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UNIFORMLY NON-SQUARE POINTS AND REPRESENTATION OF FUNCTIONALS OF ORLICZ-BOCHNER SEQUENCE SPACES

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ABSTRACT. In this work, a representation of functionals and a necessary and sufficient condition for uniformly non-square points of Orlicz-Bochner sequence spaces endowed with the Orlicz norm are given.

1. Introduction. Uniform non-squareness of Banach spaces has been defined by James [**12, 13**] in 1964 as the geometric property which implies super-reflexivity. Thus, after proving this property for a Banach space, we know, without any characterization of the dual space, that it is super-reflexive, and reflexive as well. Recently, García-Falset, et al., [**6**] have shown that uniformly non-square Banach spaces have the fixed point property. Therefore, it is natural and interesting to look for criteria of non-squareness properties in various well-known classes of Banach spaces. In 2013, Foralewski, et al., [**5**] presented criteria for uniform non-squareness and locally uniform non-squareness of Orlicz-Lorentz sequence spaces. Among a great number of papers concerning this topic, see [**1, 3, 4, 7, 8, 10, 11, 20**].

In 1985, the problem of uniform non-squareness of Orlicz-Bochner spaces was initiated by Hudzik [**9**]. He gave criteria for uniform non-squareness of Orlicz-Bochner function spaces equipped with the Luxemburg norm under the condition that the generating Orlicz function is uniformly convex. Also, in 1985, Kamińska and Turett [**14**] removed the restriction of the generating Orlicz function being uniformly convex and obtained necessary and sufficient conditions of uniform non-squareness of Orlicz-Bochner function spaces equipped with

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the Luxemburg norm. In 2009, Zhang [**22**] studied the non-squareness of Orlicz-Bochner function spaces endowed with both the Luxemburg and the Orlicz norms and gave criteria for uniform non-squareness of Orlicz-Bochner function spaces equipped with the Orlicz norm. He also gave a necessary and sufficient condition for non-square points of Orlicz-Bochner sequence spaces equipped with the Orlicz norm. In 2014, Shang and Cui [**17**] repeated the same result by using the same techniques as Zhang [**22**].

Usually, the pointwise expression describes the local structure more finely than the spatial expression in a Banach space. It is convenient to obtain a geometric property of a Banach space from the corresponding pointwise property. For example, each point on the unit sphere being extreme implies that the space is rotund. Therefore, Wang, et al., [**21**] introduced notions of non-square and uniformly non-square points in Banach spaces and obtained criteria for them in Orlicz spaces. Recently, Shi and Wang [**18, 19**] studied non-square and uniformly non-square points of Orlicz-Bochner spaces endowed with the Luxemburg norm. As is well known, a Banach space is locally uniformly non-square if every point on the unit sphere is a uniformly non-square point. Thus, we can readily obtain the criteria for locally uniform non-squareness of a Banach space by using the property of the uniformly non-square points. However, Orlicz-Bochner sequence spaces are complicated structures since they are an interaction effect of the Bochner and Orlicz sequence spaces, thus making it a real struggle to obtain criteria for uniformly non-square points in Orlicz-Bochner sequence spaces.

For the Orlicz-Bochner sequence space endowed with the Luxemburg norm, the criterion of the uniformly non-square point is given [**19**]. To date, for the Orlicz-Bochner sequence space endowed with the Orlicz norm, the criterion of the uniformly non-square point has not been given. This is due to the fact that its structure is complicated under the Orlicz norm. In this paper, we give necessary and sufficient conditions of uniformly non-square points and locally uniformly non-squareness of Orlicz-Bochner sequence spaces equipped with the Orlicz norm. In order to get these criteria, we first establish the representation of the continuous linear functional, which is a very useful tool and also plays an important role in many other fields. Furthermore, we easily obtain the criterion of the classical Orlicz sequence spaces from the corresponding one of the Bochner type.

2. Preliminaries. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. Let $(X, \|\cdot\|)$ represent a real Banach space and $S(X)$ the unit sphere of X. The dual space of X is denoted by X^*. Let M be an $\mathcal{N}$ function and N the complementary function of M, i.e., $N(v) = \sup_{u\in\mathbb{R}}\{u|v| - M(u)\}$. Young's inequality $uv \le M(u) + N(v), u, v \in \mathbb{R}$, holds, and the equality in the Young's inequality holds if and only if $v \in [p_-(|u|)\operatorname{sgn} u, p_+(|u|)\operatorname{sgn} u]$ or $u \in [q_-(|v|)\operatorname{sgn} v, q_+(|v|)\operatorname{sgn} v]$, where $p_-(|u|)$ and $p_+(|u|)$ are the left and right derivatives of M at $|u|$ and $q_-(|v|)$ and $q_+(|v|)$ are the left and right derivatives of N at $|v|$, respectively. We say that M satisfies the condition δ_2 (write $M \in \delta_2$) if there exist $u_0 > 0$ and $K > 2$ such that $M(2u) \le KM(u), |u| \le u_0$. For details, see **[2, 15]**.

Given a sequence $u = (u(1), u(2), \ldots, u(n), \ldots)$ with $u(i) \in X$ for all $i \in \mathbb{N}$, set $|u| = (\|(u(1)\|, \|u(2)\|, \ldots, \|u(n)\|, \ldots)$, and define a modular $\rho_M(u)$ of u by the formula $\rho_M(u) = \rho_M(|u|) = \Sigma_{i=1}^{\infty} M(\|u(i)\|)$. Let

$$\widetilde{l_M}(X) = \{u : \text{there exists } \lambda > 0 \text{ such that } \rho_M(\lambda u) < \infty\}.$$

The linear set $\widetilde{l_M}(X)$ equipped with the Orlicz norm

$$\|u\|_M = \sup_{\rho_N(v)\le 1} \sum_{i=1}^{\infty} \langle u(i), v(i)\rangle,$$

where $v \in \widetilde{l_N}(X^*)$, or with the Luxemburg norm

$$\|u\|_{(M)} = \inf\{\lambda > 0 : \rho_M\left(\frac{u}{\lambda}\right) < \infty\}$$

becomes a Banach space which is called the Orlicz-Bochner sequence space, denoted by

$$l_M(X) = (\widetilde{l_M}(X), \|\cdot\|_M), l_{(M)}(X) = (\widetilde{l_M}(X), \|\cdot\|_{(M)}).$$

Let

$$\widetilde{h_M}(X) = \{u : \text{for all } \lambda > 0 \text{ such that } \rho_M(\lambda x) < \infty\}.$$

Then, under the Orlicz or the Luxemburg norms, $\widetilde{h_M}(X)$ is a closed subspace of $l_M(X)$ or $l_{(M)}(X)$, denoted by $h_M(X)$ or $h_{(M)}(X)$, respectively.

Lemma 2.1 ([**2**]). *For* $u \in l_M$*, we have* $\lim_{n\to\infty} \|u - u_n\|_{(M)} = \lim_{n\to\infty} \|u - u_n\|_M = \theta(u)$*, where* l_M *denotes the classical Orlicz sequence space,* $\theta = \theta(u) = \inf\{\lambda > 0 : \rho_M(u/\lambda) < \infty\}$ *and* $u_n = (u(1), \ldots, u(n), 0, \ldots)$.

Lemma 2.2 ([**2**]). *For* $u \in l_M(X)$*, we have*

$$\|u\|_M = \inf_{k>0} \frac{1}{k}[1 + \rho_M(ku)] = \frac{1}{k}[1 + \rho_M(ku)]$$

if and only if $k \in K(u)$*, where* $K(u) = [k^*, k^{**}]$*,* $k^{**} = k^{**}(u) = \sup\{k > 0 : \rho_N(p_+(k|u|)) \le 1\}$ *and* $k^* = k^*(u) = \inf\{k > 0 : \rho_N (p_+(k|u|)) \ge 1\}$.

Lemma 2.3 ([**2**]). *For* $\|u\|_M \le 1$ *or* $\|u\|_{(M)} \le 1$*, we have* $\rho_M(u) \le \|u\|_M$ *or* $\rho_M(u) \le \|u\|_{(M)}$*, respectively.*

Lemma 2.4 ([**2**]). *Hölder's inequality*:

$$\sum_{i=1}^{\infty} \langle u(i), v(i)\rangle \le \|u\|_M \|v\|_{(N)}, \quad u \in l_M(X), v \in l_{(N)}(X^*).$$

Lemma 2.5 ([**2**]). *For any* $u \in l_M(X)$*, we have*

$$\|u\|_M = \sup_{\rho_N(v)\le 1} \sum_{i=1}^{\infty} \|u(i)\|\|v(i)\|,$$

where $v \in \widetilde{l_N}(X^*)$.

Lemma 2.6 ([**2**]). *Suppose that* $N \in \delta_2$*. Then, for any positive constant* C*, the set* $\{k \in K(u) : u \in l_M, \|u\|_M \ge C\}$ *is bounded.*

Lemma 2.7 ([**2, 14**]). *The following statements are equivalent*:

(i) $N \in \delta_2$.
(ii) *For all* $l > 1$*, and for all* $v_0 > 0$ *there exists a* $K > 1$ *such that, for all* $0 < v < v_0$*, we have* $N(lv) \le KN(v)$.
(iii) *There exist* $l_1 > 1$*,* $v_1 > 0$*,* $K_1 > 1$ *such that, for all* $0 < v < v_1$*, we have* $N(l_1 v) \le K_1 N(v)$.
(iv) *For all* $l_2 > 1$ *and* $v_2 > 0$ *there exists a* $\delta > 1$ *such that, for all* $0 < v < v_0$*, we have* $N(\delta v) \le l_2 \delta N(v)$.

(v) *For all* $\alpha \in (0,1)$ *and* $u_0 > 0$ *there exists a* $\beta \in (0,1)$ *such that, for all* $0 < u \le u_0$*, we have* $M(\alpha u) \le \alpha\beta M(u)$*.*

In Lemma 2.7 above, for $u_0 > 0$, $\alpha \in (0,1)$, we see that $\beta = \beta(\alpha)$ is non-decreasing with respect to α, where

$$\beta(\alpha) = \sup_{0<u\le u_0} \frac{M(\alpha u)}{\alpha M(u)}.$$

Definition 2.8 ([**21**]). $x \in S(X)$ is called a *uniformly non-square point* if there exists a $\delta \in (0,2)$ such that $\min\{\|x+y\|, \|x-y\|\} < 2-\delta$ for all $y \in S(X)$.

Lemma 2.9 ([**14**]). $x/\|x\|$ *is a uniformly non-square point of* X *if and only if there exists a* $\delta \in (0,1)$*, for every* $y \in X$*, we have*

$$\min\{\|x+y\|, \|x-y\|\} \le (\|x\| + \|y\|)\left(1 - \frac{2\delta \min\{\|x\|, \|y\|\}}{\|x\| + \|y\|}\right).$$

Lemma 2.10. *If* $N \in \delta_2$*, for each* $x \in B_\eta(X)$ *with* $x/\|x\|$ *a uniformly non-square point, we have a positive number* $r \in (0,1)$ *such that*

$$M\left(\frac{kh}{k+h}\|x+y\|\right) + M\left(\frac{kh}{k+h}\|x-y\|\right) \le 2r\left[\frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|)\right]$$

holds for all $k, h \in (1,b)$ *and* $y \in B_\eta(X)$*, where* u_0 *and* b *are arbitrary constants,* $u_0 > 0$*,* $b > 1$*,* $\eta = u_0/b$ *and* $B_\eta(X) = \{x \in X : \|x\| \le \eta\}$*.*

Proof. From Lemma 2.7, for $u_0 > 0$, and for all $\alpha \in (0,1)$, there exists a $\beta = \beta(\alpha) \in (0,1)$ such that

$$M(\alpha u) \le \alpha\beta M(u) \tag{2.1}$$

for all $u \in (0, u_0]$, where β is non-decreasing with respect to α. For $\alpha = (b+1)/(b+2)$, denoting

$$\beta_0 = \beta\left(\frac{b+1}{b+2}\right),$$

we have

$$M\left(\frac{b+1}{b+2}u\right) \le \frac{b+1}{b+2}\beta_0 M(u)$$

for all $u \in (0, u_0]$. Let

$$d = \min\left\{\frac{1}{b(b+2)}, \frac{1}{2}\left(\frac{1}{\beta_0} - 1\right)\right\}.$$

The remainder of the proof shall be given in several cases.

Case 1. $\|x\| \ge \|y\|$.

(i) $\|y\| \ge d\|x\|$. From Lemma 2.9, we know that:

(a)

$$\|x+y\| \le (\|x\| + \|y\|)\left(1 - \frac{2\delta\|y\|}{\|x\| + \|y\|}\right)$$

or

(b)

$$\|x-y\| \le (\|x\| + \|y\|)\left(1 - \frac{2\delta\|y\|}{\|x\| + \|y\|}\right)$$

holds.

In case (a), we have

$$\begin{aligned}
&M\left(\frac{kh}{k+h}\|x+y\|\right) + M\left(\frac{kh}{k+h}\|x-y\|\right) \\
&\quad \le M\left(\frac{kh}{k+h}\left(1 - \frac{2\delta\|y\|}{\|x\|+\|y\|}\right)(\|x\|+\|y\|)\right) + M\left(\frac{kh}{k+h}(\|x\|+\|y\|)\right) \\
&\quad \le \left(1 - \frac{2\delta\|y\|}{\|x\|+\|y\|}\right)\left[\frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|)\right] \\
&\qquad + \frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|) \\
&\quad = 2\left(1 - \frac{\delta\|y\|}{\|x\|+\|y\|}\right)\left[\frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|)\right] \\
&\quad \le 2\left(1 - \frac{\delta d\|x\|}{\|x\|+d\|x\|}\right)\left[\frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|)\right] \\
&\quad = 2\left(1 - \frac{\delta d}{1+d}\right)\left[\frac{h}{k+h}M(k\|x\|) + \frac{k}{k+h}M(h\|y\|)\right].
\end{aligned}$$

In case (b), in the same manner, we obtain

$$M\left(\frac{kh}{k+h}\|x+y\|\right)+M\left(\frac{kh}{k+h}\|x-y\|\right) \leq 2\left(1-\frac{\delta d}{1+d}\right)\left[\frac{h}{k+h}M(k\|x\|)+\frac{k}{k+h}M(h\|y\|)\right]$$

(ii) $\|y\| < d\|x\|$. Since

$$\frac{h}{k+h}(1+d) \leq \frac{b}{1+b}\left(1+\frac{1}{b(b+2)}\right)=\frac{b+1}{b+2},$$

then

$$\beta\left(\frac{h}{k+h}(1+d)\right) \leq \beta\left(\frac{b+1}{b+2}\right)=\beta_0. \tag{2.2}$$

Since $\|x\| \leq u_0/b$, $k \in (1,b)$, then $k\|x\| \leq u_0$, and, from (2.1) and (2.2), we have

$$\begin{aligned}
&M\left(\frac{kh}{k+h}\|x+y\|\right)+M\left(\frac{kh}{k+h}\|x-y\|\right)\\
&\leq 2M\left(\frac{kh}{k+h}(\|x\|+\|y\|)\right) \leq 2M\left(\frac{kh}{k+h}(1+d)\|x\|\right)\\
&\leq 2\frac{h}{k+h}(1+d)\beta\left(\frac{h}{k+h}(1+d)\right)M(k\|x\|)\\
&\leq 2\left[1+\frac{1}{2}\left(\frac{1}{\beta_0}-1\right)\right]\beta_0\frac{h}{k+h}M(k\|x\|)\\
&= 2\left(\frac{1}{2}\beta_0+\frac{1}{2}\right)\frac{h}{k+h}M(k\|x\|)\\
&\leq 2\left(\frac{1}{2}\beta_0+\frac{1}{2}\right)\left[\frac{h}{k+h}M(k\|x\|)+\frac{k}{k+h}M(h\|y\|)\right].
\end{aligned}$$

Setting

$$r=\max\left\{\frac{1}{2}\beta_0+\frac{1}{2}, 1-\frac{\delta d}{1+d}\right\},$$

we have that $r \in (0,1)$. Thus, if $\|x\| \geq \|y\|$, then we have

$$M\left(\frac{kh}{k+h}\|x+y\|\right)+M\left(\frac{kh}{k+h}\|x-y\|\right) \leq 2r\left[\frac{h}{k+h}M(k\|x\|)+\frac{k}{k+h}M(h\|y\|)\right].$$

Case 2. $\|x\| < \|y\|$. As in Case 1, we can prove that $r \in (0,1)$, and therefore,

$$M\left(\frac{kh}{k+h}\|x+y\|\right)+M\left(\frac{kh}{k+h}\|x-y\|\right) \leq 2r\left[\frac{h}{k+h}M(k\|x\|)+\frac{k}{k+h}M(h\|y\|)\right]. \quad \square$$

3. Main results.

Lemma 3.1. *Denote*

$$\widehat{l_N}(X^*) = \Big\{v : \textit{for all } u \in \widetilde{l_M}(X) \textit{ with } \rho_M(u) < \infty; \ \Big|\sum_{i=1}^{\infty}\langle u(i), v(i)\rangle\Big| < \infty \textit{ holds}\Big\}$$

and, for any $v \in \widehat{l_N}(X^*)$, *define*

$$\|v\| = \sup\left\{\sum_{i=1}^{\infty}\langle u(i), v(i)\rangle : \textit{for all } u \in \widetilde{l_M}(X) \textit{ with } \rho_M(u) \leq 1\right\}.$$

Then, $\|v\| < \infty$.

Proof. Suppose that there exists a $v \in \widehat{l_N}(X^*)$ such that $\|v\| = \infty$. For each n, we choose $u_n \in \widetilde{l_M}(X)$ with $\rho_M(u_n) \leq 1$ and $\langle u_n(i), v(i)\rangle \geq 0$ for all $i \in \mathbb{N}$ such that $\Sigma_{i=1}^{\infty}\langle u_n(i), v(i)\rangle > 2^n$. Letting $g_n = \Sigma_{m=1}^{n} u_m/2^m$, we deduce

$$\rho_M(g_n) = \sum_{i=1}^{\infty} M(\|g_n(i)\|) = \sum_{i=1}^{\infty} M\left(\sum_{m=1}^{n}\left\|\frac{u_m(i)}{2^m}\right\|\right) \leq \sum_{i=1}^{\infty}\sum_{m=1}^{n}\frac{M(\|u_m(i)\|)}{2^m} = \sum_{m=1}^{n}\frac{1}{2^m}\sum_{i=1}^{\infty} M(\|u_m(i)\|) < 1.$$

Since $|\Sigma_{i=1}^{\infty}\langle u_m(i), v(i)\rangle| < \infty$, for all $m \in \mathbb{N}$, then

$$\sum_{i=1}^{\infty}\langle g_n(i), v(i)\rangle = \sum_{i=1}^{\infty}\Big\langle \sum_{m=1}^{n} \frac{u_m(i)}{2^m}, v(i)\Big\rangle = \sum_{i=1}^{\infty}\sum_{m=1}^{n} \frac{1}{2^m}\langle u_m(i), v(i)\rangle$$
$$= \sum_{m=1}^{n} \frac{1}{2^m}\sum_{i=1}^{\infty}\langle u_m(i), v(i)\rangle > \sum_{m=1}^{n} \frac{1}{2^m}2^m = n.$$

Let $g = \Sigma_{n=1}^{\infty}u_n/2^n$, in which $g(i) = \Sigma_{n=1}^{\infty}u_n(i)/2^n$ converges in the norm of X for each $i \in \mathbb{N}$. From Levi's lemma [**16,** page 87, Theorem 10], we get

$$\rho_M(g) = \sum_{i=1}^{\infty} M(\|g(i)\|) = \sum_{i=1}^{\infty} M\Bigg(\Bigg\|\sum_{n=1}^{\infty}\frac{u_n(i)}{2^n}\Bigg\|\Bigg)$$
$$\le \sum_{i=1}^{\infty} M\Bigg(\sum_{n=1}^{\infty}\frac{\|u_n(i)\|}{2^n}\Bigg) = \lim_{n\to\infty}\sum_{i=1}^{\infty} M\Bigg(\sum_{m=1}^{n}\frac{\|u_m(i)\|}{2^m}\Bigg)$$
$$\le \lim_{n\to\infty}\sum_{i=1}^{\infty}\sum_{m=1}^{n}\frac{M(\|u_m(i)\|)}{2^m} = \lim_{n\to\infty}\sum_{m=1}^{n}\frac{1}{2^m}\sum_{i=1}^{\infty}M(\|u_m(i)\|) \le 1.$$

However, since $\langle u_n(i), v(i)\rangle \ge 0$ for all n, $i \in \mathbb{N}$, again by Levi's lemma, it follows that

$$\sum_{i=1}^{\infty}\langle g(i), v(i)\rangle = \sum_{i=1}^{\infty}\lim_{n\to\infty}\langle g_n(i), v(i)\rangle = \lim_{n\to\infty}\sum_{i=1}^{\infty}\langle g_n(i), v(i)\rangle \ge \lim_{n\to\infty} n = \infty,$$

a contradiction to $v \in \widehat{l_N}(X^*)$. □

Lemma 3.2. *If* $v \in \widehat{l_N}(X^*)$ *and* $\|v\| \le 1$, *then* $\rho_N(v) \le \|v\|$.

Proof. First, we will prove that $\Sigma_{i=1}^{\infty}M(q_+(\|v(i)\|)) \le 1$, if $\|v\| \le 1$. Otherwise, suppose

$$\sum_{i=1}^{\infty} M(q_+(\|v(i)\|)) > 1,$$

such that there exist an $n_0 \in \mathbb{N}$ and a positive D such that

$$1 < \sum_{i=1}^{n_0} M(q_+(\|v(i)\|)) < D < \infty.$$

Given any $\epsilon > 0$, for $i \in \{1, 2, \dots, n_0\}$, there exists a $w(i) \in S(X)$ such that

$$\langle w(i), v(i)\rangle > \|v(i)\| - \frac{\epsilon}{2^i M^{-1}(D)};$$

thus,

$$\begin{aligned}\langle q_+(\|v(i)\|)w(i), v(i)\rangle &> q_+(\|v(i)\|)\|v(i)\| - q_+(\|v(i)\|)\frac{\epsilon}{2^i M^{-1}(D)}\\ &\geq q_+(\|v(i)\|)\|v(i)\| - \frac{\epsilon}{2^i}.\end{aligned}$$

And,

$$\begin{aligned}\sum_{i=1}^{n_0} M\left(\left\|\frac{q_+(\|v(i)\|)w(i)}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\right\|\right) &= \sum_{i=1}^{n_0} M\left(\frac{q_+(\|v(i)\|)}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\right)\\ &\leq \sum_{i=1}^{n_0} \frac{M(q_+(\|v(i)\|)}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))} = 1.\end{aligned}$$

Hence,

$$\begin{aligned}\|v\| &\geq \sum_{i=1}^{n_0} \left\langle \frac{q_+(\|v(i)\|)w(i)}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}, v(i)\right\rangle\\ &= \sum_{i=1}^{n_0} \frac{1}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\langle q_+(\|v(i)\|)w(i), v(i)\rangle\\ &\geq \frac{1}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\sum_{i=1}^{n_0}\left(q_+(\|v(i)\|)\|v(i)\| - \frac{\epsilon}{2^i}\right)\\ &\geq \frac{1}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\left(\sum_{i=1}^{n_0} M(q_+(\|v(i)\|)) + \sum_{i=1}^{n_0} N(\|v(i)\|) - \epsilon\right).\end{aligned}$$

Taking $\epsilon \to 0$, we obtain

$$\begin{aligned}\|v\| &\geq \frac{1}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\left(\sum_{i=1}^{n_0} M(q_+(\|v(i)\|)) + \sum_{i=1}^{n_0} N(\|v(i)\|)\right)\\ &> \frac{1}{\sum_{i=1}^{n_0} M(q_+(\|v(i)\|))}\sum_{i=1}^{n_0} M(q_+(\|v(i)\|)) = 1,\end{aligned}$$

a contradiction to $\|v\| \leq 1$. Hence, $\Sigma_{i=1}^{\infty} M(q_+(\|v(i)\|)) \leq 1$.

Given any $\epsilon > 0$, for each $i \in \mathbb{N}$, there exists a $w(i) \in S(X)$ such that

$$\langle w(i), v(i)\rangle > \|v(i)\| - \frac{\epsilon}{2^i M^{-1}(1)},$$

Since $\Sigma_{i=1}^{\infty} M(\|q_+(\|v(i)\|)w(i)\|) = \Sigma_{i=1}^{\infty} M(q_+(\|v(i)\|)) \le 1$, then

$$\begin{aligned}\|v\| &\ge \sum_{i=1}^{\infty} \langle q_+(\|v(i)\|)w(i), v(i)\rangle \\ &\ge \sum_{i=1}^{\infty} q_+(\|v(i)\|)\left(\|v(i)\| - \frac{\epsilon}{2^i M^{-1}(1)}\right) \\ &\ge \sum_{i=1}^{\infty} q_+(\|v(i)\|)\|v(i)\| - \epsilon.\end{aligned}$$

Taking $\epsilon \to 0$, we obtain

$$\begin{aligned}\|v\| &\ge \sum_{i=1}^{\infty} q_+(\|v(i)\|)\|v(i)\| \\ &= \sum_{i=1}^{\infty} M(q_+(\|v(i)\|)) + \sum_{i=1}^{\infty} N(\|v(i)\|) \\ &\ge \sum_{i=1}^{\infty} N(\|v(i)\|) = \rho_N(v).\end{aligned}$$

□

Theorem 3.3. *The space $(h_M(X))^*$ is isometrically isomorphic to $l_{(N)}(X^*)$.*

Proof. For any $v \in l_{(N)}(X^*)$, let $f_v(u) = \Sigma_{i=1}^{\infty}\langle u(i), v(i)\rangle$ for all $u \in h_M(X)$. Thus, f_v is linear. From Lemma 2.4, we know that

$$f_v(u) = \sum_{i=1}^{\infty}\langle u(i), v(i)\rangle \le \|u\|_M \|v\|_{(N)},$$

whence $f_v \in (h_M(X))^*$ and $\|f_v\| \le \|v\|_{(N)}$.

Note that $\lim_{i\to\infty} \|v(i)\| = 0$ since $v \in l_N(X^*)$. We have that, for any $v \ne 0$, there exists an $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$,

$$\frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{1}{2^n} < 1,$$

whence

$$\frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{1}{n2^n} < \frac{1}{n}.$$

For all n, $i \in \mathbb{N}$, take $u_n(i) \in S(X)$ such that

$$\langle u_n(i), v(i)\rangle \ \geq \|v(i)\| - \frac{1}{n2^n},$$

and set

$$\widetilde{u_n} = \left(q_+\left(\frac{\|v(1)\|}{\|f_v\|}\right)u_n(1), \ldots, q_+\left(\frac{\|v(n)\|}{\|f_v\|}\right)u_n(n), 0, \ldots\right).$$

Then, $\widetilde{u_n} \in h_M(X)$ and

$$\frac{1}{\|f_v\|}f_v(\widetilde{u_n}) \leq \frac{1}{\|f_v\|}\|f_v\|\|\widetilde{u_n}\|_M = \|\widetilde{u_n}\|_M.$$

From Young's inequality and Lemma 2.2, we have

$$\begin{aligned}
\|\widetilde{u_n}\|_M &\geq \frac{1}{\|f_v\|}f_v(\widetilde{u_n}) \\
&= \frac{1}{\|f_v\|}\sum_{i=1}^{n}\left\langle q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)u_n(i), v(i)\right\rangle \\
&\geq \frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\left(\|v(i)\| - \frac{1}{n2^n}\right) \\
&= \sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{\|v(i)\|}{\|f_v\|} - \frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{1}{n2^n} \\
&= \sum_{i=1}^{n} M\left(q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\right) + 1 + \sum_{i=1}^{n} N\left(\frac{\|v(i)\|}{\|f_v\|}\right) \\
&\quad - 1 - \frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{1}{n2^n} \\
&= \sum_{i=1}^{n} M\left(\left\|q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)u_n(i)\right\|\right) + 1 + \sum_{i=1}^{n} N\left(\frac{\|v(i)\|}{\|f_v\|}\right)
\end{aligned}$$

$$- 1 - \frac{1}{\|f_v\|}\sum_{i=1}^{n} q_+\left(\frac{\|v(i)\|}{\|f_v\|}\right)\frac{1}{n2^n}$$
$$\geq \|\widetilde{u_n}\|_M + \sum_{i=1}^{n} N\left(\frac{\|v(i)\|}{\|f_v\|}\right) - 1 - \frac{1}{n},$$

that is, $\Sigma_{i=1}^{n} N(\|v(i)\|/\|f_v\|) \leq 1 + 1/n$.

Taking $n \to \infty$, we obtain

$$\sum_{i=1}^{\infty} N\left(\frac{\|v(i)\|}{\|f_v\|}\right) \leq 1,$$

whence, from the definition of $\|v\|_{(N)}$, we know that $\|v\|_{(N)} \leq \|f_v\|$.

Given any $f \in (h_M(X))^*$, for all $i \in \mathbb{N}$ and $x \in X$, define $\langle x, f(i)\rangle = f(0, \ldots, \underset{i}{x}, 0, \ldots)$. It is easy to see that $f(i) \in X^*$, $i \in \mathbb{N}$. Let $v_f = (f(1), f(2), \ldots)$. Before proving $v_f \in l_{(N)}(X^*)$, we first claim that $\widetilde{l_N}(X^*) = \widehat{l_N}(X^*)$. In fact, for $v \in \widetilde{l_N}(X^*)$, there exists a $\lambda > 0$ such that $\rho_N(\lambda v) < \infty$. Then, for $u \in \widetilde{l_M}(X)$ with $\rho_M(u) < \infty$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}\langle u(i), v(i)\rangle &\leq \sum_{i=1}^{\infty} \|u(i)\|\|v(i)\| \\ &= \frac{1}{\lambda}\sum_{i=1}^{\infty} \|u(i)\|\lambda\|v(i)\| \\ &\leq \frac{1}{\lambda}\left(\sum_{i=1}^{\infty} M(\|u(i)\|) + \sum_{i=1}^{\infty} N(\lambda\|v(i)\|)\right) < \infty,\end{aligned}$$

whence, by the definition of $\widehat{l_N}(X^*)$, we know that $v \in \widehat{l_N}(X^*)$. For $v \in \widehat{l_N}(X^*)$, $v \neq 0$, from Lemma 3.1, we have $0 < \|v\| < \infty$. Then,

$$\begin{aligned}\left\|\frac{v}{\|v\|}\right\| &= \sup\left\{\sum_{i=1}^{\infty}\left\langle u(i), \frac{v(i)}{\|v\|}\right\rangle : \text{for all } u \in \widetilde{l_M}(X) \text{ with } \rho_M(u) \leq 1\right\} \\ &= \frac{1}{\|v\|}\sup\left\{\sum_{i=1}^{\infty}\langle u(i), v(i)\rangle : \text{for all } u \in \widetilde{l_M}(X) \text{ with } \rho_M(u) \leq 1\right\} \\ &= \frac{\|v\|}{\|v\|} = 1.\end{aligned}$$

From Lemma 3.2, we know that $\rho_N(v/\|v\|) \le 1$. From the definition of $\widetilde{l_N}(X^*)$, we have $v \in \widetilde{l_N}(X^*)$. Thus,

$$\widetilde{l_N}(X^*) = \widehat{l_N}(X^*).$$

Second, for each $u \in l_M(X)$, if $u_n = (u(1), u(2), \ldots, u(n), 0, \ldots)$ and $\widetilde{u_n} = (u(1)\operatorname{sgn}\langle u(1), f(1)\rangle, \ldots, u(n)\operatorname{sgn}\langle u(n), f(n)\rangle, 0, \ldots)$, we see that $u_n, \widetilde{u_n} \in h_M(X)$, $\|\widetilde{u_n}\|_M \le \|u_n\|_M$ and

$$\begin{aligned}\left|\sum_{i=1}^{n}\langle u(i), f(i)\rangle\right| &\le \sum_{i=1}^{n}|\langle u(i), f(i)\rangle| \\ &= \sum_{i=1}^{n}\langle u(i), f(i)\rangle \operatorname{sgn}\langle u(i), f(i)\rangle \\ &= \sum_{i=1}^{n}\langle u(i)\operatorname{sgn}\langle u(i), f(i)\rangle, f(i)\rangle \\ &= |f(\widetilde{u_n})| \le \|f\|\|\widetilde{u_n}\|_M \\ &\le \|f\|\|u_n\|_M \le \|f\|\|u\|_M < \infty.\end{aligned}$$

Taking $n \to \infty$, we obtain

$$\left|\sum_{i=1}^{\infty}\langle u(i), f(i)\rangle\right| \le \sum_{i=1}^{\infty}|\langle u(i), f(i)\rangle| \le \|f\|\|u\|_M < \infty.$$

Therefore, $v_f \in l_{(N)}(X^*)$.

For any $u \in h_M(X)$, we have $\theta(u) = 0$, and, from Lemma 2.1, we have $\lim_{n\to\infty}\|u - u_n\|_M = \theta(u) = 0$. Therefore,

$$f(u) = \lim_{n\to\infty} f(u_n) = \lim_{n\to\infty}\sum_{i=1}^{n}\langle u(i), f(i)\rangle = \sum_{i=1}^{\infty}\langle u(i), f(i)\rangle.$$

Thus, $(h_M(X))^*$ is isometrically isomorphic to $l_{(N)}(X^*)$. $\square$

Theorem 3.4. *Given any $u \in S(l_M(X))$, u is a uniformly non-square point of $l_M(X)$ if and only if $N \in \delta_2$, and, for some $i_0 \in \operatorname{supp} u := \{i \in \mathbb{N} : u(i) \neq 0\}$, the element $u(i_0)/\|u(i_0)\|$ is a uniformly non-square point of X.*

Proof.

Necessity. First, suppose $N \notin \delta_2$. Then, by virtue of Lemma 2.7, there exists $a_n \searrow 0$ such that $N((1+1/n)a_n) > 2^{n+1}N(a_n)$ and $N(a_n) \le 1/2^n$ for all $n \in \mathbb{N}$.

For any $n \in \mathbb{N}$, take $m_n \in \mathbb{N}$ such that

$$\frac{1}{2^{n+1}} < m_n N(a_n) \le \frac{1}{2^n}, \tag{3.1}$$

and, for u, from Lemma 2.5, we choose $v_n \in S(l_{(N)}(X^*))$ such that

$$\sum_{i=1}^{\infty} \|u(i)\| \|v_n(i)\| > 1 - \frac{1}{n}.$$

Then, take $I_n \in \mathbb{N}$ such that

$$\sum_{i=1}^{I_n} \|u(i)\| \|v_n(i)\| > 1 - \frac{1}{n} \tag{3.2}$$

and $I_n > I_{n-1} + m_{n-1}$.

Let

$$z_n = (\underbrace{v_n(1), \ldots, v_n(I_n)}_{I_n}, 0, \ldots),$$

choose $x^* \in S(X^*)$ and set

$$w_n = (\underbrace{0, \ldots, 0}_{I_n}, \underbrace{a_n x^*, \ldots, a_n x^*}_{m_n}, 0, \ldots),$$

so that $w_n \in h_{(N)}(X^*)$ and

$$\rho_N(w_n) = \sum_{i=I_n+1}^{I_n+m_n} N(\|a_n x^*\|) = m_n N(a_n) \le \frac{1}{2^n},$$

$$\rho_N\left(\left(1+\frac{1}{n}\right)w_n\right) = \sum_{i=I_n+1}^{I_n+m_n} N\left(\left\|\left(1+\frac{1}{n}\right)a_n x^*\right\|\right) > 2^{n+1} m_n N(a_n) > 1.$$

Then,

$$1 \ge \|w_n\|_{(N)} \ge \frac{1}{1+1/n}.$$

Let

$$\widetilde{v_n} = \frac{1}{1+1/n}(z_n + w_n).$$

We know from Lemma 2.3 that

$$\begin{aligned}\rho_N(\widetilde{v_n}) &= \sum_{i=1}^{I_n} N\left(\frac{1}{1+1/n}\|v_n(i)\|\right) + \sum_{i=I_n+1}^{I_n+m_n} N\left(\frac{1}{1+1/n}a_n\right)\\ &\le \frac{1}{1+1/n}\left[\sum_{i=1}^{I_n} N(\|v_n(i)\|) + \sum_{i=I_n+1}^{I_n+m_n} N(a_n)\right]\\ &\le \frac{1}{1+1/n}\left(1+\frac{1}{2^n}\right)\\ &\le 1.\end{aligned}$$

From Theorem 3.3, there exists a $\{u_n\}_n \subseteq S(h_M(X))$ such that

$$\begin{aligned}\sum_{i=I_n+1}^{I_n+m_n} \|u_n(i)\|\|w_n(i)\| &\ge \sum_{i=I_n+1}^{I_n+m_n} \langle u_n(i), w_n(i)\rangle\\ &\ge \left(1-\frac{1}{n}\right)\|w_n\|_{(N)}\\ &\ge \left(1-\frac{1}{n}\right)\frac{1}{1+1/n},\end{aligned} \tag{3.3}$$

where

$$u_n = (\underbrace{0,\ldots,0}_{I_n}, \underbrace{u_n(I_n+1),\ldots,u_n(I_n+m_n)}_{m_n}, 0,\ldots).$$

From (3.1), (3.2), (3.3), Lemmas 2.3, 2.5 and Young's inequality, we have

$$\begin{aligned}\|u_n + u\|_M &\ge \sum_{i=1}^{\infty} \|u_n(i)+u(i)\|\|\widetilde{v_n}(i)\|\\ &= \frac{1}{1+1/n}\left\{\sum_{i=1}^{I_n} \|u_n(i)+u(i)\|\|z_n(i)\|\right.\\ &\qquad\left. + \sum_{i=I_n+1}^{I_n+m_n} \|u_n(i)+u(i)\|\|w_n(i)\|\right\}\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{1+1/n}\bigg\{\sum_{i=1}^{I_n}\|u(i)\|\|v_n(i)\| + \sum_{i=I_n+1}^{I_n+m_n}\|u_n(i)\|\|w_n(i)\| \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}\|u(i)\|\|w_n(i)\|\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\left(1-\frac{1}{n}\right)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}[M(\|u(i)\|)+N(\|w_n(i)\|)]\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\left(1-\frac{1}{n}\right)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}M(\|u(i)\|)-m_nN(a_n)\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\left(1-\frac{1}{n}\right)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}M(\|u(i)\|)-\frac{1}{2^n}\bigg\} \longrightarrow 2, \quad n\to\infty,
\end{aligned}
$$

and

$$
\begin{aligned}
\|u_n-u\|_M &\geq \sum_{i=1}^{\infty}\|u_n(i)-u(i)\|\|\widetilde{v_n}(i)\| \\
&= \frac{1}{1+1/n}\bigg\{\sum_{i=1}^{I_n}\|u_n(i)-u(i)\|\|z_n(i)\| \\
&\qquad\qquad + \sum_{i=I_n+1}^{I_n+m_n}\|u_n(i)-u(i)\|\|w_n(i)\|\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{\sum_{i=1}^{I_n}\|u(i)\|\|v_n(i)\| + \sum_{i=I_n+1}^{I_n+m_n}\|u_n(i)\|\|w_n(i)\| \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}\|u(i)\|\|w_n(i)\|\bigg\}
\end{aligned}
$$

$$\begin{aligned}
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\Big(1-\frac{1}{n}\Big)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n}[M(\|u(i)\|)+N(\|w_n(i)\|)]\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\Big(1-\frac{1}{n}\Big)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n} M(\|u(i)\|) - m_n N(a_n)\bigg\} \\
&\geq \frac{1}{1+1/n}\bigg\{1-\frac{1}{n}+\Big(1-\frac{1}{n}\Big)\frac{1}{1+1/n} \\
&\qquad\qquad - \sum_{i=I_n+1}^{I_n+m_n} M(\|u(i)\|) - \frac{1}{2^n}\bigg\} \longrightarrow 2, \quad n\to\infty.
\end{aligned}$$

This is a contradiction to u being a uniformly non-square point.

Second, suppose that, for any $i \in \operatorname{supp} u$, $u(i)/\|u(i)\|$ is not a uniformly non-square point. Then, for all $n, i \in \mathbb{N}$, there is a $v_n(i) \in S(X)$ such that $\|u(i)/\|u(i)\| + v_n(i)\| \geq 2-1/n$ and $\|u(i)/\|u(i)\| - v_n(i)\| \geq 2-1/n$. Equivalently, $\|u(i) + \|u(i)\|v_n(i)\| \geq (2-1/n)\|u(i)\|$ and $\|u(i) - \|u(i)\|v_n(i)\| \geq (2-1/n)\|u(i)\|$.

Set $u_n = (\|u(1)\|v_n(1), \|u(2)\|v_n(2), \ldots)$. Then, $\|u_n\|_M = \|u\|_M = 1$ since $\|\|u(i)\|v_n(i)\| = \|u(i)\|$ for all n, $i \in \mathbb{N}$. From Lemma 2.5, we have

$$\begin{aligned}
\|u_n + u\|_M &= \sup_{\rho_N(w)\leq 1} \sum_{i=1}^{\infty} \big\|\|u(i)\|v_n(i)+u(i)\big\|\|w(i)\| \\
&\geq \sup_{\rho_N(w)\leq 1} \sum_{i=1}^{\infty} \Big(2-\frac{1}{n}\Big)\|u(i)\|\|w(i)\| \\
&= \Big(2-\frac{1}{n}\Big)\|u\|_M = 2-\frac{1}{n},
\end{aligned}$$

and

$$\begin{aligned}
\|u_n - u\|_M &= \sup_{\rho_N(w)\leq 1} \sum_{i=1}^{\infty} \big\|\|u(i)\|v_n(i)-u(i)\big\|\|w(i)\| \\
&\geq \sup_{\rho_N(w)\leq 1} \sum_{i=1}^{\infty} \Big(2-\frac{1}{n}\Big)\|u(i)\|\|w(i)\| = \Big(2-\frac{1}{n}\Big)\|u\|_M = 2-\frac{1}{n}.
\end{aligned}$$

Taking $n \to \infty$ gives $\|u_n + u\|_M \to 2$ and $\|u_n - u\|_M \to 2$. This is a contradiction to u being a uniformly non-square point.

Sufficiency. From Lemma 2.6, we know that there exists a $b > 0$ such that $\sup\{h \in K(v) : \|v\|_M = 1\} < b$. For any $v \in S(l_M(X))$ and $h \in K(v)$, we have $h > 1$. Thus, for u and, any $v \in S(l_M(X))$, and, for any $k \in K(u)$, $h \in K(v)$, we have $k, h \in (1, b)$.

For $v \in S(l_M(X))$, from Lemma 2.3, we know that $\Sigma_{i=1}^{\infty} M(\|v(i)\|) \le 1$. Thus, we have

$$\|v(i)\| \le M^{-1}(1) = \frac{M^{-1}(1)b}{b}$$

for all $i \in \mathbb{N}$.

Since $u(i_0)/\|u(i_0)\|$ is a uniformly non-square point of X, we know from Lemma 2.10 that there exists an $r \in (0, 1)$ such that the inequality

$$\tag{3.4} M\left(\frac{kh}{k+h}\|u(i_0) + v(i_0)\|\right) + M\left(\frac{kh}{k+h}\|u(i_0) - v(i_0)\|\right) \le 2r\left[\frac{h}{k+h}M(k\|u(i_0)\|) + \frac{k}{k+h}M(h\|v(i_0)\|)\right]$$

holds for all $v \in S(l_M(X))$, where $k \in K(u)$ and $h \in K(v)$. Then, for every $v \in S(l_M(X))$ and $k \in K(u)$, $h \in K(v)$, from Lemma 2.2 and inequality (3.4), we have

$$\begin{aligned}
&\|u+v\|_M + \|u-v\|_M \\
&\quad \le \frac{k+h}{kh}\left\{1 + \rho_M\left(\frac{kh}{k+h}(u+v)\right)\right\} + \frac{k+h}{kh}\left\{1 + \rho_M\left(\frac{kh}{k+h}(u-v)\right)\right\} \\
&\quad = \frac{2}{k} + \frac{2}{h} + \frac{k+h}{kh}\left\{\rho_M\left(\frac{kh}{k+h}(u+v)\right) + \rho_M\left(\frac{kh}{k+h}(u-v)\right)\right\} \\
&\quad = \frac{2}{k} + \frac{2}{h} + \frac{k+h}{kh} \\
&\quad \cdot \left\{\sum_{i \ne i_0} M\left(\frac{kh}{k+h}\|u(i) + v(i)\|\right) + \sum_{i \ne i_0} M\left(\frac{kh}{k+h}\|u(i) - v(i)\|\right)\right. \\
&\qquad \left. + M\left(\frac{kh}{k+h}\|u(i_0) + v(i_0)\|\right) + M\left(\frac{kh}{k+h}\|u(i_0) - v(i_0)\|\right)\right\}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2}{k}+\frac{2}{h}+\frac{k+h}{kh}\bigg\{2\sum_{i\neq i_0}\bigg[\frac{h}{k+h}M(k\|u(i)\|)+\frac{k}{k+h}M(h\|v(i)\|)\bigg] \\
&\qquad\qquad + 2r\bigg[\frac{h}{k+h}M(k\|u(i_0)\|)+\frac{k}{k+h}M(h\|v(i_0)\|)\bigg]\bigg\} \\
&= \frac{2}{k}+\frac{2}{h}+2\frac{k+h}{kh}\sum_{i=1}^{\infty}\bigg\{\frac{h}{k+h}M(k\|u(i)\|)+\frac{k}{k+h}M(h\|v(i)\|)\bigg\} \\
&\quad - 2(1-r)\frac{k+h}{kh}\bigg\{\frac{h}{k+h}M(k\|u(i_0)\|)+\frac{k}{k+h}M(h\|v(i_0)\|)\bigg\} \\
&= 2\bigg\{\frac{1}{k}+\frac{1}{k}\sum_{i=1}^{\infty}M(k\|u(i)\|)+\frac{1}{h}+\frac{1}{h}\sum_{i=1}^{\infty}M(h\|v(i)\|)\bigg\} \\
&\quad - 2(1-r)\bigg\{\frac{1}{k}M(k\|u(i_0)\|)+\frac{1}{h}M(h\|v(i_0)\|)\bigg\} \\
&= 4-2(1-r)\bigg\{\frac{1}{k}M(k\|u(i_0)\|)+\frac{1}{h}M(h\|v(i_0)\|)\bigg\} \\
&\le 4-2(1-r)\frac{1}{k}M(k\|u(i_0)\|) \le 4-2(1-r)M(\|u(i_0)\|) \\
&= 2\{2-(1-r)M(\|u(i_0)\|)\}.
\end{aligned}$$

Let $\delta=(1-r)M(\|u(i_0)\|)$. Then, $\delta\in(0,1)$ since $M(\|u(i_0)\|), r\in(0,1)$. Thus,

$$\min\{\|u+v\|_M, \|u-v\|_M\}\le 2-\delta.$$

Hence, u is a uniformly non-square point of $l_M(X)$. □

When $X=\mathbb{R}$, if $u\in S(l_M)$, then for $i_0\in \operatorname{supp} u$, we have $u(i_0)/\|u(i_0)\|=1$ or -1, which is a uniformly non-square point of $\mathbb{R}$. Thus, we obtain

Corollary 3.5. *For $u\in S(l_M)$, u is a uniformly non-square point if and only if $N\in\delta_2$.*

Theorem 3.6. *The space $l_M(X)$ is locally uniformly non-square if and only if $N\in\delta_2$ and X is locally uniformly non-square.*

Proof.

Necessity. If $l_M(X)$ is uniformly non-square, that is, every $u \in S(l_M(X))$ is a uniformly non-square point, then from Theorem 3.4, we know that $N \in \delta_2$.

Suppose that X is not locally uniformly non-square. Then, there is an $x \in S(X)$ which is not a uniformly non-square point of X. Setting $u = (x, 0, 0, \ldots)$, we have $u \in l_M(X)$ and $\operatorname{supp} u = \{1\}$. If $u_0 := u/\|u\|_M$, then $u_0 \in S(l_M(X))$; however,

$$\frac{u_0(1)}{\|u_0(1)\|} = \frac{x/\|u\|_M}{\|x/\|u\|_M\|} = x$$

is not a uniformly non-square point of X. By Theorem 3.4, u_0 is not a uniformly non-square point of $l_M(X)$, a contradiction.

Sufficiency. For $u \in S(l_M(X))$, we see that $\operatorname{supp} u \neq \emptyset$. Since X is locally uniformly non-square, then, for all $i \in \operatorname{supp} u$, $u(i)/\|u(i)\|$ is a uniformly non-square point of X. From $N \in \delta_2$ and Theorem 3.4, we know that u is a uniformly non-square point of $l_M(X)$. □

Corollary 3.7. *The space l_M is locally uniformly non-square if and only if $N \in \delta_2$.*

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