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TENSOR PRODUCTS AND ENDOMORPHISM RINGS OF FINITE VALUATED GROUPS

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ABSTRACT. This paper discusses homological properties of a finite valuated p-group A. A category equivalence between full subcategories of the category of valuated p-groups and the category of right modules over the endomorphism ring of A is developed to study A-presented and A-valuated valuated p-groups. In particular, we show that these classes do not coincide if $|A/pA| > p$. Examples are given throughout the paper.

1. Introduction. Let p be a prime and G a p-local Abelian group. A *valuation* v on G assigns a value $v(g)$ to each $g \in G$, which is either an ordinal or ∞ subject to the rules

(i) $v(px) > v(x)$ for all $x \in G$ where $\infty > \infty$;
(ii) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in G$;
(iii) $v(nx) = v(x)$ for all x and all integers n not divisible by p [**10**].

Condition (iii) is redundant if G is a p-group, which will be our standard assumption throughout this paper. The finite valuated p-groups are the objects of the category $\mathcal{V}_p$. A $\mathcal{V}_p$-morphism $(G, v) \to (H, w)$ is a group homomorphism $\alpha : G \to H$ such that $w(\alpha(g)) \geq v(g)$ for all $g \in G$. The family of $\mathcal{V}_p$-morphisms from (G, v) to (H, w) is denoted by $\mathrm{Mor}(G, H)$.

Hunter, Richman and Walker studied valuated groups in a series of papers in the 1970s and 1980s (e.g., [**7, 9, 10**]). They showed that $\mathcal{V}_p$ is a *pre-Abelian* category, i.e., all maps have kernels and cokernels, but it is not Abelian. In particular, the kernel and the cokernel of a $\mathcal{V}_p$-morphism are its kernel and cokernel in the category $\mathcal{A}b$ of Abelian groups; however, they carry an additional valuation. Arnold discovered

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a surprising connection between finite valuated p-groups and torsion-free Abelian groups of finite rank in [**3**]. He used representation theory to investigate finite rank Butler groups, and then described valuated p-groups similarly in terms of representations of the underlying valuated p-trees. In particular, there are infinitely many isomorphism classes of indecomposable finite valuated p-groups G such that $p^4G = 0$ and $v(g) \leq 9$ for all $0 \neq g \in G$ [**3,** Example 8.2.5]. Moreover, the category of indecomposable finite valuated p-groups G such that $p^5G = 0$ and $v(g) \leq 11$ for all $0 \neq g \in G$ has wild representation type [**3,** Example 8.2.6].

This paper follows Arnold's approach by investigating valuated p-groups using tools which have traditionally been used in the discussion of torsion-free groups of finite rank. In particular, homological properties of a torsion-free group A have been successfully studied by viewing A as a left module over its endomorphism ring. Valuated groups can similarly be investigated in terms of their endomorphism ring $R = \mathrm{Mor}(A, A)$. Some of the major difficulties we encounter are that neither the 5-lemma nor the snake lemma hold in $\mathcal{V}_p$ [**9**]. Moreover, compositions of kernels (cokernels) may not be kernels (cokernels) [**9**]. Therefore, the usual homological constructions may not carry over from Abelian categories. Nevertheless, it is still possible to develop a homological algebra for pre-Abelian categories as Yakovlev showed in [**14**].

Since A is a left R-module, $H_A = \mathrm{Mor}(A, -)$ can be viewed as a functor from $\mathcal{V}_p$ to $\mathcal{M}_R$, the category of finitely generated right R-modules. In particular, $H_A(G)$ is a free right R-module if G is *A-free*, i.e. of the form A^n for some $n < \omega$. Similarly, $H_A(G)$ is projective if G is *A-projective*, i.e., a $\mathcal{V}_p$-direct summand of an A-free group. The functor H_A induces a category equivalence between the full subcategory of $\mathcal{V}_p$ consisting of the A-free groups and the category of finitely generated free right R-modules [**1**]. Its converse T_A was obtained by considering a finitely generated free right R-module F with basis $\{x_i \mid i \in I\}$. Writing $y \in F \otimes_R A$ as $y = \Sigma_{i \in I}(x_i \otimes a_i)$, the valuation v on $F \otimes_R A$ was defined as $v(y) = \min_{i \in I} v(a_i)$. Clearly, $T_A(F) = (F \otimes_R A, v)$ is A-free. Moreover, if F_1 and F_2 are finitely generated free right R-modules, then every map $\phi \in \mathrm{Hom}_R(F_1, F_2)$ induces a $\mathcal{V}_p$-morphism $T_A(\phi) : T_A(F_1) \to T_A(F_2)$ [**1**]. Unfortunately, this construction relies on the freeness of the right R-module and does not carry over even to projective modules.

As a first step towards developing a homological algebra for finite valuated p-groups and their endomorphism rings, Section 2 extends the definition of T_A to the whole category $\mathcal{M}_R$ (Theorem 2.3). Although the underlying group structure of T_A is induced by the tensor product $- \otimes_R A$, as in [**1**], we show that its exactness properties may be different from those of the classical tensor product due to the failure of the 5-lemma in $\mathcal{V}_p$ (Proposition 2.2). Section 3 describes the classes of A-solvable and A-presented valuated p-groups, i.e., finite valuated p-groups which arise as cokernels of maps in $\mathrm{Mor}(A^n, A^m)$ for some $n, m < \omega$, and investigates the splitting of monomorphisms $A^n \to A^m$ (Theorem 3.6). Section 4 concerns the case where A is a simply presented group and shows that, for such a group A, these two classes coincide if and only if A is cyclic (Theorem 4.3).

2. Tensor products and valuations. We begin our discussion with a few technical results. If α is a kernel in $\mathcal{V}_p$, then $\alpha = \ker(\mathrm{coker}(\alpha))$ [**13**], and a similar result holds for cokernels. If U is a valuated subgroup of a finite valuated group G, then the cokernel of the inclusion $U \subseteq G$ is the group G/U with a valuation v defined by

$$v(g + U) = \max\{v(g + u) \mid u \in U\}$$

[**10**]. A sequence $A \overset{\alpha}{\to} B \overset{\beta}{\to} C$ of valuated p-groups is *left-exact* if α is a $\mathcal{V}_p$-kernel for β, and right exact if β is a $\mathcal{V}_p$-cokernel for α. It is *exact* if α is a kernel for β and β is a cokernel for α [**10**]. The functor H_A is left-exact since

$$(*) \qquad 0 \longrightarrow H_A(U) \overset{H_A(\alpha)}{\longrightarrow} H_A(B) \overset{H_A(\beta)}{\longrightarrow} H_A(C)$$

is an exact sequence of right R-modules whenever $0 \to U \overset{\alpha}{\to} B \overset{\beta}{\to} C$ is a left-exact sequence of valuated p-groups. Finally, the *forgetful functor* $\mathcal{F} : \mathcal{V}_p \to \mathcal{A}b$ strips a valuated group (G, v) of its valuation.

Lemma 2.1. *Let A, B and C be finite valuated p-groups.*

(a) *Let $\alpha \in \mathrm{Mor}(A, B)$ be an epimorphism and $\beta \in \mathrm{Mor}(B, C)$. If $\beta\alpha$ is a cokernel of $\delta \in \mathrm{Mor}(G, A)$ for some $G \in \mathcal{V}_p$, then β is a cokernel for $\alpha\delta$.*

(b) *Let $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ be a sequence of valuated p-groups which is exact in $\mathcal{A}b$.*

(i) *If there is a* $\delta \in \mathrm{Mor}(C, B)$ *such that* $\beta\delta = 1_C$*, then* $\beta = \operatorname{coker} \alpha$.

(ii) *If there is a* $\gamma \in \mathrm{Mor}(B, A)$ *such that* $\gamma\alpha = 1_A$*, then* $\alpha = \ker \beta$.

Proof.

(a) Suppose that ϕ satisfies $\phi\alpha\delta = 0$. Since $\beta\alpha$ is a cokernel for δ, there is a map ψ such that $\psi\beta\alpha = \phi\alpha$. Since α is an epimorphism, $\phi = \psi\beta$. In addition, since β is an epimorphism, ψ is unique with this property.

(b) We only show (i). Let X be a valuated group, and $\phi \in \mathrm{Mor}(B, X)$ with $\phi\alpha = 0$. Since $\beta(1_B - \delta\beta) = 0$, every $b \in B$ can be written as $b = \delta\beta(b) + \alpha(a)$ for some $a \in A$ since the given sequence is exact in $\mathcal{A}b$. Thus, $\phi(b) = (\phi\delta)\beta(b)$. Moreover, ϕ is unique since β is an epimorphism. $\square$

Lemma 2.1 (b) raises the question of whether the splitting of β in $\mathcal{V}_p$ guarantees the splitting of α in $\mathcal{V}_p$, and vice-versa. However, this need not be the case unless the sequence

$$0 \longrightarrow G \xrightarrow{\alpha} H \xrightarrow{\beta} K \longrightarrow 0$$

is $\mathcal{V}_p$-exact. In order to see this, let $A_1 = \langle x \rangle$ be a cyclic group of order p with a valuation such that $v(x) > 0$ and $A_2 = \mathbb{Z}/p\mathbb{Z}$ with the height valuation. The sequence

$$0 \longrightarrow A_2 \xrightarrow{\alpha} A_1 \oplus A_2 \xrightarrow{\beta} A_2 \longrightarrow 0,$$

in which α is the embedding into the first coordinate and β is the projection onto the second coordinate, is $\mathcal{A}b$-exact. Although β splits in $\mathcal{V}_p$, this is not true for α.

This example highlights a significant difference between the category $\mathcal{V}_p$ and the quasi-category $\mathbb{Q}\mathcal{A}b$ of torsion-free groups of finite rank. In the latter, there is a natural monomorphism from $\mathrm{Hom}(G, H)$ to $\mathrm{Mor}_{\mathbb{Q}\mathcal{A}b}(G, H)$ for all $G, H \in \mathbb{Q}\mathcal{A}b$, while such a map does usually not exist in $\mathcal{V}_p$. In particular, $\mathcal{A}b$-decompositions of a group induce $\mathbb{Q}\mathcal{A}b$-decomposition, while there exist indecomposable finite valuated p-groups which are acyclic.

Consider the functor $t_A : \mathcal{M}_R \to \mathcal{A}b$ defined by $t_A = - \otimes_R A$ for all $M \in \mathcal{M}_R$. In order to define a valuation on $t_A(M)$, choose a free resolution $F_1 \overset{\alpha}{\to} F_0 \overset{\beta}{\to} M \to 0$ of M. Applying t_A induces an exact sequence

$$T_A(F_1) \overset{t_A(\alpha)}{\longrightarrow} T_A(F_0) \overset{t_A(\beta)}{\longrightarrow} t_A(M) \longrightarrow 0,$$

where $t_A(\alpha) = T_A(\alpha)$ is a $\mathcal{V}_p$-map by **[1]**. Since $\mathcal{V}_p$ is pre-Abelian, there is a unique valuation v on $t_A(M)$ such that $t_A(\beta)$ is the $\mathcal{V}_p$-cokernel of $T_A(\alpha)$ **[10]**. Let $T_A(M) = (t_A(M), v)$. Although $t_A = \mathcal{F}T_A$, exactness properties of t_A do not necessarily carry over to T_A, as the following example shows:

Proposition 2.2. *Let $A = \langle a \rangle$ be a valuated cyclic group of order p^n for some $1 < n < \omega$ with a valuation v such that $v(a) \neq \infty$. Then, A is free as a left R-module, and there exists a monomorphism $\alpha : R/J \to R$ such that the induced map $T_A(\alpha) : T_A(R/J) \to T_A(R)$ is not a $\mathcal{V}_p$-kernel.*

Proof. Observe that $R = \mathbb{Z}/p^n\mathbb{Z}$ and $J = p\mathbb{Z}/p^n\mathbb{Z}$. The left R-module R/J fits into the exact sequence $R \overset{\phi}{\to} R \overset{\pi}{\to} R/J \to 0$ where $\phi(1_R) = p1_R$ and $\pi(1_R) = 1_R + J$. Setting $v((1_R + J) \otimes a) = v(a)$ defines the cokernel valuation on $T_A(R/J)$.

The map $\gamma : R/J \to E$, defined by $\gamma(1_R + J) = p^{n-1} + p^n\mathbb{Z}$, induces a monomorphism $T_A(\gamma) : T_A(R/J) \to T_A(E)$ such that $\mathrm{im}(T_A(\gamma)) = \langle p^{n-1}a \rangle$. Since

$$v((1_R + J) \otimes a) = v(a) < v(p^{n-1}a)$$

and $v(a) \neq \infty$, the map $T_A(\gamma)$ does not preserve valuations. Therefore, it is not a kernel, and the induced sequence

$$0 \longrightarrow T_A(R/J) \overset{\gamma}{\longrightarrow} T_A(R) \longrightarrow T_A(R/\operatorname{im}\gamma) \longrightarrow 0$$

of Abelian groups is not $\mathcal{V}_p$-exact. $\square$

Although the classical adjoint functor theorem and its associated exactness properties apply to the pair (H_A, t_A), they may not be applicable to (H_A, T_A) since homological arguments that work in an Abelian category may not carry over to a pre-Abelian category like $\mathcal{V}_p$ **[14]**.

Theorem 2.3. *Let A be a finite valuated p-group.*

(a) *T_A is a right exact functor from $\mathcal{V}_p$ to $\mathcal{M}_R$.*

(b) *The evaluation map* $\theta_G : T_AH_A(G) \to G$ *defined by* $\theta_G(\alpha \otimes a) = \alpha(a)$ *is a natural* $\mathcal{V}_p$*-map for all valuated p-groups G such that*

(i) θ_F *is an isomorphism for all A-free groups F.*

(ii) $\theta_{T_A(M)}$ *is an epimorphism for all right R-modules M.*

(c) *The natural map* $\Phi_M : M \to \mathrm{Hom}(A, T_A(M))$*, defined by* $[\Phi_M(x)](a) = x \otimes a$*, is a natural transformation, and* Φ_F *is an isomorphism for all free right R-modules F.*

Proof.

(a) Let $M, N \in \mathcal{M}_R$, and choose exact sequences $F_1 \xrightarrow{\alpha} F_0 \xrightarrow{\beta} M \to 0$ and $P_1 \xrightarrow{\gamma} P_0 \xrightarrow{\delta} N \to 0$ of right R-modules in which the P_i's and the F_i's are free. For every $\phi : M \to N$, applying the functor t_A induces a commutative diagram

$$\begin{array}{ccccccc}
T_A(F_1) & \xrightarrow{T_A(\alpha)} & T_A(F_0) & \xrightarrow{T_A(\beta)} & T_A(M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle T_A(\phi_1)} & & \downarrow{\scriptstyle T_A(\phi_0)} & & \downarrow{\scriptstyle t_A(\phi)} & & \\
T_A(P_1) & \xrightarrow{T_A(\gamma)} & T_A(P_0) & \xrightarrow{T_A(\delta)} & T_A(N) & \longrightarrow & 0
\end{array}$$

of Abelian groups in which the maps ϕ_0 and ϕ_1 are induced by ϕ using standard homological arguments. Here, the usage of the symbol T_A indicates that the corresponding map has already been shown to be a $\mathcal{V}_p$-map. Since t_A is right-exact, the rows of the diagram are exact as sequences of Abelian groups, and

$$[T_A(\delta)T_A(\phi_0)]T_A(\alpha) = t_A(\phi)T_A(\beta)T_A(\alpha) = 0.$$

Since $T_A(\delta)T_A(\phi_0)$ is a $\mathcal{V}_p$-map and $T_A(\beta) : T_A(F_0) \to T_A(M)$ is a cokernel for $T_A(\alpha)$, there is a $\mathcal{V}_p$-map $\lambda : T_A(M) \to T_A(N)$ with $\lambda T_A(\beta) = T_A(\delta)T_A(\phi_0)$. On the other hand, $T_A(\delta)T_A(\phi_0) = t_A(\phi)T_A(\beta)$ in $\mathcal{A}b$. Since $T_A(\beta)$ is an epimorphism of Abelian groups, $t_A(\phi) = \lambda$ is a $\mathcal{V}_p$-map. In particular, if $M = N$ and $\phi = 1_M$, then we denote the valuation on $T_A(M)$ induced by the F_i's by v and the one induced by the P_i's by w. Since $t_A(1_M)$ is a $\mathcal{V}_p$-map, $w(x) = w(T_A(1_M))(x) \geq v(x)$ for all $x \in T_A(M)$.

Now, consider an exact sequence $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ in $\mathcal{M}_R$. Standard homological arguments based on the Horseshoe lemma **[11,** Lemma 6.20] and an application of T_A yield the commutative diagram:

$$\begin{tikzcd}
 & T_A(F_1) \arrow[r, "T_A(\beta_1)"] \arrow[d, "T_A(\sigma_2)"] & T_A(P_1) \arrow[r] \arrow[d, "T_A(\sigma_3)"] & 0 \\
T_A(Q_0) \arrow[r, "T_A(\alpha_0)"] \arrow[d, "T_A(\tau)"] & T_A(F_0) \arrow[r, "T_A(\beta_0)"] \arrow[d, "T_A(\tau_2)"] & T_A(P_0) \arrow[r] \arrow[d, "T_A(\tau_3)"] & 0 \\
T_A(L) \arrow[r, "T_A(\alpha)"] & T_A(M) \arrow[r, "T_A(\beta)"] \arrow[d] & T_A(N) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

of $\mathcal{V}_p$-maps whose rows and columns are exact in $\mathcal{A}b$. Here, the F_i's, P_i's and Q_i's are free modules obtained by the horseshoe lemma, and the R-module homomorphisms β_0 and β_1 split. The definition of T_A ensures that $T_A(\tau_i)$ is a cokernel of $T_A(\sigma_i)$ for $i = 2, 3$. Since $T_A(\beta_0)$ and $T_A(\beta_1)$ split, they are cokernels for $T_A(\alpha_0)$ and $T_A(\alpha_1)$, respectively, by Lemma 2.1 (b).

Clearly, $T_A(\beta)T_A(\alpha) = 0$. in order to see that $T_A(\beta)$ is a cokernel for $T_A(\alpha)$, assume that $\phi T_A(\alpha) = 0$ for some valuated p-group G and a map $\phi \in \mathrm{Mor}(T_A(M), G)$. Then, $[\phi T_A(\tau_2)]T_A(\alpha_0) = \phi T_A(\alpha)T_A(\tau) = 0$. Since $T_A(\beta_0)$ is a cokernel for $T_A(\alpha_0)$, we can find $\lambda_1 \in \mathrm{Mor}(T_A(P_0), G)$ such that $\phi T_A(\tau_2) = \lambda_1 T_A(\beta_0)$. Now,

$$\lambda_1 T_A(\sigma_3)T_A(\beta_1) = \lambda_1 T_A(\beta_0)T_A(\sigma_2) = \phi T_A(\tau_2)T_A(\sigma_2) = 0.$$

Since $T_A(\beta_1)$ is an epimorphism, $\lambda_1 T_A(\sigma_3) = 0$. However, $T_A(\tau_3)$ is a cokernel for $T_A(\sigma_3)$. Hence, there is a $\lambda \in \mathrm{Mor}(T_A(N), G)$ with $\lambda T_A(\tau_3) = \lambda_1$. Then,

$$\lambda T_A(\beta)T_A(\tau_2) = \lambda T_A(\tau_3)T_A(\beta_0) = \lambda_1 T_A(\beta_0) = \phi T_A(\tau_2).$$

Since $T_A(\tau_2)$ is an epimorphism, $\phi = \lambda T_A(\beta)$.

Finally, if $\phi = \lambda_1 T_A(\beta) = \lambda_2 T_A(\beta)$, then $\lambda_1 = \lambda_2$ since $T_A(\beta)$ is an epimorphism. Therefore, $T_A(\beta)$ is a cokernel of $T_A(\alpha)$.

(b) The map θ_G is well defined since $\mathrm{Mor}(G, H) \subseteq \mathrm{Hom}_{\mathbb{Z}}(G, H)$. Arguing as in the case of Abelian groups yields that this map is natural. From [**1**], θ_F is a $\mathcal{V}_p$-isomorphism if F is an A-free group.

It remains to show that θ_G is a $\mathcal{V}_p$-morphism for all $G \in \mathcal{V}_p$. For this, let I indicate the finite set $H_A(G)$. For each $\phi \in I$, choose $A_\phi = A$ and define $F = \oplus_I A$ and $\pi : F \to G$ by $\pi(\Sigma_I a_\phi) = \Sigma_I \phi(a_\phi)$. Then, there is a $\phi_0 \in I$ such that

$$v(\Sigma_I \phi(a_\phi)) \geq \min\{v(\phi(a_\phi))\} = v(\phi_0(a_{\phi_0})) \geq v(a_{\phi_0}) \geq v(\Sigma_I a_\phi),$$

and π is a $\mathcal{V}_p$-map.

Since A is finite, $H_A(F)$ is a free R-module [**1**]. Applying H_A induces an R-module epimorphism $H_A(\pi) : H_A(F) \to H_A(G)$ as in the case of Abelian groups. Select a free R-module P and an epimorphism $\tau : P \to \ker(H_A(\pi))$ to obtain the exact sequence

$$P \xrightarrow{\tau} F \xrightarrow{H_A(\pi)} H_A(G) \longrightarrow 0$$

of right R-modules. Applying the functor T_A yields the commutative diagram

$$\begin{tikzcd} T_A(P) \arrow[r, "T_A(\tau)"] & T_AH_A(F) \arrow[r, "T_AH_A(\pi)"] \arrow[d, "\theta_F", "\wr"'] & T_AH_A(G) \arrow[r] \arrow[d, "\theta_G"] & 0 \\ & F \arrow[r, "\pi"] & G & \end{tikzcd}$$

of Abelian p-groups in which θ_F is a $\mathcal{V}_p$-isomorphism by [**1**]. Then

$$[\pi\theta_F]T_A(\tau) = \theta_G T_A H_A(\pi) T_A(\tau) = 0.$$

Since $\pi\theta_F$ is a $\mathcal{V}_p$-map, there is a $\mathcal{V}_p$-map $\lambda : T_AH_A(G) \to G$ with $\lambda T_AH_A(\pi) = \pi\theta_F$ since $T_AH_A(\pi) : T_A(F) \to T_AH_A(G)$ is a $\mathcal{V}_p$-cokernel of $T_A(\tau)$. On the other hand, $\pi\theta_F = \theta_G T_A(\pi)$ holds in $\mathcal{A}b$. Arguing as before, $\theta_G = \lambda$ is a $\mathcal{V}_p$-morphism.

Let M be a finitely generated right R-module. Since $T_A(M)$ is an epimorphic image of an A-free group, we can argue as in the case of Abelian groups that there exist an A-free valuated group P_1 and an epimorphism $\alpha : P_1 \to T_A(M)$ such that $H_A(\alpha) : H_A(P_1) \to H_AT_A(M)$ is onto. We therefore obtain the commutative diagram:

$$\begin{tikzcd} T_AH_A(P_1) \arrow[r, "T_AH_A(\alpha)"] \arrow[d, "\theta_{P_1}", "\wr"'] & T_AH_AT_A(M) \arrow[r] \arrow[d, "\theta_{T_A(M)}"] & 0 \\ P_1 \arrow[r, "\alpha"] & T_A(M) \arrow[r] & 0 \end{tikzcd}$$

of Abelian groups whose rows are exact. Thus, $\theta_{T_A(M)}$ is an epimorphism.

(c) Consider a finitely generated right R-module M. We must show that $\Phi_M(x)$ is a $\mathcal{V}_p$-map for all $x \in M$. In order to see this, select a free resolution $F_1 \to F_0 \xrightarrow{\pi} M \to 0$ where $F_0 = \oplus_M R_x$ with $R_x = R$ and $\pi(\delta_x(1)) = x$, where the δ_x's are the coordinate embeddings. Since the valuation on $T_A(M)$ is the cokernel valuation,

$$v(x \otimes a) = v(\pi\delta_x(1) \otimes a) \geq v(\delta_x(1) \otimes a) = v(a)$$

for all $x \in M$ and all $a \in A$. Finally, Φ_F is an isomorphism by **[1]** whenever F is a free right R-module. □

For A-free groups F and P, consider $\alpha \in \mathrm{Mor}(P, F)$. The last result induces a valuation on $T_A(M)$ where $M = H_A(F)/im(H_A(\alpha))$, making the latter the cokernel of $T_A H_A(\alpha)$. On the other hand, α also has a $\mathcal{V}_p$-cokernel $\beta : F \to G$ for some valuated p-group (G, w). We consider the commutative diagram

$$\begin{array}{ccccccc}
T_A H_A(P) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(F) & \longrightarrow & T_A(M) & \longrightarrow & 0 \\
\wr\big\downarrow \theta_P & & \wr\big\downarrow \theta_F & & & & \\
P & \xrightarrow{\alpha} & F & \xrightarrow{\beta} & G & \longrightarrow & 0
\end{array}$$

in which all maps are $\mathcal{V}_p$-maps. Since the rows are $\mathcal{V}_p$-exact and $T_A(\pi)$ is a cokernel by the last theorem, there is an isomorphism $\delta : (T_A(M), w) \to (G, v)$. Thus, the valuation given by the last result coincides with the cokernel valuation defined in **[10]**.

Example 2.4. The application of the functor $T_A H_A$ to a valuated p-group G does not preserve the original valuation on G. For instance, if $A = \mathbb{Z}/p\mathbb{Z}$ with the height valuation, and $G = \mathbb{Z}/p\mathbb{Z}$ with the valuation which assigns the value ∞ to all elements of G, then $T_A H_A(G) \cong A$ in $\mathcal{V}_p$, but $G \ncong A$ in $\mathcal{V}_p$. In particular, θ_G need not be a $\mathcal{V}_p$-cokernel.

Although the focus of this paper is on finite valuated p-groups, we want to mention that the results of this section carry over to the more general case of a self-small p-local group.

3. A-presented groups. An epimorphism $G \to H$ of valuated p-groups is *A-balanced* if the induced map $H_A(\alpha) : H_A(G) \to H_A(H)$ is surjective. A valuated p-group is *A-generated* if we can find an A-balanced $\mathcal{V}_p$-exact sequence $\oplus_I A \xrightarrow{\beta} G \to 0$. It is weakly A-generated if there is an epimorphism $\oplus_I A \to G$ for some index-set I. As in the case of Abelian groups, we can choose this epimorphism to be A-balanced. When working in $\mathcal{A}b$, it is not necessary to distinguish between A-generated and weakly A-generated groups. However, the next result shows that weakly A-generated groups which are not A-generated arise regularly in $\mathcal{V}_p$:

Proposition 3.1. *Let A be a finite valuated group such that $v(a) \neq \infty$ for all $0 \neq a \in A$. Then, there exists a weakly A-generated valuated group G which is not A-generated.*

Proof. Select an ordinal λ such that $v(a) < \lambda$ for all non-zero $a \in A$, and let G be the group A equipped with a valuation w such that $w(a) > \lambda$ for all $a \in A$. Then, $1_A : A \to G$ is a $\mathcal{V}_p$-epimorphism which is not a cokernel, for, otherwise, it would be an isomorphism.

If G were A-generated, then we could find an A-balanced cokernel $\pi : A^n \to G$ for some $n < \omega$. Since A^n carries the co-product valuation, $v(x) < \lambda$ for all non-zero $x \in A^n$. If $U = \ker \pi$, then $v(x + U) = v(x + u_0) < \lambda$ for some $u_0 \in U$ since U is finite. Thus, $v(g) < \lambda$ for all non-zero $g \in G$, which contradicts the construction of G. $\square$

A valuated p-group G is *A-presented* if there is an exact sequence

$$0 \longrightarrow U \longrightarrow F \longrightarrow G \longrightarrow 0 \qquad \text{(E)}$$

of valuated p-groups such that F is A-free and U is weakly A-generated. An A-presented group G is *A-solvable* if (E) can be chosen to be A-balanced. Since A is projective with respect to split-exact sequences, every A-projective group is A-solvable. We begin our discussion with a technical result which we use frequently throughout this paper as a substitute for the 5-lemma:

Proposition 3.2. *Let A and G be finite valuated p-groups.*

(a) *For every $\mathcal{V}_p$-exact sequence $0 \to U \to H \to G \to 0$ such that θ_H is an isomorphism, there exists a commutative diagram*

$$\begin{array}{ccccccc} T_A H_A(U) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(H) & \xrightarrow{T_A H_A(\beta)} & T_A(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \theta_U} & & \wr\downarrow{\scriptstyle \theta_H} & & \downarrow{\scriptstyle \theta} & & \\ U & \xrightarrow{\alpha} & H & \xrightarrow{\beta} & G & \longrightarrow & 0 \end{array}$$

of valuated groups and $\mathcal{V}_p$-maps in which $M = \operatorname{im} H_A(\beta)$ is a submodule of $H_A(G)$ and $\theta : T_A(M) \to G$ is the evaluation map. Moreover, θ is a cokernel, and $\theta = \theta_G T_A(\iota)$, where $\iota : M \to H_A(G)$ is the embedding.

(b) *If G is A-generated, then an A-balanced sequence $0 \to U \to \oplus_I A \to G \to 0$ induces a diagram as in* (a) *with $M = H_A(G)$ and $\theta = \theta_G$. In particular, θ_G is a cokernel.*

Proof.

(a) Since H_A is left-exact, every exact sequence $0 \to U \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$ of valuated groups induces an exact sequence

$$0 \longrightarrow H_A(U) \xrightarrow{H_A(\alpha)} H_A(H) \xrightarrow{H_A(\beta)} M \longrightarrow 0$$

of right R-modules where $M = \operatorname{im}(H_A(\beta))$ is a submodule of $H_A(G)$. From Theorem 2.3, the induced sequence

$$T_A H_A(U) \xrightarrow{T_A H_A(\alpha)} T_A H_A(H) \xrightarrow{T_A H_A(\beta)} T_A(M) \longrightarrow 0$$

is exact. Theorem 2.3 also yields the commutativity of the diagram in $\mathcal{A}b$ and the fact that θ_U and θ_G are $\mathcal{V}_p$-maps. Since θ_H is a $\mathcal{V}_p$-isomorphism such that $\theta = \theta_H T_A(\iota)$, we obtain that θ is a $\mathcal{V}_p$-map since this holds for $T_A(\iota)$ by Theorem 2.3. The diagram shows $\theta[T_A H_A(\beta)\theta_H^{-1}] = \beta$. Since $T_A H_A(\beta)$ is a cokernel by Theorem 2.3, Lemma 2.1 yields that θ is a cokernel too.

(b) follows directly from (a) since $M = H_A(G)$ in this case. $\square$

Using the last result, we obtain the following description of A-presented and A-solvable valuated p-groups:

Theorem 3.3. *Let A be a finite valuated p-group.*

(a) *A finite valuated p-group G is A-presented if and only if $G \cong T_A(M)$ for some finitely generated right R-module M.*

(b) *A finite valuated p-group G is A-solvable if and only if the map θ_G is a $\mathcal{V}_p$-isomorphism.*

Proof.

(a) Since G is A-presented, there is an exact sequence $0 \to U \stackrel{\alpha}{\to} A^n \stackrel{\beta}{\to} G \to 0$ such that U is weakly A-generated. Proposition 3.2 yields the commutative diagram

$$\begin{array}{ccccccc}
T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(A^n) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta_U} & & \wr\downarrow{\scriptstyle\theta_{A^n}} & & \downarrow{\scriptstyle\theta} & & \\
U & \xrightarrow{\alpha} & A^n & \xrightarrow{\beta} & G & \longrightarrow & 0
\end{array}$$

of $\mathcal{V}_p$-maps in which θ_U is an epimorphism and $M = \operatorname{im} H_A(\beta)$. The snake lemma in $\mathcal{A}b$ shows that θ is an isomorphism of Abelian groups. Therefore, $\theta T_AH_A(\beta)\theta_A^{-1} = \beta$ yields $T_AH_A(\beta)\theta_A^{-1}\alpha = 0$. Since β is a cokernel of α, there is a $\lambda \in \operatorname{Mor}(G, T_A(M))$ such that $\lambda\beta = T_AH_A(\beta)\theta_A^{-1}$. Hence,

$$\theta\lambda\beta = \theta T_AH_A(\beta)\theta_A^{-1} = \beta$$

so that $\theta\lambda = 1_G$ because β is onto. Moreover, λ is a $\mathcal{V}_p$-isomorphism since it is an isomorphism of Abelian groups such that $v(g) = v(\theta\lambda(g)) \geq v(\lambda(g)) \geq v(g)$ for all $g \in G$. Hence, θ is a $\mathcal{V}_p$-isomorphism. The converse is obvious.

(b) Since G is A-solvable, there is an A-balanced sequence $0 \to U \stackrel{\alpha}{\to} A^n \stackrel{\beta}{\to} G \to 0$ such that U is weakly A-generated group. Proposition 3.2 (b) yields the commutative diagram

$$\begin{array}{ccccccc}
T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(\oplus_I A) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(G) & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta_U} & & \wr\downarrow{\scriptstyle\theta_{\oplus_I A}} & & \downarrow{\scriptstyle\theta_G} & & \\
U & \xrightarrow{\alpha} & \oplus_I A & \xrightarrow{\beta} & G & \longrightarrow & 0.
\end{array}$$

Observe that θ_U is an epimorphism since there is an exact sequence $\oplus_J A \to U \to 0$. Looking at the diagram in $\mathcal{A}b$, we get $\ker\theta_G = 0$. Since θ_G is a cokernel, the valuation on G is the smallest valuation v such that $v(x) \geq w(y)$ for all elements y of $T_AH_A(G)$ with $\theta_G(y) = x$, where w is the valuation on $T_AH_A(G)$ given by Theorem 2.3. On the other hand, setting $v_1(x) = w(\theta_G^{-1}(x))$ defines a valuation on G with

$v_1(x) \geq w(y)$. Thus, $w(\theta_G^{-1}(x)) \geq v(x)$ and θ_G^{-1} is a $\mathcal{V}_p$-map. Hence, θ_G is a $\mathcal{V}_p$-isomorphism.

Conversely, if θ_G is an isomorphism, then $H_A(\theta_G)$ is an isomorphism such that

$$[H_A(\theta_G)\Phi_{H_A(G)}(\sigma)](a) = [\theta_G\Phi_{H_A(G)}(\sigma)](a) = \theta_G(\sigma \otimes a) = \sigma(a)$$

for all $\sigma \in H_A(G)$ and all $a \in A$. Consequently, $\Phi_{H_A(G)}$ is an isomorphism.

We consider an exact sequence $P \xrightarrow{\alpha} F \xrightarrow{\beta} H_A(G) \to 0$ of right R-modules in which P and F are projective. It induces the exact sequence

$$T_A(P) \xrightarrow{T_A(\alpha)} T_A(F) \xrightarrow{T_A(\beta)} T_AH_A(G) \longrightarrow 0.$$

The $\mathcal{V}_p$-kernel $K \subseteq T_A(F)$ of $T_A(\beta)$ is weakly A-generated since it is an image of $T_A(P)$ which carries the valuation induced by $T_A(F)$. Also, since $T_A(\beta)$ is a cokernel, we have $T_A(\beta) = \text{Coker}(\ker(T_A(\beta))$. Therefore, the sequence

$$0 \longrightarrow K \xrightarrow{\iota} T_A(F) \xrightarrow{T_A(\beta)} T_AH_A(G) \longrightarrow 0$$

is $\mathcal{V}_p$-exact where $\iota : K \to T(F)$ is the inclusion map.

In order to see that the last sequence is A-balanced, consider the commutative diagram

$$\begin{tikzcd}
H_AT_A(F) \arrow[r, "H_AT_A(\beta)"] & H_AT_AH_A(G) & \\
F \arrow[u, "\Phi_F", "\wr"'] \arrow[r, "\beta"] & H_A(G) \arrow[u, "\Phi_{H_A(G)}", "\wr"'] \arrow[r] & 0
\end{tikzcd}$$

in which the vertical maps are isomorphisms by what has already been shown. Therefore, $H_AT_A(\beta)$ is onto, and $T_AH_A(G)$ is an A-solvable group. The same holds for $G \cong T_AH_A(G)$. □

Our next result summarizes the most important homological properties of A-solvable groups. The reader is reminded that a left module A is faithful if $A \neq IA$ for all proper right ideals I of R. Since the Jacobson radical $J = J(S)$ is nilpotent for all Artinian rings S, every indecomposable finite valuated p-groups is faithful as an R-module.

Corollary 3.4. *Let A be a finite valuated p-group.*

(a) *If G is an A-solvable group and $\alpha : A^n \to G$ is an A-balanced epimorphism, then α is a cokernel.*

(b) *If $U \to G \stackrel{\beta}{\to} H \to 0$ is an A-balanced exact sequence of valuated p-groups such that U is weakly A-generated and G is A-solvable, then H is A-solvable.*

(c) *Let $0 \to U \stackrel{\alpha}{\to} C \stackrel{\beta}{\to} H$ be a left exact sequence of valuated p-groups such that U is weakly A-generated, C is A-solvable and $\beta : C \to H$ is an A-balanced epimorphism. If $\pi : C \to G$ is a cokernel for α, then there is a monomorphism $G \to H$, and the sequence $0 \to U \stackrel{\alpha}{\to} C \stackrel{\pi}{\to} G \to 0$ is A-balanced.*

(d) *Suppose that A is faithful as a left R-module.*

(i) *If G is weakly A-generated and H is A-solvable, then every epimorphism $G \to H$ is A-balanced.*

(ii) *An A-generated group G is A-solvable if it fits into an exact sequence $0 \to U \stackrel{\alpha}{\to} G \stackrel{\beta}{\to} H \to 0$ of valuated groups in which U and H are A-solvable.*

Proof. In order to see (a), let $U = \ker \alpha$, and consider the induced diagram

$$\begin{array}{ccccccc}
T_A H_A(U) & \xrightarrow{T_A H_A(\iota)} & T_A H_A(A^n) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(G) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_U} & & \wr\downarrow{\scriptstyle \theta_{A^n}} & & \wr\downarrow{\scriptstyle \theta_G} & & \\
U & \xrightarrow{\iota} & A^n & \xrightarrow{\alpha} & G & \longrightarrow & 0,
\end{array}$$

in which $T_A H_A(\alpha)$ is a cokernel by Theorem 2.3. Since the last two vertical maps are $\mathcal{V}_p$-isomorphisms, α is a cokernel as well.

(b) Successively applying the functors H_A and T_A induces the top-row of the commutative diagram

$$\begin{array}{ccccccc}
T_A H_A(U) & \longrightarrow & T_A H_A(G) & \xrightarrow{T_A H_A(\beta)} & T_A H_A(H) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_U} & & \wr\downarrow{\scriptstyle \theta_G} & & \downarrow{\scriptstyle \theta_H} & & \\
U & \longrightarrow & G & \xrightarrow{\beta} & H & \longrightarrow & 0.
\end{array}$$

Although the 5-lemma fails in $\mathcal{V}_p$, we can apply it in $\mathcal{A}b$ to obtain that θ_H is an isomorphism of Abelian groups. However, it is also a cokernel by Lemma 2.1. This is only possible if it is a $\mathcal{V}_p$-isomorphism.

(c) Since $\pi : C \to G$ is a cokernel of α, we obtain a $\mathcal{V}_p$-map $\gamma : G \to H$ such that $\beta = \gamma\pi$. Obviously, γ is an isomorphism of Abelian groups, and hence a $\mathcal{V}_p$-monomorphism. By (b), it suffices to show that the sequence $0 \to U \xrightarrow{\alpha} C \xrightarrow{\pi} G \to 0$ is A-balanced. For this, consider $\phi \in H_A(G)$, and select $\psi \in H_A(C)$ such that $\gamma\phi = \beta\psi = \gamma\pi\psi$. Since γ is one-to-one, $\phi = \pi\psi$.

(d) Suppose that A is faithful as a left R-module, and consider the submodule $M = \operatorname{im} H_A(\beta)$ of $H_A(H)$ together with the induced commutative diagram

$$\begin{array}{ccccc} T_AH_A(G) & \xrightarrow{T_AH_A(\beta)} & T_A(M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\theta_G} & & \downarrow{\scriptstyle\theta} & & \\ G & \xrightarrow{\beta} & H & \longrightarrow & 0 \end{array}$$

of Abelian groups. Since β and θ_G are epimorphisms, the same holds for θ. However, θ also fits into the commutative diagram

$$\begin{array}{ccccccc} T_A(M) & \xrightarrow{T_A(\iota)} & T_AH_A(H) & \longrightarrow & T_A(H_A(H)/M) & \longrightarrow & 0 \\ \downarrow{\scriptstyle\theta} & & \wr\downarrow{\scriptstyle\theta_H} & & & & \\ H & \xrightarrow{1_H} & H & & & & \end{array}$$

of Abelian groups where $\iota : M \to H_A(H)$ is the inclusion map. Since θ_H is an isomorphism, $T_A(\iota)$ is onto and $T_A(H_A(H)/M) = 0$. Since A is a faithful R-module, $M = H_A(H)$, and (i) holds.

Moreover, the given sequence is A-balanced by (b) and induces the exact sequence

$$0 \longrightarrow H_A(U) \xrightarrow{H_A(\alpha)} H_A(G) \xrightarrow{H_A(\beta)} H_A(H) \longrightarrow 0.$$

As in Proposition 3.2, we obtain the commutative diagram

$$\begin{array}{ccccccc}
T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(G) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(H) & \longrightarrow & 0 \\
\wr\big\downarrow \theta_U & & \big\downarrow \theta_G & & \wr\big\downarrow \theta_H & & \\
U & \xrightarrow{\alpha} & G & \xrightarrow{\beta} & H & \longrightarrow & 0.
\end{array}$$

Since the 5-lemma is valid in $\mathcal{A}b$, the map θ_G is an isomorphism of Abelian groups. However, θ_G is a cokernel by Proposition 3.2 (b). Arguing as before, we obtain that θ_G is a $\mathcal{V}_p$-isomorphism. By Theorem 3.3, G is A-solvable. □

We continue consideration of groups of the form $T_A(M)$ by examining the case where M is a module of projective dimension 1. A ring R has *right finistic dimension* 0 if every finitely generated right R-module is either projective or has infinite projective dimension [**4**]. In particular, every local Artinian ring has right and left finistic dimension 0.

A valuated group A has the *Szele property* if every monomorphism $\alpha : P \to F$ such that P and F are A-projective groups splits. If A and G are valuated p-groups, then the *A-radical of G* is the valuated subgroup

$$R_A(G) = \cap\{\ker\phi \mid \phi \in \mathrm{Mor}(G, A)\}$$

of G. As in the case of Abelian groups, $R_A(G)$ is the kernel of the $\mathcal{V}_p$-morphism $\Psi_G : G \to A^I$ defined by $\Psi(g) = (\alpha(g))_{\alpha\in I}$, where $I = \mathrm{Mor}(G, A)$. In particular, Ψ_G induces a $\mathcal{V}_p$-monomorphism $G/R_A(G) \to A^I$ whenever G is a valuated p-group.

Before giving a description of the structure of finite valuated groups with the Szele-property, we remind the reader of the following technical result from [**1**]:

Lemma 3.5. *Let $G, H \in \mathcal{V}_p$ be such that $H = U \oplus V$ in $\mathcal{V}_p$. If $\alpha \in \mathrm{Mor}(G, H)$ is a $\mathcal{V}_p$-monomorphism such that the restriction of α to U splits in $\mathcal{V}_p$ as $G = \alpha(U) \oplus C$ with projection $\pi : G \to C$ and the map $\pi\alpha : V \to C$ splits in $\mathcal{V}_p$, then the map α splits in $\mathcal{V}_p$ as well.*

Theorem 3.6. *Let A be a finite valuated p-group.*

(a) *R has right finistic dimension* 0 *if and only if*

(i) *A has the Szele property, and*

(ii) $\mathrm{Tor}_1^R(M,A)=0$ *whenever* M *is a finitely generated right* R-*module with* $p.d.(M_R)\le 1$.

(b) A *has the Szele-property if and only if*

(i) $R_{A_i}(A_j)\ne 0$ *for* $i\ne j$.

(ii) $\cap_{j=1}^n U_j\ne 0$ *for all non-zero valuated subgroups* $U_1,\dots,U_n$ *of* A_i *such that* $R_{A_j}(A_i/U_j)=0$ *for* $j=1,\dots,n$.

Proof.

(a) Suppose that R has right finistic dimension 0. If $0\to P\overset{\alpha}{\to} F$ is exact with P and F A-projective, then the sequence

$$0\longrightarrow H_A(P)\overset{H_A(\alpha)}{\longrightarrow} H_A(F)\longrightarrow M\longrightarrow 0$$

of right R-modules splits since $p.d.(M)\le 1$. Thus, the top-row of the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & T_AH_A(P) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(F)\\ & & \wr\big\downarrow \theta_P & & \wr\big\downarrow \theta_F\\ 0 & \longrightarrow & P & \xrightarrow{\alpha} & F\end{array}$$

splits, and the same holds for the bottom. Clearly, (ii) holds.

Conversely, consider an exact sequence $0\to P\to F\to M\to 0$ in which P and F are finitely generated projective right R-modules. An application of T_A yields the sequence $0=\mathrm{Tor}_1^R(M,A)\to T_A(P)\to T_A(F)$, which splits since A has the Szele property. Using an argument similar to that used in (i) $\Rightarrow$ (ii), we obtain that M is projective.

(b) Suppose that A has the Szele property. If we can find $i,j\in\{1,\dots,n\}$ such that $R_{A_i}(A_j)=0$, then there exists a monomorphism $\alpha: A_j\to A_i^k$ for some $k<\omega$. This sequence splits since A has the Szele property. Since direct decompositions of finite valuated p-groups into indecomposables are unique, $A_j\cong A_i$.

If there are non-zero valuated subgroups $U_1,\dots,U_n$ of A_i with $R_{A_j}(A_i/U_j)=0$ for $j=1,\dots,n$ and $\cap_{j=1}^n U_j=0$, then the map $\pi: A_i\to\oplus_j A_i/U_j$, defined by $\pi(a_i)=(a_i+U_1,\dots,a_i+U_n)$, is a monomorphism. Moreover, we can select a monomorphism $\alpha_j: A_i/U_j\to A_j^{k_j}$ for each j. The α_j coordinatewise induce a monomorphism $\alpha:\oplus_j A_i/U_j\to\oplus_{j=1}^n A_j^{k_j}$. Clearly, $\alpha\pi: A_i\to\oplus_{j=1}^n A_j^{k_j}$ is a

$\mathcal{V}_p$-monomorphism which splits by (a), say, $\beta\alpha\pi = 1_{A_i}$. Thus, A_i is isomorphic to an indecomposable direct summand of $\oplus_j[A_i/U_j]$, and we can find $j \in \{1, \dots, n\}$ such that A_i is isomorphic to a direct summand of A_i/U_j contradicting $|A_i/U_j| < |A_i|$.

Conversely, let F and P be A-projective groups, and let $\alpha : P \to F$ be a monomorphism. We have $P = A_1^{r_1} \oplus \cdots \oplus A_n^{r_n}$ and $F = A_1^{s_1} \oplus \cdots \oplus A_n^{s_n}$ for some $r_1, \dots, r_n, s_1, \dots, s_n < \omega$. We establish the splitting of the map α by induction on $r = r_1 + \cdots + r_n$. If $r = 1$, then we may assume that $r_1 = 1$. Let $\pi_i : F \to A_i^{s_i}$ be the projection onto the ith-coordinate, and set $U_i = \ker \pi_i\alpha$. If $U_i = 0$ for some $i \geq 2$, then $R_{A_i}(A_1) = 0$, which contradicts (i). Since $\cap_{i=1}^n U_i = 0$ due to the fact that α is one-to-one, $U_1 = 0$ by (ii). By (a), A_1 has the Szele property since its has a local endomorphism ring, and $\pi_1\alpha$ splits, say $\beta\pi_1\alpha = 1_{A_1}$. Thus, α splits.

Now, suppose that $r > 1$. As before, we may assume $r_1 > 0$. For the decomposition $D = A_1^{r_1-1} \oplus \cdots \oplus A_n^{r_n}$, let $\delta_1 : D \to P$ and $\delta_2 : A_1 \to P$ be the embeddings associated with the decomposition $P = A_1 \oplus D$. By the induction hypothesis, there is a map $\beta_1 : F \to D$ such that $\beta_1\alpha\delta_1 = 1_D$. Thus, $F = \alpha(D) \oplus C$ for some valuated subgroup of F. Let $\pi : F \to C$ denote the projection with kernel $\alpha(D)$. If $\pi\alpha(a_1) = 0$ for some $a_1 \in A_1$, then $\alpha(a_1) \in \alpha(D)$, which is a contradiction unless $a_1 = 0$. Hence, $\pi\alpha\delta_2 : A_1 \to C$ is a monomorphism, which splits by the last paragraph, say, $\beta_2\pi\alpha\delta_2 = 1_{A_1}$ for a map $\beta_2 : C \to A_1$. By Lemma 3.5, α splits. $\square$

4. Simply presented groups. A (p-)*valuated tree* (X, v) is a set X with a (partial) multiplication by p and a map v defined on X subject to the following rules

(i) $v(x)$ is an ordinal or ∞ for all $x \in X$.

(ii) If $p^n x = x$ for some $0 < n < \omega$, then $px = x$, and there is exactly one element in X with this property, called the root of X.

(iii) $v(px) > v(x)$ whenever px is defined.

We can associate a *simply presented* valuated p-group $S(X)$ with a rooted tree X by setting $S(X) = F_X/R_X$, where F_X is a free $\mathbb{Z}_p$-module with basis $\{[x] | x \in X\}$ and R_X is generated by the elements $p[x] - [px]$. Every non-zero element $g \in S(X)$ can be uniquely written as $g = \Sigma_{x \in X} n_x([x] + R_X)$. We now define the valuation on $S(X)$ by $v(g) = \min\{v(x) \mid n_x \neq 0\}$.

A map $\psi : X \to Y$, where X and Y are valuated trees, is a *tree map* if

(i) $\psi(px) = p\psi(x)$ whenever px exists and

(ii) $v(\psi(x)) \geq v(x)$ for all $x \in X$.

Every tree map $\psi : X \to Y$ induces a $\mathcal{V}_p$-map $\overline{\psi} : S(X) \to S(Y)$. A tree map $r : X \to X$ is a retraction if $r^2 = r$. In particular, there is an order preserving retraction from $S(X)$ onto X for all valuated trees [**7**].

If $A = S(T)$ and $G = S(X)$ are simply presented groups, then any monomorphism $\alpha : S(T) \to S(X)$ induces a one-to-one tree map $r\alpha \mid T : T \to X$, where $r : S(X) \to X$ is the retraction introduced in [**7**]. We now show that a similar result need not hold for epimorphisms $\pi : A^n \to G$:

Example 4.1. Let A be a cyclic group of order p^3 with the height valuation. We consider $F = A \oplus A$ and its valuated subgroup U generated by $(p^2x, -p^2x)$. There exists a simply presented valuated p-group G generated by x_1 and x_2 such that $v(x_1) = 1$, $v(px_1) = 4$, $v(x_2) = 2$, $v(px_2) = 3$, $p_1^3 = 0$ and $p^2x_1 = p^2x_2$ has value 5. The group G is indecomposable, and it is generated by x_1 and $x_1 - x_2$. Sending $(1, 0)$ to x_1 and $(0, 1)$ to $x_1 - x_2$ yields an epimorphism of $F = A \oplus A$ onto $G = S(X)$ where X is the valuated tree given by x_1 and x_2. If $r : S(X) \to X$ is the order-preserving retraction from [**7**], then $r(x_1) = x_2$ and $r(x_1 - x_2)$ has order p^2. Therefore, $x_2 \notin r\phi(T \cup T)$.

The next result illustrates Corollary 3.4 (c) further:

Example 4.2. Let A be a cyclic valuated group of order p^3, and choose generator x of A such that $v(x) = 1$, $v(px) = 3$ and $v(p^2x) = 5$. We consider $F = A \oplus A$ and its valuated subgroup U generated by $(p^2x, -p^2x)$.

The cokernel $\pi : F \to G$ of the embedding $U \subseteq F$ has order p^5. If we consider the induced sequence

$$0 \longrightarrow H_A(U) \longrightarrow H_A(F) \overset{H_A(\pi)}{\longrightarrow} H_A(G),$$

then $|H_A(U)| = p$ and $|H_A(F)| = p^6$. Clearly, U is weakly A-generated. As an Abelian group, we have $G \cong \mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$ since it is generated by two elements and has order p^5. Therefore,

$$p^5 \leq |H_A(G)| \leq |\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^3\mathbb{Z}, \mathbb{Z}/p^3\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z})| = p^5,$$

and $H_A(\pi)$ is onto. By Proposition 3.4 (a), G is A-solvable and a $\mathcal{V}_p$-direct sum of cyclics.

On the other hand, there exists a simply presented valuated p-group H generated by x_1 and x_2 such that $v(x_1) = 1$, $v(px_1) = 4$, $v(x_2) = 2$, $v(px_2) = 3$, $p_1^3 = 0$ and $p^2x_1 = p^2x_2$ has value 5. The group H is indecomposable, and there is an epimorphism $\phi : F \to H$ with kernel U. Arguing as before yields that

$$0 \longrightarrow H_A(U) \longrightarrow H_A(F) \overset{H_A(\phi)}{\longrightarrow} H_A(H) \longrightarrow 0$$

is exact. Since G is the cokernel of $\ker \phi$, there is a map $\lambda : G \to H$ with $\phi = \lambda\pi$. Another order argument establishes that λ is an isomorphism of groups, and hence, a $\mathcal{V}_p$-monomorphism. However, H is not A-solvable by Corollary 3.4 (b).

We now show that usually there exist A-presented groups which are not A-solvable unless A is cyclic.

Theorem 4.3. *Let A be an indecomposable finite valuated p-group:*

(a) *If $|A/pA| > p$, then A/pA is not A-solvable.*

(b) *If $|A/JA| > p$, then A/JA is not A-solvable.*

(c) *The following are equivalent if A is simply presented:*

(i) *A is cyclic.*

(ii) *Every A-presented group is A-solvable.*

Proof.

(a) Let J denote the Jacobson-radical of R, and

$$S = \operatorname{Mor}_{\mathcal{V}_p}(A/pA, A/pA)$$

the $\mathcal{V}_p$-endomorphism ring of A/pA, where A/pA has the cokernel valuation. Define a map $\lambda : R \to S$ by $\lambda(\alpha)(a + pA) = \alpha(a) + pA$ for all $\alpha \in R$. As in the case of Abelian groups, $\lambda(\alpha)$ is well defined. If

$a \in A$, then we can find $a_0 \in A$ such that $v(a+pA) = v(a+pa_0)$ **[10]**. Select $a_1 \in A$ with $v(\alpha(a)+pA) = v(\alpha(a)+pa_1)$. Then, $v(a+pA) = v(a+pa_0) \leq v(\alpha(a)+p\alpha(a_0)) \leq v(\alpha(a)+pa_1) = v(\lambda(\alpha)(a+pA))$. Hence, $\lambda(\alpha) \in S$. As in the case of Abelian groups, λ is a ring-morphism with $\ker\lambda = \mathrm{Mor}_{\mathcal{V}_p}(A, pA) = H_A(pA)$. Since $H_A(pA)$ is a proper ideal of R, we obtain $H_A(pA) \subseteq J$, and $R/H_A(pA)$ is a local ring since A is indecomposable. On the other hand, A/pA is a valuated direct sum of cyclic groups of order p. Since $|A/pA| > p$, its endomorphism ring contains at least one non-trivial idempotent and cannot be local. Therefore, λ cannot be onto.

On the other hand, consider the exact sequence $0 \to pA \to A \to A/pA \to 0$ of valuated groups. By Proposition 3.2, it induces the commutative diagram

$$\begin{tikzcd}
T_AH_A(pA) \arrow[r, "T_AH_A(\alpha)"] \arrow[d, "\theta_pA"] & T_AH_A(A) \arrow[r, "T_AH_A(\beta)"] \arrow[d, "\theta_A", "\wr"'] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
pA \arrow[r, "\alpha"] & A \arrow[r, "\beta"] & A/pA \arrow[r] & 0
\end{tikzcd}$$

for some submodule M of $H_A(A/pA)$. The evaluation map θ is a cokernel and satisfies $\theta = \theta_{A/pA}T_A(\iota)$ where $\iota : M \to H_A(A/PA)$ is the inclusion map. If A/pA were A-solvable, then $\theta_{A/pA}$ would be an isomorphism, and $T_A(\iota)$ would be onto. As before, this yields that $T_A(H_A(A/pA)/M) = 0$, from which we obtain $M = H_A(A/pA)$ since A is faithful. Therefore, the sequence

$$0 \longrightarrow H_A(pA) \longrightarrow H_A(A) \longrightarrow H_A(A/pA) \longrightarrow 0$$

is exact. We also have the induced sequence

$$0 \longrightarrow \mathrm{Mor}_{\mathcal{V}_p}(A/pA, A/pA) \longrightarrow \mathrm{Mor}_{\mathcal{V}_p}(A, A/pA).$$

Clearly, this map is onto, too, so that

$$|\lambda(R)| = |H_A(A)/H_A(pA)| = |H_A(A/pA)| = |S| < \infty.$$

Hence, λ is onto, which is not possible by the last paragraph. Consequently, A/pA is not A-solvable.

(b) In order to simplify our notation, we write $G = T_A(R/J)$. Assume that we can find an A-balanced exact sequence $0 \to U \to F \to G \to 0$ with F A-free and U weakly A-generated. This induces the commutative diagram

$$\begin{array}{ccccccc} T_AH_A(U) & \xrightarrow{T_AH_A(\alpha)} & T_AH_A(\oplus_I A) & \xrightarrow{T_AH_A(\beta)} & T_AH_A(G) & \longrightarrow & 0 \\ \downarrow \theta_U & & \wr\downarrow \theta_{\oplus_I A} & & \downarrow \theta_G & & \\ & \xrightarrow{\alpha} & \oplus_I A & \xrightarrow{\beta} & G & \longrightarrow & 0, \end{array}$$

in which θ_U is an epimorphism since U is weakly A-generated. Looking at this diagram as a diagram in $\mathcal{A}b$, we obtain that θ_G is an isomorphism of Abelian groups.

Since A is indecomposable and finite, R/J is a field of characteristic p. Thus, $pR \subseteq J$, and $G = T_A(R/J)$ is a valuated direct sum of cyclic groups of order p [**8**], say, $G = G_1 \oplus \cdots \oplus G_n$. We obtain the diagram

$$\begin{array}{ccccccc} T_AH_A(G_i) & \longrightarrow & T_AH_A(G) & \longrightarrow & T_AH_A(\oplus_{j\neq i}G_j) & \longrightarrow & 0 \\ \downarrow \theta_{G_i} & & \downarrow \theta_G & & \downarrow \theta_{\oplus_{j\neq i}G_j} & & \\ G_i & \longrightarrow & \oplus_I A & \longrightarrow & \oplus_{j\neq i}G_j & \longrightarrow & 0 \end{array}$$

of valuated p-groups in which the rows split in $\mathcal{V}_p$. Viewing it as a diagram in $\mathcal{A}b$, we obtain that θ_{G_i} is an isomorphism of Abelian groups since the same holds for θ_G.

In order to see that this is not possible, consider a cyclic valuated group H of order p such that θ_H is an isomorphism of Abelian groups. Since $H_A(H) \neq 0$, we can find $0 \neq \alpha \in H_A(H)$. Clearly, α is onto. If V denotes the kernel of α, then the sequence $0 \to V \to A \stackrel{\alpha}{\to} H$ is left exact, and the same is true for the induced sequence

$$0 \longrightarrow H_A(V) \longrightarrow H_A(A) \stackrel{H_A(\alpha)}{\longrightarrow} H_A(H).$$

Let $M = \mathrm{im}(H_A(\alpha)) \subseteq H_A(H)$ and $\iota : M \to H_A(H)$ be the corresponding embedding. Observe that $H_A(V)$ is a proper right ideal of R since, otherwise, $U = A$, a contradiction. Therefore, $H_A(V) \subseteq J$. We obtain an exact sequence

$$M \cong H_A(A)/H_A(V) \longrightarrow R/J \longrightarrow 0$$

of right R-modules. Applying T_A yields

$$|T_A(M)| \geq |T_A(R/J)| = |A/JA| > p.$$

Arguing as before, we obtain the commutative diagram

$$\begin{tikzcd}
 & T_A H_A(V) \arrow[r, "T_A H_A(\alpha)"] \arrow[d, "\theta_V"] & T_A H_A(A) \arrow[r, "T_A(\alpha)"] \arrow[d, "\theta_A", "\wr"'] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & V \arrow[r, "\alpha"] & A \arrow[r, "\beta"] & H \arrow[r] & 0
\end{tikzcd}$$

of Abelian groups. Since the snake lemma is valid in $\mathcal{A}b$, the evaluation map θ is an epimorphism of Abelian groups. Arguing as before, we obtain $M = H_A(H)$. Therefore, $|T_A(M)| = |T_A H_A(H)| = |H| = p$, a contradiction.

(c) (i) $\Rightarrow$ (ii). Let $G \cong T_A(M)$ for some right R-module M. Since A is a cyclic group, $R \cong \mathbb{Z}/p^n\mathbb{Z}$ for some $n < \omega$. Hence, every finitely generated R-module is a direct sum of cyclic submodules. By Theorem 3.3 (a), $G \cong T_A(M)$ for some finite right R-module M. Since M is a direct sum of cyclic modules, G is a direct sum of valuated subgroups of the form $T_A(U_i)$, where each U_i is a cyclic right R-module, and it suffices to show that $T_A(M)$ admits an A-balanced exact sequence of the desired form whenever M is cyclic.

In order to see this, consider an exact sequence $0 \to p^k R \to R \xrightarrow{\pi} M \to 0$ for some $k < n$. To establish that the induced sequence $0 \to p^k A \to A \xrightarrow{T_A(\pi)} T_A(M) \to 0$ is A-balanced, observe that M is isomorphic to a right ideal of R. We obtain the commutative diagram

$$\begin{tikzcd}
 & H_A T_A(M) \arrow[r] & H_A T_A(R) \\
0 \arrow[r] & M \arrow[r] \arrow[u, "\Phi_M"'] & R \arrow[u, "\Phi_R"', "\wr"]
\end{tikzcd}$$

which yields that Φ_M is a monomorphism. On the other hand, $T_A(M)$ is a cyclic group of order at most p^{n-k} since it is an epimorphic image of $M^+ \otimes_{\mathbb{Z}} A \cong \mathbb{Z}/p^{n-k}\mathbb{Z}$. Thus, $p^{n-k} = |M| \leq |H_A T_A(M)| \leq p^{n-k}$, and Φ_M is an isomorphism. On the other hand, the commutative diagram

$$\begin{tikzcd}
H_A T_A(R) \arrow[r, "T_A H_A(\pi)"] & H_A T_A(M) & \\
R \arrow[r, "\pi"] \arrow[u, "\Phi_R"', "\wr"] & M \arrow[r] \arrow[u, "\Phi_M"'] & 0
\end{tikzcd}$$

yields that $T_A H_A(\pi)$ is onto. Clearly, $p^k A$ is weakly A-generated.

(ii) $\Rightarrow$ (i). Suppose that A is not cyclic, and choose a finite valuated p-tree X such that $A = S(X)$. Suppose that $x_1, \ldots, x_n$ are the elements of highest order in X. Computing as in the case of Abelian groups, we can show that $\{x_1 + pA, \ldots, x_n + pA\}$ is a p-basis for A/pA. Since A is acyclic, $n > 1$ and $|A/pA| > p$. From (a), A/pA is an A-presented group which is not A-solvable. $\square$

We want to contrast the last result with **[12,** Lemma 9], which shows that $|R/J| = p$ if A is an indecomposable simply presented group. Moreover, $T_A(M)$ may be A-solvable for some finitely generated R-module M, although $T_A(X)$ is not A-solvable for all composition factors X of M. For instance, if A is indecomposable, then the only possible composition factors of a finitely generated R-module are isomorphic to R/J. Thus, $T_A(M)$ can be A-solvable although $T_A(R/J)$ may not be by part (b) of the last theorem.

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