_j) = \varepsilon_1$ and $w(\varepsilon_i) = \varepsilon_{i+1}$ for $i = 1, \ldots, m$. This yields $w(\alpha_1) = \varepsilon_2 - \varepsilon_3, \ldots, w(\alpha_{m-1}) = \varepsilon_m - \varepsilon_{m+1}, w(\alpha_m) = \varepsilon_1 - \varepsilon_2$, and the claim follows.

In the second case, choosing $w \in W$ with $w(\varepsilon_i) = \varepsilon_i$ for $i = 1, \ldots, m$ and $w(\varepsilon_j) = \varepsilon_{m+1}$. This yields $w(\alpha_1) = \varepsilon_1 - \varepsilon_2, \ldots, w(\alpha_{m-1}) = \varepsilon_{m-1} - \varepsilon_m, w(\alpha_m) = \varepsilon_m - \varepsilon_{m+1}$, and Claim 4.7 is proved. □

It follows from the claim that the poset associated to the sum $\alpha_{[m]}$ is the same up to isomorphism as that associated to the sum $\beta_{[m]}$ where $\beta_1 = \varepsilon_1 - \varepsilon_2, \beta_2 = \varepsilon_2 - \varepsilon_3, \ldots, \beta_m = \varepsilon_m - \varepsilon_{m+1}$. This poset is isomorphic to the collection of nonempty intervals $\{[i,j] \mid 1 \leq i < j \leq n\}$ inside $[n]$, ordered by containment, which is easily seen to be a lattice.

4.2. Root system of type B_n ([1, Chapter VI]). Let $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ denote the cannonical basis of $\mathbb{R}^n$. The set of vectors $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \subset \mathbb{R}^n$ forms a root system of type B_n.

Example 4.8. A sum of roots in a root system of type B_4 where the associated poset is not a lattice.

We have

$$\varepsilon_1 + \varepsilon_2 = (\varepsilon_1 - \varepsilon_2) + 2(\varepsilon_2 - \varepsilon_3) + 2(\varepsilon_3 - \varepsilon_4) + 2\varepsilon_4 \in \Phi.$$

Letting $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \alpha_3 = \varepsilon_2 - \varepsilon_3, \alpha_4 = \alpha_5 = \varepsilon_3 - \varepsilon_4, \alpha_6 = \alpha_7 = \varepsilon_4$, we see that $\alpha_{[7]} \in \Phi$. Figure 1, generated by the `POSETS` package for `Mathematica` [4], shows the interval $[\alpha_7, \alpha_{[7]}]$ in the associated poset C.

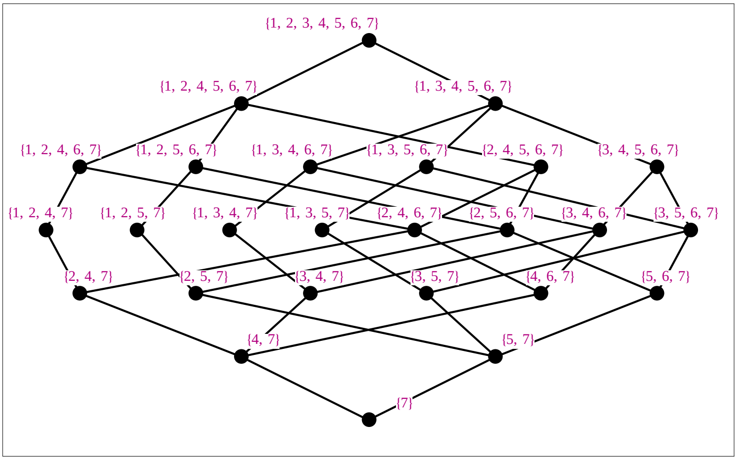

FIGURE 1. Note on sums of roots.

We see that the subsets $I = \{4, 7\}$ and $J = \{5, 7\}$ do not have a least upper bound $I \vee J$. The highlighted nodes in Figure 2 show the subsets of C containing I and J, respectively.

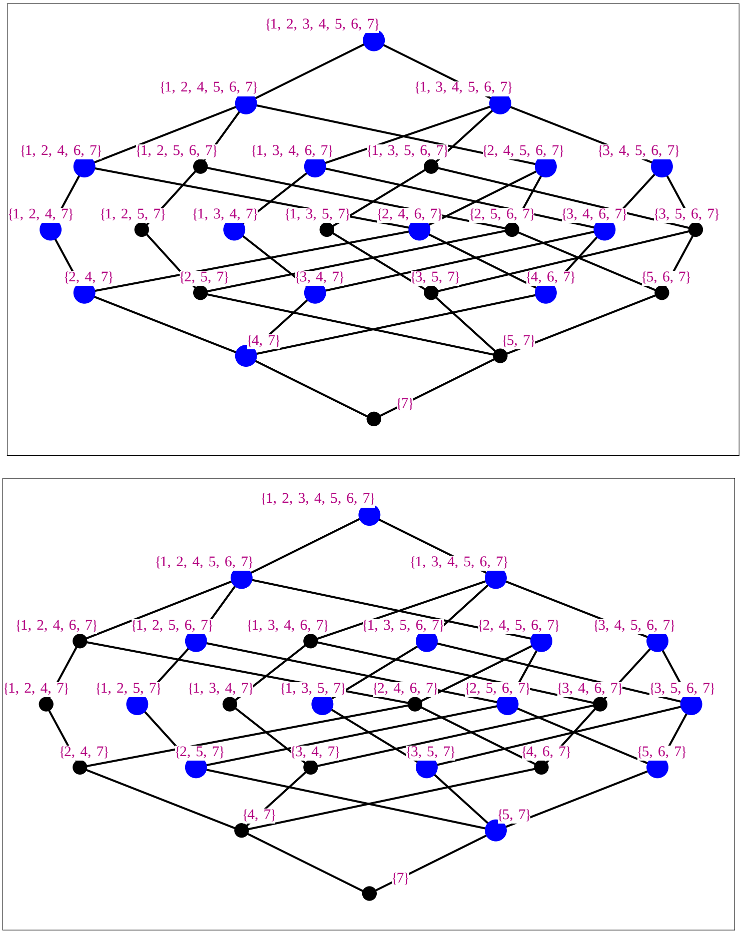

Figure 2.

We see that I and J are both bounded above by the sets $\{2, 4, 5, 6, 7\}$ and $\{3, 4, 5, 6, 7\}$, and there is no subset $K \subset [7] \in C$ with $I, J \subseteq K$ and $|K| < 5$.

Example 4.9. The root system of type BC_2.

This is the unique irreducible, non-reduced root system of rank 2. The positive roots may be chosen to be

$$\Phi^+ = \{\varepsilon_1, \varepsilon_2, 2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_2 - \varepsilon_1\} \subset \mathbb{R}^2.$$

Consider the roots $\{\alpha_1 = \varepsilon_2 - \varepsilon_1, \alpha_2 = \varepsilon_1 + \varepsilon_2, \alpha_3 = -\varepsilon_2\}$. Then, $\alpha_I \in \Phi$ for all I with $\emptyset \subsetneq I \subseteq [3]$. The corresponding poset is the Boolean lattice of all subsets of [3]. Note that, in contrast to the root system of type A_n above, we cannot transform the set $\{\alpha_1, \alpha_2, \alpha_3\}$ here to a set of positive roots using an element of the associated Weyl group.

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Bethel College, Department of Mathematics, 1001 Bethel Circle, Mishawaka, IN 46545
Email address: **timothy.ferdinands@bethelcollege.edu**

University of Notre Dame, Department of Mathematics, Room 255 Hurley Building, Notre Dame, IN 46556
Email address: **Pilkington.4@nd.edu**