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ALMOST COMPATIBLE FUNCTIONS AND INFINITE LENGTH GAMES

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ABSTRACT. $\mathcal{A}'(\kappa)$ asserts the existence of pairwise almost compatible finite-to-one functions $A \to \omega$ for each countable subset A of κ. The existence of winning 2-Markov strategies in several infinite-length games, including the Menger game on the one-point Lindelöfication $\kappa^\dagger$ of κ, are guaranteed by $\mathcal{A}'(\kappa)$. $\mathcal{A}'(\kappa)$ is implied by the existence of cofinal Kurepa families of size κ, and thus, holds for all cardinals less than $\aleph_\omega$. It is consistent that $\mathcal{A}'(\aleph_\omega)$ fails; however, there must always be a winning 2-Markov strategy for the second player in the Menger game on $\omega_\omega^\dagger$.

1. Introduction.

Definition 1.1. Two functions f, g are *almost compatible*, that is, $f \sim g$ when $\{a \in \operatorname{dom} f \cap \operatorname{dom} g : f(a) \neq g(a)\}$ is finite.

Scheepers used almost compatible functions in **[11]** in order to study the existence of limited information strategies on a variation of the meager-nowhere dense game he introduced in **[12]**.

Game 1.2. Let $\mathrm{Sch}^{\cup,\subsetneq}(\kappa)$ denote *Scheepers' strict countable-finite union game* with two players $\mathscr{C}$, $\mathscr{F}$. In round 0, $\mathscr{C}$ chooses $C_0 \in [\kappa]^{\leq\omega}$, followed by $\mathscr{F}$ choosing $F_0 \in [\kappa]^{<\omega}$. In round $n+1$, $\mathscr{C}$ chooses $C_{n+1} \in [\kappa]^{\leq\omega}$ such that $C_{n+1} \supset C_n$, followed by $\mathscr{F}$ choosing $F_{n+1} \in [\kappa]^{<\omega}$.

$$\mathscr{F} \text{ wins the game if } \bigcup_{n<\omega} F_n \supseteq \bigcup_{n<\omega} C_n; \text{ otherwise, } \mathscr{C} \text{ wins.}$$

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Of course, with perfect information, this game is trivial: during round n player $\mathscr{F}$ simply chooses n ordinals from each of the n countable sets played by $\mathscr{C}$. However, if $\mathscr{F}$ is limited to using information from the last k moves by $\mathscr{C}$ during each round, the task becomes more difficult. Such a strategy is called a *k-tactical strategy* or *k-tactic*; if using the round number is allowed, then the strategy is called a *k-Markov strategy* or a *k-mark*.

Definition 1.3. The statement $\mathcal{A}(\kappa)$ (given as $S(\kappa, \aleph_0, \omega)$ in **[11]**) claims that there exist one-to-one functions $f_A : A \to \omega$ for each $A \in [\kappa]^{\leq\aleph_0}$ such that the collection $\{f_A : A \in [\kappa]^{\leq\aleph_0}\}$ is pairwise almost compatible.

In **[11]**, Scheepers noted that $\mathcal{A}(\omega_1)$ holds in ZFC, and that it is possible to force $\mathfrak{c}$ to be arbitrarily large while preserving $\mathcal{A}(\mathfrak{c})$; however, it was also shown that $\mathcal{A}(\mathfrak{c}^+)$ always fails. This axiom may be applied to obtain a winning 2-tactic for $\mathscr{F}$ in the countable-finite game.

In **[1]**, Clontz related this game to a game which may be used to characterize the Menger covering property of a topological space.

Game 1.4. Let $\mathrm{Men}(X)$ denote the *Menger game* with players $\mathscr{C}$, $\mathscr{F}$. In round n, $\mathscr{C}$ chooses an open cover $\mathcal{U}_n$, followed by $\mathscr{F}$ choosing a subset F_n of X which may be finitely covered by $\mathcal{U}_n$.

$$\mathscr{F} \text{ wins the game if } X = \bigcup_{n<\omega} F_n, \text{ and } \mathscr{C} \text{ wins otherwise.}$$

This characterization is slightly different than the typical characterization in which the second player first chooses a specific finite subcollection $\mathcal{F}_n$ of the cover itself and lets

$$F_n = \bigcup \mathcal{F}_n,$$

denoted as $G_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ in **[13]**. However, it is easily seen that these games are equivalent for perfect information strategies (so both characterize the Menger property in the same way), and this characterization is more convenient for our concerns.

Definition 1.5. Let $\kappa^\dagger = \kappa \cup \{\infty\}$ where κ is discrete and ∞'s neighborhoods are the co-countable sets which contain it.

The relationship between $\mathrm{Sch}^{\cup,\subsetneq}(\kappa)$ and $\mathrm{Men}(\kappa^\dagger)$ is strong; in both games, $\mathscr{C}$ essentially chooses a countable subset of κ followed by $\mathscr{F}$ choosing a finite subset of that choice, and it is easy to see the winning perfect information strategy for $\mathscr{F}$ in both games. In addition, it was shown in [**1**] that, when $\mathcal{A}(\kappa)$ holds, $\mathscr{F}$ has a winning 2-Markov strategy in $\mathrm{Men}(\kappa^\dagger)$.

One source of motivation is to make progress on the following open question:

Question 1.6. *Does there exist a topological space X for which $\mathscr{F} \uparrow \mathrm{Men}(X)$ but*

$$\mathscr{F} \underset{2\text{-}mark}{\not\uparrow} \mathrm{Men}(X)\,?$$

In other words, the second player can win the Menger game on X with perfect information but not with 2-Markov information.

2. One-to-one and finite-to-one almost compatible functions.

We may weaken Scheeper's $\mathcal{A}(\kappa)$ as follows:

Definition 2.1. The statement $\mathcal{A}'(\kappa)$ weakens $\mathcal{A}(\kappa)$ by only requiring the witnessing almost-compatible functions $f_A : A \to \omega$ to be finite-to-one.

Proposition 2.2. *$\mathcal{A}(\kappa)$ and $\mathcal{A}'(\kappa)$ need only be witnessed by functions $\{f_A : A \in \mathcal{S}\}$ for some family $\mathcal{S}$ cofinal in $[\kappa]^{\leq\aleph_0}$.*

Proof. For each $A \in [\kappa]^{\leq\aleph_0}$, choose $A' \supseteq A$ from $\mathcal{S}$, and let $g_A = f_{A'} \restriction A$. □

In the final section, we will show that $\mathcal{A}'(\kappa)$ is sufficient for many applications to the Scheepers and Menger games. In the meantime, we will demonstrate that $\mathcal{A}'(\kappa)$ is strictly weaker than $\mathcal{A}(\kappa)$.

Recall the following.

Definition 2.3. A *Kurepa family* $\mathcal{K} \subseteq [\kappa]^{\aleph_0}$ on κ satisfies that

$$\mathcal{K}\restriction A = \{K \cap A : K \in \mathcal{K}\}$$

is countable for each $A \in [\kappa]^{\aleph_0}$. Let $\mathcal{K}(\kappa)$ be the statement claiming there exists a Kurepa family on κ cofinal in $[\kappa]^{\aleph_0}$.

Theorem 2.4. *$\mathcal{K}(\kappa) \Rightarrow \mathcal{A}'(\kappa)$.*

Proof. Let $\mathcal{K} = \{K_\alpha : \alpha < \theta\}$ be a cofinal Kurepa family on κ. We first define $f_\alpha : K_\alpha \to \omega$ for each $\alpha < \theta$.

Suppose that we have already defined pairwise almost compatible finite-to-one functions $\{f_\beta : \beta < \alpha\}$. In order to define f_α, we first recall that $\mathcal{K} \restriction K_\alpha$ is countable, so we may choose $\beta_n < \alpha$ for $n < \omega$ such that

$$\{K_\beta : \beta < \alpha\} \restriction K_\alpha \setminus \{\emptyset\} = \{K_\alpha \cap K_{\beta_n} : n < \omega\}.$$

Let $K_\alpha = \{\delta_{i,j} : i \leq \omega, j < w_i\}$ where $w_i \leq \omega$ for each $i \leq \omega$,

$$K_\alpha \cap \left(K_{\beta_n} \setminus \bigcup_{m<n} K_{\beta_m} \right) = \{\delta_{n,j} : j < w_n\},$$

and

$$K_\alpha \setminus \bigcup_{n<\omega} K_{\beta_n} = \{\delta_{\omega,j} : j < w_\omega\}.$$

Then, let $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ for $n < \omega$ and $f_\alpha(\delta_{\omega,j}) = j$ otherwise.

We should show that f_α is finite-to-one. Let $n < \omega$. Since $f_\alpha(\delta_{m,j}) \geq m$, we only consider the finite cases where $m \leq n$. Since each f_{β_m} is finite-to-one, $f_{\beta_m}(\delta_{m,j}) \leq n$ for only finitely many j. Thus, $f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$ maps to n for only finitely many j.

We now want to demonstrate that $f_\alpha \sim f_{\beta_n}$ for all $n < \omega$. Note that $\delta_{m,j} \in K_{\beta_n}$ implies $m \leq n$. For $m = n$, we have $f_\alpha(\delta_{n,j}) = \max(n, f_{\beta_n}(\delta_{n,j}))$ which differs from $f_{\beta_n}(\delta_{n,j})$ for only the finitely many j which are mapped below n by f_{β_n}. For $m < n$ and $\delta_{m,j} \in K_{\beta_n}$, we have

$$f_\alpha(\delta_{m,j}) = \max(m, f_{\beta_m}(\delta_{m,j}))$$

which can only differ from $f_{\beta_n}(\delta_{m,j})$ for only the finitely many j which are mapped below m by f_{β_m} or the finitely many j for which the almost compatible $f_{\beta_n} \sim f_{\beta_m}$ differ.

Finally, for any $\beta < \alpha$, we may conclude that $f_\alpha \sim f_\beta$ since there is some β_n with

$$K_\alpha \cap K_\beta = K_\alpha \cap K_{\beta_n}, \qquad f_\alpha \sim f_{\beta_n} \text{ and } f_{\beta_n} \sim f_\beta. \qquad \square$$

We now make use of a topology on ω_n for each $n < \omega$ that witnesses a Kurepa family of size $\aleph_n$ [**6**].

Definition 2.5. A topological space is said to be ω-*bounded* if each countable subset of the space has compact closure. As in [**6**], we call a T_2, locally countable, ω-bounded space *splendid*, and let $\mathcal{S}(\kappa)$ represent the claim that there exists a splendid space of cardinality κ.

Theorem 2.6 ([**6**]). *$\mathcal{S}(\aleph_k)$ for $k < \omega$.*

Lemma 2.7. *The family of compact open sets in a locally countable, ω-bounded topological space X is a Kurepa family cofinal in $[X]^\omega$, that is,*

$$\mathcal{S}(\kappa) \Longrightarrow \mathcal{K}(\kappa).$$

Proof. Let $\mathcal{K}$ collect all compact open subsets of X. Of course, every Lindelöf set in a locally countable space is countable, and the closure of every countable set is a compact countable set; thus, $\mathcal{K}$ is cofinal in $[X]^\omega$. It is Kurepa since every countable set is contained in a countable compact open subspace of X. This subspace has a countable base of compact open sets, which, closed under finite unions, enumerates all compact open subsets of the subspace. □

Corollary 2.8. *$\mathcal{K}(\aleph_k)$ for all $k < \omega$.*

Alternatively, the previous corollary may be obtained via an observation of Todorcevic communicated by Dow in [**3**]: if every Kurepa family of size at most κ extends to a cofinal Kurepa family, then the same is true of κ^+.

Nyikos pointed out [**10**] that a cofinal Kurepa family may be used to construct a locally metrizable, ω-bounded, zero-dimensional space with appropriate cardinality; whether this can be strengthened to locally countable and ω-bounded (as asked in [**6**]) remains an open question.

Also left open is this extension of the question asked in [**6, 10**] on the possible equivalence of $\mathcal{S}(\kappa)$ and $\mathcal{K}(\kappa)$.

Question 2.9. *Can any of the implications in the theorem*

$$\mathcal{S}(\kappa) \Longrightarrow \mathcal{K}(\kappa) \Longrightarrow \mathcal{A}'(\kappa)$$

be reversed?

Regardless, we have obtained our desired result.

Corollary 2.10. *$\mathcal{A}'(\aleph_k)$ for all $k < \omega$.*

3. Consistency results. As noted in [**3**], Jensen's one gap two-cardinal theorem under $V = L$ introduced in [**5**] implies that $\mathcal{K}(\kappa)$, and therefore, $\mathcal{A}'(\kappa)$, holds for all cardinals κ.

Corollary 3.1. *Assume the covering lemma over the Core Model holds. Then $\mathcal{A}'(\kappa)$ holds for all cardinals κ.*

Proof. Juhász and Weiss note [**7,** page 186] that the covering lemma over the Core Model guarantees $\mathcal{S}(\kappa)$, and therefore, $\mathcal{K}(\kappa)$ and $\mathcal{A}'(\kappa)$, when $\operatorname{cf}\kappa > \omega$. $\square$

As noted earlier, Scheepers proved [**11**] that $\neg\mathcal{A}(\mathfrak{c}^+)$ is a theorem of ZFC, showing $\mathcal{A}'(\kappa)$ is not equivalent to $\mathcal{A}(\kappa)$.

We now demonstrate that CH is not required to have $\mathcal{A}(\aleph_2)$ fail. The forcing extension of a model M by a poset $\mathbb{P} \in M$ is simply obtained by evaluating all $\mathbb{P}$-names from M by a generic filter G. A set τ is a $\mathbb{P}$-name if τ is a (possibly empty) set of ordered pairs (σ, p) where $p \in \mathbb{P}$ and σ is also itself a $\mathbb{P}$-name. If G is a $\mathbb{P}$-generic filter, then $\operatorname{val}_G(\tau)$ is defined to equal

$$\{\operatorname{val}_G(\sigma) : (\text{there exists a } p \in G)\ (\sigma, p) \in \tau\}.$$

If $x \in M$, then the canonical $\mathbb{P}$-name, $\check{x}$, is generally, and recursively, taken to be $\{(\check{y}, 1) : y \in x\}$, where 1 is the maximum element of $\mathbb{P}$. However, it will be convenient to consider, when the context is clear, (x, p) (for any $p \in \mathbb{P}$) as a type of $\mathbb{P}$-name. In particular, if $\tau \subset X \times \mathbb{P}$, for some fixed $X \in M$, then we may let

$$\tau[G] = \{x : (\text{there exists a } p \in G)(x, p) \in \tau\}.$$

Thus, $\operatorname{val}_G(\tau)$ will denote the recursive evaluation by G and $\tau[G]$ will be defined as above. In fact, if $\tau \in M$ is any set, then each of $\operatorname{val}_G(\tau)$ and $\tau[G]$ are well defined. It is a standard convention to use a dotted letter, such as $\dot{x}$, to indicate that we are discussing a $\mathbb{P}$-name.

It may be stated that a condition $p \in \mathbb{P}$ forces a statement φ to hold, denoted $p \Vdash \varphi$, if that statement holds in $M[G]$ for all $\mathbb{P}$-generic filters with $p \in G$. The forcing theorem states that, if $M[G] \models \varphi$, then there is some $p \in G$ forcing that φ holds. The following is an immediate consequence of the forcing theorem.

Lemma 3.2. *If $X \in M$ and $\dot{x}$ is a $\mathbb{P}$-name, then there is a $\tau \subset X \times \mathbb{P}$, such that for any generic G, $\tau[G] = X \cap \mathrm{val}_G(\dot{x})$.*

In other words, the family of subsets of any $X \in M$ in the extension $M[G]$ is equal to

$$\{\tau[G] : \tau \subset X \times \mathbb{P},\ \tau \in M\}.$$

We will be using the forcing poset $\mathrm{Fn}(\omega_2, 2)$. The elements of this poset are all of the finite partial functions from ω_2 into 2 ordered by reverse inclusion. It follows that, for any $\lambda \in \omega_2$, each of $\mathrm{Fn}(\lambda, 2)$ and $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$ are subposets. For any $\mathrm{Fn}(\omega_2, 2)$-generic filter G, it easily follows that $G_\lambda = G \cap \mathrm{Fn}(\lambda, 2)$ and $G^\lambda = G \cap \mathrm{Fn}(\omega_2 \setminus \lambda, 2)$ are also generic filters. However, a much stronger statement is true.

Lemma 3.3 ([**8**]). *Assume that $G \subset \mathrm{Fn}(\omega_2, 2)$ is a generic filter, and let $\lambda \in \omega_2$. Then, the final model $M[G]$ is equal to $(M[G_\lambda])[G^\lambda]$ in the sense that G^λ is a $\mathrm{Fn}(\omega_2 \setminus \lambda, 2)$-generic filter over the model $M[G_\lambda]$.*

In addition, for each $X \in M$ and name $\dot{A} \subset X \times \mathrm{Fn}(\omega_2, 2)$, we obtain $(\dot{A}(G_\lambda))[G^\lambda] = \dot{A}[G]$ where

$$\dot{A}(G_\lambda) = \{(x, p \restriction [\lambda, \omega_2)) : (x, p) \in \dot{A} \textit{ and } p \restriction \lambda \in G_\lambda\}.$$

With these lemmas at hand, we are ready to prove the theorem. The idea of the proof comes from Kunen's result regarding no ω_2 length mod finite chains of subsets of ω. We consider any family of names of suitable one-to-one functions from countable subsets of ω_2 into ω. We identify a large enough $\lambda \in \omega_2$ such that a pattern has emerged, and we pass to the model $M[G_\lambda]$. We then show that this pattern cannot continue out to ω_2.

Theorem 3.4. *There exists a model of* ZFC *for which $\mathfrak{c} = \aleph_2$ and $\neg\mathcal{A}(\aleph_2)$.*

Proof. We start with a model M of GCH and suppose that G is an $\mathrm{Fn}(\omega_2, 2)$-generic filter. The argument takes place in M. Let $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ be a family of names (in M) such that, for any generic G and each $A \in [\omega_2]^\omega \cap M$, $\dot{f}_A[G]$ is a one-to-one function from A into ω. We also assume that, whenever $B \subset A$ are members of $[\omega_2]^\omega$, we have that $\dot{f}_B[G] \subset^* \dot{f}_A[G]$. If we now obtain a contradiction, then we will have shown that $\mathcal{A}(\aleph_2)$ fails.

From [**2,** 1.5], there is a set $H \subset H(\aleph_3)$ such that the family $\{\dot{f}_A : A \in [\omega_2]^\omega\}$ is an element of H, H is an elementary submodel of $H(\aleph_3)$,

H has cardinality $\aleph_1$ and $H^\omega \subset H$ (every countable subset of H is an element of H).

Let $\lambda = H \cap \omega_2$, the same as the supremum of $H \cap \omega_2$. Consider the name $\dot{f}_{[\lambda,\lambda+\omega)}$. What is such a name? By Lemma 3.2, we can assume that it is a set of pairs of the form $((\lambda + k, m), p)$ where $p \in Fn(\omega_2, 2)$ and, of course, $k, m \in \omega$. Furthermore, for each k and m, it is enough (see [**8,** 5.11, 5.12]) to take a countable set of such p to get an equivalent (*nice*) name. Given any such nice name $\dot{f}$, let $\text{supp}(\dot{f})$ denote the union of the domains of conditions p appearing in the name.

Now, let Y equal $\text{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \setminus \lambda$. Furthermore, fix any $\mu \in \lambda \subset H$ such that $\text{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda$ is contained in μ. Let $\delta \in \omega_1$ denote the order type of Y, and let $\varphi_{\mu,\lambda}$ be the order-preserving function from $\mu \cup Y$ onto the ordinal $\mu + \delta$. This lifts canonically to an order-preserving bijection

$$\varphi_{\mu,\lambda} : \text{Fn}(\mu \cup Y, 2) \longmapsto \text{Fn}(\mu + \delta, 2).$$

We can similarly make sense of the name $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$, call it F_H. Here, simply, for each tuple $(\,(\lambda + k, m), p) \in \dot{f}_{[\lambda,\lambda+\omega)}$, we have that $(\,(\mu + k, m), \varphi_{\mu,\lambda}(p))$ is in F_H. Again, let $\varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$ be interpreted in the above sense as giving F_H, which is an element of H.

Other values replacing $\lambda > \mu$ will result in their own set Y and canonical map $\varphi_{\mu,\lambda}$. Now, the object F_H is an element of H, and H supposes this statement is true:

$$\text{(for all } \beta \in \omega_2\text{)(there exists a } \lambda \in \omega_2 \setminus \beta\text{)}\ \ \text{supp}(\dot{f}_{[\lambda,\lambda+\omega)}) \cap \lambda \subset \mu$$

and $F_H = \varphi_{\mu,\lambda}(\dot{f}_{[\lambda,\lambda+\omega)})$. However, now, this means that not only is there an $\alpha \in H$, $F_H = \varphi_{\mu,\alpha}(\dot{f}_{[\alpha,\alpha+\omega)})$ but also that there is an increasing sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset \lambda$ of such αs satisfying that, for each ξ, we have that $\text{supp}(\dot{f}_{[\alpha_\xi,\alpha_\xi+\omega)})$ is contained in $\alpha_{\xi+1}$.

Choose such a sequence. This means that, if we let

$$A = \bigcup_{n>0} [\alpha_n, \alpha_n + \omega),$$

we have the name $\dot{f}_A$ in H. This then means that all of the $((\beta, m), p)$ appearing in (the nice name) $\dot{f}_A$ have the property that $\text{dom}(p)$ is contained in H. There is, also within H, a name $\dot{g}$ satisfying that

$$\dot{f}_A(\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + k) \quad \text{for all } k > \dot{g}(n),$$

or more precisely, $\dot{g} \subset (\omega \times \omega) \times \mathrm{Fn}(\omega_2, 2)$ satisfies that $\dot{g}[G] \in \omega^\omega$ and

$$\dot{f}_A[G](\alpha_n + k) = \dot{f}_{[\alpha_n,\alpha_n+\omega)}[G](\alpha + k) \quad \text{for all } k > \dot{g}[G](n).$$

We now apply Lemma 3.3 and work in the extension $M[G_\mu]$ for a contradiction. Something special has now happened, namely, the supports of the names

$$\{\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu) : 0 < n < \omega\}$$

are pairwise disjoint and also disjoint from the support of the name $\dot{f}_{[\lambda,\lambda+\omega)}(G_\mu)$. Further, these names are pairwise isomorphic (in a manner that they all map to F_H).

Since A is disjoint from $[\lambda, \lambda + \omega)$, there must be an integer ℓ together with a condition $q \in Fn(\omega_2 \setminus \mu, 2)$ satisfying that, for all $n > \ell$, q forces that, "if $k > \dot{g}(n)$ then $(\dot{f}_{[\alpha_n,\alpha_n+\omega)}(G_\mu))(\alpha_n + k) \neq (\dot{f}_{[\lambda,\lambda+\omega)}(G_\mu))(\lambda + k)$."

Choose $n > \ell$ large enough such that $\mathrm{dom}(q) \cap [\alpha_n, \alpha_{n+1})$ is empty. Choose $q_1 < q \restriction \lambda$ (in H) so that

$$\varphi_{\mu,\alpha_n}(q_1 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}) = \varphi_{\mu,\lambda}(q \restriction \mathrm{supp}(\dot{f}_{[\lambda,\lambda+\omega)})$$

and then (again in H) choose $q_2 < q_1$ so that it both forces a value L on $\ell + \dot{g}(n)$ and subsequently forces a value m on $\dot{f}_{[\alpha_n,\alpha_n+\omega)}(\alpha_n + L + 1)$. However, now, again calculate

$$q_3 = \varphi^{-1}_{\mu,\lambda} \circ \varphi_{\mu,\alpha_n}(q_2 \restriction \mathrm{supp}(\dot{f}_{[\alpha_n,\alpha_n+\omega)}))$$

and, by the isomorphisms, we have that q_3 forces that

$$\dot{f}_{[\lambda,\lambda+\omega)}(\lambda + L + 1) = m.$$

Technically (or with more specificity) all of this takes place in the poset $\mathrm{Fn}(\omega_2 \setminus \mu, 2)$, which means that q_3 and q are with each other. In order to verify this, it suffices to consider $q(\beta) = e$ and to assume that $q_3(\beta)$ is defined. Since $q_3(\beta)$ is defined, we have that there is a $\beta' \in \mathrm{dom}(q_2)$ such that $\varphi_{\mu,\lambda}(\beta) = \varphi_{\mu,\alpha_n}(\beta')$ and that $q_3(\beta) = q_2(\beta')$. However, by the definition of q_1, $\beta' \in \mathrm{dom}(q_1)$ and even $q_1(\beta') = q(\beta)$.

Then, since $q_2 < q_1$, we have that $q_2(\beta') = q_1(\beta') = q(\beta)$. This completes the argument that $q_3(\beta) = q(\beta)$.

Finally, our contradiction is that $q_3 \cup q_2 \cup q$ forces $k = L+1$, which violates the quoted statement above. $\square$

We are also able to force $\mathcal{A}'(\kappa)$ to fail for every cardinal other than the first ω, which was just substantiated. Large cardinals are necessary to find $\kappa > \aleph_\omega$ with $\operatorname{cf}\kappa > \omega$ where $\mathcal{S}(\kappa)$ fails.

Theorem 3.5. *It follows from the existence of a 2-huge cardinal that there is a model of* ZFC *for which* $\neg\mathcal{A}'(\aleph_\omega)$.

Proof. We need the model constructed in **[9]**, in which an instance of Chang's conjecture

$$(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$$

is shown to hold.

It may be assumed **[9,** Theorem 5] that we have a model V of GCH in which there are regular limit cardinals $\kappa < \lambda$ satisfying that

$$(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega}).$$

This means that, if L is a countable language with at least one unary relation symbol R, and M is a model of L with base set $\lambda^{+\omega+1}$ in which the interpretation of R has cardinality $\lambda^{+\omega}$, then M has an elementary submodel N of cardinality $\kappa^{+\omega+1}$ in which $R \cap N$ has cardinality $\kappa^{+\omega}$ (of course, $R \cap N$ is the interpretation of R in N since $N \prec M$).

The interested reader should know that it is shown in **[9]** that, if κ is a 2-huge cardinal and j is the 2-huge embedding with critical point κ, then, with $\lambda = j(\kappa)$, it is obtained that $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$ holds. There is no loss of generality to assume in addition that GCH holds in this model.

Let $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ be a scale in $\Pi\{\lambda^{+n+1} : n \in \omega\}$, ordered by the usual mod finite coordinatewise ordering. For convenience, we may assume that $h_\xi(n) \geq \lambda^{+n}$ for all ξ and all n. For each integer m, the cofinality of the mod finite ordering on

$$\Pi\{\lambda^{+n+1} : m < n \in \omega\}$$

is the same as it is for the entire product

$$\Pi\{\lambda^{+n+1} : n \in \omega\}.$$

If P is any poset of cardinality less than λ^{+m}, then, in the forcing extension by P, every function in $\Pi\{\lambda^{+n+1} : m < n \in \omega\}$ is bounded above by a ground model function. It therefore follows easily that, in the forcing extension by P, the sequence

$$\{h_\xi : \xi \in \lambda^{+\omega+1}\}$$

remains cofinal in $\Pi\{\lambda^{+n+1} : n \in \omega\}$.

The forcing notion $\mathbb{P}_0$ is simply the finite condition collapse of $\kappa^{+\omega}$, i.e., $\mathbb{P}_0 = (\kappa^{+\omega})^{<\omega}$. In the forcing extension by $\mathbb{P}_0$, it is now obtained that the ordinal $\kappa^{+\omega+1}$ from V is the first uncountable cardinal $\aleph_1$. Then, in this forcing extension, we let $\mathbb{P}_1$ be the countable condition Levy collapse, $\mathrm{Lv}(\lambda, \omega_2)$, which collapses all cardinals less than λ to have cardinality at most $\aleph_1$. The poset $\mathbb{P}_1$ has cardinality λ. We treat $\mathbb{P}_0 * \mathbb{P}_1$ as containing $\mathbb{P}_0$ as a subposet by identifying each $(p_0, 1)$ with p_0. After forcing with $\mathbb{P}_0 * \mathbb{P}_1$, we will have that ω_1 is the ordinal $(\kappa^{+\omega+1})^V$, ω_2 is the ordinal λ and ω_ω is the ordinal $(\lambda^{+\omega})^V$.

Now, assume that we have an assignment $\dot{f}_{\dot{A}}$ of a $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from $\dot{A}$ into ω for each $\mathbb{P}_0 * \mathbb{P}_1$-name of a countable subset of $\lambda^{+\omega+1}$. We will obtain a contradiction to the claim of coherence.

Let $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ be an enumeration of all of the nice $\mathbb{P}_0$-names of countable subsets of $\lambda^{+\omega}$. For each $\xi \in \lambda^{+\omega+1}$, let $\dot{f}_\xi$ be another notation for $\dot{f}_{\dot{A}_\xi}$. Since $\mathbb{P}_0$ forces that $\mathbb{P}_1$ be countably closed, the collection of all nice $\mathbb{P}_0$-names will produce all of the countable sets in the extension by $\mathbb{P}_0 * \mathbb{P}_1$; however, $\mathbb{P}_0 * \mathbb{P}_1$ can introduce new enumerations of these names. For each $\xi \in \lambda^{+\omega+1}$, there is a minimal ζ_ξ so that $\dot{A}_{\zeta_\xi}$ is the canonical name for the range of h_ξ. This means that $\dot{f}_{\zeta_\xi} \circ h_\xi$ is simply the $\mathbb{P}_0 * \mathbb{P}_1$-name of a finite-to-one function from ω to ω. For each $\xi \in \lambda^{+\omega+1}$, choose any $p_\xi \in \mathbb{P}_0 * \mathbb{P}_1$ such that there is a nice $\mathbb{P}_0$-name $\dot{H}_\xi$ that is forced by p_ξ to equal $\dot{f}_{\zeta_\xi} \circ h_\xi$. Choose $\Lambda \subset \lambda^{+\omega+1}$ of cardinality $\lambda^{+\omega+1}$ and such that there is a pair $p, \dot{H}$ satisfying that $p_\xi = p$ and $\dot{H}_\xi = \dot{H}$ for all $\xi \in \Lambda$. We may assume that p is in a generic filter G.

Let $\{x_\xi : \xi \in \lambda^{+\omega+1}\}$ be any enumeration of $H(\lambda^{+\omega+1})$ such that $\{x_\xi : \xi \in \lambda^{+\omega}\}$ is also equal to $H(\lambda^{+\omega})$. We choose this enumeration in such a way that $x_\xi \in x_\eta$ implies $\xi < \eta$. We use the relation symbol

R_0 to code (and well order) $(H(\lambda^{+\omega+1}), \in)$ as follows: $(\xi, \eta) \in R_0$ if and only if $x_\xi \in x_\eta$. Let R_1 be a binary relation on $\kappa^{+\omega}$ so that $(\kappa^{+\omega}, R_1)$ is isomorphic to $\mathbb{P}_0$. Let R_2 be a binary relation on λ so that $R_2 \cap (\kappa^{+\omega} \times \kappa^{+\omega}) = R_1$ and (λ, R_2) is isomorphic to $\mathbb{P}_0 * \mathbb{P}_1$. Let ψ be the poset isomorphism from (λ, R_2) to $\mathbb{P}_0 * \mathbb{P}_1$.

The coding is continued by coding the sequence $\{h_\xi : \xi \in \lambda^{+\omega+1}\}$ as another binary relation R_3 on $\lambda^{+\omega+1}$ where

$$R_3 \cap (\{\xi\} \times \lambda^{+\omega+1}) = \{(\xi, h_\xi(n)) : n \in \omega\}$$

for each $\xi \in \lambda^{+\omega+1}$. The relation symbol R_4 can code the sequence $\{\dot{A}_\xi : \xi \in \lambda^{+\omega+1}\}$ where $(\xi, \alpha, \zeta) \in R_4$ if and only if $(\check{\alpha}, \psi(\zeta))$ is in the name $\dot{A}_\xi$. Let R_5 code this collection, i.e., $(\gamma, n, m, \eta) \in R_5$ if and only if $((\check{n, m}), \psi(\eta)) \in \dot{H}_\gamma$. In addition, let R_6 code (equal) the set Λ. Finally, use the relation symbol R_7 to similarly code the sequence

$$\{\dot{f}_\xi : \xi \in \lambda^{+\omega+1}\} : (\xi, \alpha, n, \zeta) \in R_7$$

if and only if $((\check{\alpha, n}), \psi(\zeta))$ is in the name $\dot{f}_\xi$.

It is evident that, the unary relation symbol R is interpreted as the set $\lambda^{+\omega}$ for the application of $(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega})$. Now, we have defined our model M of the language $L = \{\in, R, R_0, \ldots, R_7\}$, and we choose an elementary submodel N witnessing

$$(\lambda^{+\omega+1}, \lambda^{+\omega}) \twoheadrightarrow (\kappa^{+\omega+1}, \kappa^{+\omega}).$$

Of course, N is really just a $\kappa^{+\omega+1}$ sized subset of $\lambda^{+\omega+1}$ with the additional property that $N \cap \lambda^{+\omega}$ has cardinality $\kappa^{+\omega}$. In the forcing extension, N has cardinality ω_1 and $A = N \cap \lambda^{+\omega}$ is countable.

We will need the following claim from **[9]**.

Claim 3.6. *We may assume that N satisfies that $N \cap \kappa^{+\omega+1}$ is transitive, i.e., an initial segment.*

Proof of Claim 3.6. Suppose that our originally supplied N fails the conclusion of the claim. We know that $\kappa^{+\omega} \in N$, via R_1, in which case so is $\kappa^{+\omega+1}$.

Then, set $\beta_0 = \sup(N \cap \kappa^{+\omega+1})$, and consider the Skolem closure $\mathrm{Hull}(N \cup \beta_0, M)$. Somewhat informally (in that we must formalize the enumeration of formulas as per Gödel coding), let $\{\varphi_n : n \in \omega\}$

be an enumeration of all formulas in the language L, and let ℓ_n be the minimal integer such that the free variables of φ_n are among $\{v_0, \ldots, v_{\ell_n}\}$. Then, for each tuple $\langle \xi_1, \ldots, \xi_{\ell_n} \rangle$ of elements of $\lambda^{+\omega+1}$, we define $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be the minimal $\xi_0 \in \lambda^{+\omega+1}$ such that $M \models \varphi_n(\xi_0, \ldots, \xi_{\ell_n})$. If there is no such ξ_0, in other words, if

$$M \models \neg \exists x \; \varphi_n(x, \xi_1, \ldots, \xi_{\ell_n}),$$

then set $f_n(\xi_1, \ldots, \xi_{\ell_n})$ to be 0. Now, $\mathrm{Hull}(N \cup \beta_0, M)$ is just the minimal superset X of $N \cup \beta_0$ that satisfies that $f_n[X^{\{1,\ldots,\ell_n\}}] \subset X$ for all n. Since this is simply a large algebra, we can generate all of the terms t of the algebraic operations $\{f_n : n \in \omega\}$. It is easily seen that, for each $\zeta \in X$, there is a term $t(v_1, \ldots, v_m)$ such that $\zeta = t(\delta_1, \ldots, \delta_m)$ for some sequence $\langle \delta_1, \ldots, \delta_m \rangle$ with each $\delta_i \in N \cup \beta_0$. Assume that $\zeta \in \kappa^{+\omega+1}$. By re-indexing the variables in the term we can assume that there is an $n \leq m$ so that $\delta_i < \beta_0$ for $1 \leq i \leq n$ and $\kappa^{+\omega+1} \leq \delta_i$ for $n < i \leq m$. Let $\vec{a}$ denote the tuple $\langle \delta_{n+1}, \ldots, \delta_m \rangle$. Choose $\eta \in N \cap \kappa^{+\omega+1}$ large enough so that $\{\delta_1, \ldots, \delta_n\}$ is contained in η. Since set-membership in M is coded by R_0 rather than $\in$, we argue a little less naturally. Consider the set

$$s_0(\eta, \vec{a}) = \{t(\gamma_1, \ldots, \gamma_n, \vec{a}) : \{\gamma_1, \ldots, \gamma_n\} \in [\eta]^{\leq n}\}.$$

Clearly, $s_0(\eta, \vec{a})$ is a member of $H(\lambda^{+\omega+1})$. Now, define $s_1(\eta, \vec{a})$ to be $\{x_\alpha : \alpha \in s_0(\eta, \vec{a})\}$, and choose the unique $\zeta_1 \in \lambda^{+\omega+1}$ such that $x_{\zeta_1} = s_1(\eta, \vec{a})$. We claim that $\zeta_1 \in N$. Note that $\alpha R_0 \zeta_1$ holds if and only if $\alpha \in s_0(\eta, \vec{a})$, and therefore,

$$\begin{aligned} M \models (\forall \alpha)[\alpha R_0 \zeta_1 \text{ if and only if } (\exists \, \gamma_1 \in \eta) \\ \cdots (\exists \, \gamma_n \in \eta)(\alpha = t(\gamma_1, \ldots, \gamma_n, \vec{a}))]. \end{aligned}$$

By elementarity, then, we have that $\zeta_1 \in N$, and by similar reasoning, the supremum ζ_0 of $\zeta_1 \cap \kappa^{+\omega+1}$ is also in N. This, of course, means that $\zeta < \beta_0$. $\square$

The elementarity of N is used to deduce properties of the families $\{\dot{A}_\xi : \xi \in N\}$ and $\{\dot{f}_\xi : \xi \in N\}$. In particular, the collection we are most interested in is the family

$$\{h_\xi : \xi \in \Lambda \cap N\}.$$

Now, we need a result from Shelah's pcf theory, proven in [**4,** 24.9]. Since $\aleph_1 = \mathfrak{c} < \kappa^{+\omega+1}$, there is a function $\langle \varrho_n : n \in \omega \rangle$ in $\Pi_n \lambda^{+\omega}$ such that the sequence $\{h_\xi : \xi \in N\}$ is unbounded mod finite in $\Pi_n \varrho_n$ For each n, $\varrho_n \leq \sup(N \cap \lambda^{+n+2})$. Since $\mathbb{P}_0$ has cardinality $\kappa^{+\omega}$, and so, less than $|N| = \kappa^{+\omega+1}$, a standard argument (analogous to the fact that adding a Cohen real does not add a dominating real) shows that the sequence $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded mod finite in $\Pi_n \varrho_n$ (and in $\Pi_n(\varrho_n \cap N)$).

Next, pass to the extension by $G \cap \mathbb{P}_0$, and let H be the function $\mathrm{val}_G(\dot{H})$. Recall that $f_{\zeta_\xi}(h_\xi(n)) = H(n)$ for all $n \in \omega$ and $\xi \in \Lambda$. Now, pass to the full extension $V[G]$, and again, since $\mathbb{P}_1$ was forced to be countably closed, the family $\{h_\xi : \xi \in \Lambda \cap N\}$ remains unbounded in $\Pi_n(\varrho_n \cap N)$ (no new elements were added). Let A be the countable set $N \cap \lambda^{+\omega}$, and, for each $\xi \in \Lambda \cap N$, there is an n_ξ such that $f_\xi(h_\xi(m)) = f_A(h_\xi(m))$ for all $m > n_\xi$. There is a single n so that $\Lambda_n = \{\xi \in \Lambda \cap N : n_\xi = n\}$ has cardinality ω_1, and thus, $\{h_\xi : \xi \in \Lambda_n \cap N\}$ is also unbounded in $\Pi_n(\rho_n \cap N)$. This certainly implies that there is an $m > n$ such that

$$\{h_\xi(m) : \xi \in \Lambda_n \cap N\}$$

is infinite. This completes the proof since $f_A(h_\xi(m)) = H(m)$ for all $\xi \in \Lambda_n \cap N$. $\square$

4. Applications to infinite length games. We introduce three variations of Scheeper's game which was defined in the introduction.

Game 4.1. Let $\mathrm{Sch}^{\cup,\subseteq}(\kappa)$ denote the *Scheepers' countable-finite union game* which proceeds analogously to $\mathrm{Sch}^{\cup,\subsetneq}(\kappa)$, except that $\mathscr{C}$'s restriction in round $n+1$ is weakened to

$$C_{n+1} \supseteq C_n.$$

Game 4.2. Let $\mathrm{Sch}^{1,\subseteq}(\kappa)$ denote the *Scheepers' countable-finite initial game* which proceeds analogously to $\mathrm{Sch}^{\cup,\subseteq}(\kappa)$, except that $\mathscr{F}$'s winning condition is weakened to

$$\bigcup_{n<\omega} F_n \supseteq C_0.$$

Game 4.3. Let $\mathrm{Sch}^{\cap}(\kappa)$ denote the *Scheepers' countable-finite intersection game* which proceeds analogously to $\mathrm{Sch}^{1,\subseteq}(\kappa)$, except that $\mathscr{C}$ may choose any $C_n \in [\kappa]^{\leq\omega}$ each round, and $\mathscr{F}$'s winning condition is weakened to

$$\bigcup_{n<\omega} F_n \supseteq \bigcap_{n<\omega} C_n.$$

In [**1**], Clontz extended Scheepers' application of almost-compatible injections to these game variants as well as $\mathrm{Men}(\kappa^\dagger)$. However, when considering Markov strategies, finite-to-one functions suffice.

Theorem 4.4. *Figure* 1.

$$\begin{array}{ccccc}
\mathcal{A}(\kappa) & & \Longrightarrow & \mathcal{A}'(\kappa) & \\
\Big\Downarrow & & & \Downarrow & \\
 & & \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Men}\left(\kappa^\dagger\right) & \Longleftrightarrow & \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\kappa) \\
 & & & \Downarrow & \\
\mathscr{F} \underset{\text{2-tact}}{\uparrow} \mathrm{Sch}^{\cup,\subsetneq}(\kappa) & \Longrightarrow & \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{1,\subseteq}(\kappa) & \Longleftrightarrow & \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cup,\subseteq}(\kappa) \\
 & & & \Downarrow & \\
 & & & \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cup,\subsetneq}(\kappa) &
\end{array}$$

FIGURE 1. Diagram of Scheeper/Menger game implications with $\mathcal{A}(\kappa)$ and $\mathcal{A}'(\kappa)$.

Proof.

$$\mathcal{A}(\kappa) \Longrightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} \mathrm{Sch}^{\cup,\subsetneq}(\kappa)$$

was shown in [**11**], see Theorem 4.5 below. Most of the other results in the figure were proven in [**1**], with the exception that $\mathcal{A}'(\kappa)$ was not considered at the time. The following proof that

$$\mathcal{A}'(\kappa) \Longrightarrow \mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\kappa)$$

is a trivial modification of the proof presented in **[1]** assuming $\mathcal{A}(\kappa)$; however, since that paper is under review at the time of this writing, we provide it here.

Let f_A for $A \in [\kappa]^{\leq\omega}$ witness $\mathcal{A}'(\kappa)$. A 2-mark σ for $\mathrm{Sch}^{\cap}(\kappa)$ is defined as follows:

$$\sigma(\langle A\rangle, 0) = \{\alpha \in A : f_A(\alpha) = 0\}$$
$$\sigma(\langle A, B\rangle, n+1) = \{\alpha \in A \cap B : f_B(\alpha) \leq n+1 \text{ or } f_A(\alpha) \neq f_B(\alpha)\}.$$

For any attack $\langle A_0, A_1, \ldots\rangle$ by $\mathscr{C}$ and

$$\alpha \in \bigcap_{n<\omega} A_n,$$

either $f_{A_n}(\alpha)$ is constant for all n, or $f_{A_n}(\alpha) \neq f_{A_{n+1}}(\alpha)$ for some n; either way, α is covered. □

We include the following proof from **[11]** to point out the reason $\mathcal{A}'(\kappa)$ seems insufficient for providing $\mathscr{F}$ a winning 2-tactic in $\mathrm{Sch}^{\cup,\subsetneq}(\kappa)$, despite that it witnesses a winning 2-mark.

Theorem 4.5 ([**11**])**.**

$$\mathcal{A}(\kappa) \Longrightarrow \mathscr{F} \underset{\text{2-tact}}{\uparrow} \mathrm{Sch}^{\cup,\subsetneq}(\kappa).$$

Proof. Let $\{f_A : A \in [\kappa]^{\leq\aleph_0}\}$ witness $\mathcal{A}(\kappa)$, and define $g_A : A \to \omega$ by

$$g_A(\alpha) = f_A(\alpha) - |\{\beta \in A : f_A(\beta) < f_A(\alpha)\}|.$$

We claim that

$$\{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

must be finite since it is bounded above by

$$\max\{M, f_A(\alpha), f_B(\alpha) : f_A(\alpha) \neq f_B(\alpha)\},$$

where $M = f_B(\alpha)$ for some $\alpha \in B \setminus A$. In order to see this, let $f_A(\alpha) = f_B(\alpha) = N > M$, and assume that $f_A(\beta) \neq f_B(\beta)$ implies

$f_A(\beta), f_B(\beta) < N$. Then,

$$\begin{aligned} g_A(\alpha) &= N - |\{\beta \in A : f_A(\beta) < N\}| \\ &> N - |\{\beta \in B : f_B(\beta) < N\}| \\ &= g_B(\alpha) \end{aligned}$$

with the strictness of the inequality witnessed by $f_B(\alpha) = M < N$ for some $\alpha \in B \setminus A$. As a result,

$$\sigma(\langle A, B\rangle) = \{\alpha \in A : g_A(\alpha) \leq g_B(\alpha)\}$$

is a legal 2-tactic for $\mathscr{F}$. Let $C = \langle C_0, C_1, \dots\rangle$ be a strictly increasing sequence of countable sets and $\alpha \in C_n$. Noting that f_A is an injection (not merely finite-to-one), $0 \leq g_{C_{n+m}}(\alpha)$ for all $m < \omega$, and it follows that $g_{C_{n+m}}(\alpha) \leq g_{C_{n+m+1}}(\alpha)$ for some $m < \omega$. Therefore,

$$\alpha \in \sigma(\langle C_{n+m}, C_{n+m+1}\rangle). \qquad \square$$

While the above proof cannot be trivially modified to utilize the finite-to-one functions witnessed by $\mathcal{A}'(\kappa)$ in constructing a winning 2-tactical strategy for $\mathrm{Sch}^{\cup,\subsetneq}(\kappa)$, whether $\mathcal{A}'(\kappa)$ is sufficient for

$$\mathscr{F} \underset{\text{2-tact}}{\uparrow} \mathrm{Sch}^{\cup,\subsetneq}(\kappa)$$

after all does remain open:

Question 4.6. *Can the previous theorem be improved by replacing $\mathcal{A}(\kappa)$ with $\mathcal{A}'(\kappa)$?*

We would like to demonstrate that $\mathcal{A}'(\kappa)$ is unnecessary for constructing winning 2-Markov strategies in $\mathrm{Sch}^{\cap}(\kappa)$.

Theorem 4.7. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If*

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\aleph_{\beta_n})$$

for all $n < \omega$, then

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\aleph_\alpha).$$

Proof. Let σ_n be a winning 2-mark for $\mathscr{F}$ in $\mathrm{Sch}^{\cap}(\aleph_{\beta_n})$. Define the 2-mark σ for $\mathscr{F}$ in $\mathrm{Sch}^{\cap}(\aleph_\alpha)$ as follows:

$$\sigma(\langle C\rangle, 0) = \sigma_0(\langle C \cap \aleph_{\beta_0}\rangle, 0)$$

$$\begin{aligned}\sigma(\langle C, D\rangle, n+1) = {} & \sigma_{n+1}(\langle D \cap \aleph_{\beta_{n+1}}\rangle, 0) \\ & \cup \bigcup_{m\leq n} \sigma_m(\langle C \cap \aleph_{\beta_m}, D \cap \aleph_{\beta_m}\rangle, n-m+1).\end{aligned}$$

Let $\langle C_0, C_1, \ldots\rangle$ be an attack by $\mathscr{C}$ in $\mathrm{Sch}^{\cap}(\aleph_\alpha)$ and

$$\alpha \in \bigcap_{n<\omega} C_n.$$

Choose $N < \omega$ with $\alpha < \aleph_{\beta_{N+1}}$. Consider the attack

$$\langle C_{N+1} \cap \aleph_{\beta_{N+1}}, C_{N+2} \cap \aleph_{\beta_{N+1}}, \ldots\rangle$$

by $\mathscr{C}$ in $\mathrm{Sch}^{\cap}(\aleph_{\beta_{N+1}})$. Since σ_{N+1} is a winning 2-mark and

$$\alpha \in \bigcap_{n<\omega} C_{N+n+1} \cap \aleph_{\beta_{N+1}},$$

either $\alpha \in \sigma_{N+1}(\langle C_{N+1} \cap \aleph_{\beta_{N+1}}\rangle, 0)$, and thus,

$$\alpha \in \sigma(\langle C_N, C_{N+1}\rangle, N+1),$$

or

$$\alpha \in \sigma_{N+1}(\langle C_{N+M+1} \cap \aleph_{\beta_{N+1}}, C_{N+M+2} \cap \aleph_{\beta_{N+1}}\rangle, M+1)$$

for some $M < \omega$. Therefore,

$$\alpha \in \sigma(\langle C_{N+M+1}, C_{N+M+2}\rangle, N+M+2).$$

Thus, σ is a winning 2-mark. □

Theorem 4.8. *Let α be the limit of increasing ordinals β_n for $n < \omega$. If*

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{1,\subseteq}(\aleph_{\beta_n})$$

for all $n < \omega$, then

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{1,\subseteq}(\aleph_\alpha).$$

Proof. The proof proceeds nearly identically to the previous proof: substitute $\alpha \in C_0$ in place of

$$\alpha \in \bigcap_{n<\omega} C_n,$$

and proceed. □

Corollary 4.9. *It is consistent that $\mathcal{A}'(\aleph_\omega)$ fails, but since $\mathcal{A}'(\aleph_k)$ holds in* ZFC *for all $k < \omega$, both*

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\aleph_\omega)$$

and

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{1,\subseteq}(\aleph_\omega)$$

hold in ZFC*.*

We conclude by returning our attention to Question 1.6, which asks whether there exists a space for which the second player $\mathscr{F}$ in the game $\mathrm{Men}(X)$ has a winning strategy without a winning 2-mark.

Question 4.10. *Does*

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cap}(\kappa)$$

hold for all cardinals κ in ZFC*?*

If not, the model producing $\kappa > \aleph_\omega$, where

$$\mathscr{F} \underset{\text{2-mark}}{\not\uparrow} \mathrm{Sch}^{\cap}(\kappa)$$

yields a positive answer to Question 1.6: $X = \kappa^\dagger$. On the other hand, under $V = L$ Corollary 3.1 shows that $\mathcal{A}'(\kappa)$, and therefore,

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Men}(\kappa^\dagger)$$

for every cardinal κ; thus, a more exotic example than $X = \kappa^\dagger$ would be required to answer Question 1.6 in ZFC.

Solving the following, weaker, question would not answer Question 1.6 in itself; however, a solution would be nonetheless interesting.

Question 4.11. *Does*

$$\mathscr{F} \underset{\text{2-mark}}{\uparrow} \mathrm{Sch}^{\cup,\subseteq}(\kappa)$$

hold for all cardinals κ in ZFC*?*

Whether the previous two questions are even distinct remains open.

Question 4.12. *Can a winning* 2*-Markov strategy in* $\mathrm{Sch}^{\cup,\subseteq}(\kappa)$ *be used to construct a winning* 2*-Markov strategy in* $\mathrm{Sch}^{\cap}(\kappa)$*?*

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REFERENCES

1. S. Clontz, *Applications of limited information strategies in Menger's game*, Comm. Math. Univ. Carolin. **58** (2017), 225-239.

2. A. Dow, *An introduction to applications of elementary submodels to topology*, Topol. Proc. **13** (1988), 17–72.

3. ———, *Set theory in topology*, in *Recent progress in general topology*, North-Holland, Amsterdam, 1992.

4. T. Jech, *Set theory*, Springer Mono. Math. (2003).

5. R.B. Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

6. I. Juhász, Zs. Nagy and W. Weiss, *On countably compact, locally countable spaces*, Math. Hungar. **10** (1979), 193–206.

7. I. Juhász and W. Weiss, *Good, splendid and Jakovlev*, in *Open problems in topology*, II, Elsevier, Amsterdam, 2007.

8. K. Kunen, *Set theory*, Stud. Logic Found. Math. **102** (1980).

9. J.-P. Levinski, M. Magidor and S. Shelah, *Chang's conjecture for* $\aleph_\omega$, Israel J. Math. **69** (1990), 161–172.

10. P. Nyikos, *Generalied Kurepa and mad families and topology*, `oai:CiteSeerX.psu:10.1.1.404.5308`.

11. M. Scheepers, *Concerning n-tactics in the countable-finite game*, J. Symb. Log. **56** (1991), 786–794.

12. ———, *Meager-nowhere dense games*, I, *n-tactics*, Rocky Mountain J. Math. **22** (1992), 1011–1055.

13. M. Scheepers, *Combinatorics of open covers*, I, *Ramsey theory*, Topol. Appl. **69** (1996), 31–62.

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