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FIXED POINTS OF AUGMENTED GENERALIZED HAPPY FUNCTIONS

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ABSTRACT. An augmented generalized happy function $S_{[c,b]}$ maps a positive integer to the sum of the squares of its base b digits plus c. In this paper, we study various properties of the fixed points of $S_{[c,b]}$; count the number of fixed points of $S_{[c,b]}$ for $b \geq 2$ and $0 < c < 3b-3$; and prove that, for each $b \geq 2$, there exist arbitrarily many consecutive values of c for which $S_{[c,b]}$ has no fixed point.

1. Introduction. The concept of a happy number, defined in [**5**] and popularized by Guy [**3**], was generalized in [**2**] by allowing for varying bases and exponents in the defining function. In [**1**], this was further generalized, altering the defining function with the addition of a constant. Specifically, for the integers $c \geq 0$ and $b \geq 2$, the augmented generalized happy function $S_{[c,b]} : \mathbb{Z}^+ \to \mathbb{Z}^+$ is defined by

$$S_{[c,b]}\left(\sum_{i=0}^{n} a_i b^i\right) = c + \sum_{i=0}^{n} a_i^2, \tag{1.1}$$

where $0 \leq a_i \leq b-1$ and $a_n \neq 0$. Thus, for a positive integer a denoted $a_n \cdots a_1 a_0$ in base b,

$$S_{[c,b]}(a_n \cdots a_1 a_0) = c + a_n^2 + \cdots + a_1^2 + a_0^2.$$

A positive integer a is a happy number if, for some $k \in \mathbb{Z}^+$, $S_{[0,10]}^k(a) = 1$. Although the sole fixed point of $S_{[0,10]}$ is 1, as shown in [**2**], for $b \neq 10$, $S_{[0,b]}$ may have additional fixed points. Similarly, as

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shown in [**1**], when $c > 0$ (and $b \geq 2$), $S_{[c,b]}$ may have multiple fixed points.

In this work, we study the fixed points of the functions $S_{[c,b]}$. First, in Section 2, we prove some preliminary results providing properties of the fixed points and consecutive fixed points of an arbitrary, but fixed, $S_{[c,b]}$. Then, in Section 3, we discuss the exact number of fixed points of $S_{[c,b]}$, in terms of c and b. Finally, in Section 4, we let $c \geq 0$ vary and prove that, for each $b \geq 2$, there are arbitrarily long sequences of consecutive values of c for which $S_{[c,b]}$ has no fixed point.

2. Fixed point characteristics. Here, we discuss a variety of results concerning relationships between the fixed points of a single $S_{[c,b]}$, where $c \geq 0$ and $b \geq 2$ are arbitrary integers. The first theorem concerns consecutive fixed points of $S_{[c,b]}$.

Theorem 2.1. *Fix $c \geq 0$ and $b \geq 2$.*

1. *If $a \in \mathbb{Z}^+$ is a multiple of b, then a is a fixed point of $S_{[c,b]}$ if and only if $a+1$ is a fixed point of $S_{[c,b]}$.*
2. *Every consecutive pair of fixed points of $S_{[c,b]}$ has a multiple of b as its first member.*
3. *There is no consecutive triplet of fixed points of $S_{[c,b]}$.*

Proof. We begin with the proof of Part 1. Since a is a multiple of b, we have $S_{[c,b]}(a+1) = S_{[c,b]}(a) + 1^2 = S_{[c,b]}(a) + 1$. Thus, $S_{[c,b]}(a) = a$ if and only if $S_{[c,b]}(a+1) = a+1$.

For Part 2, assume that a and $a+1$ are both fixed points of $S_{[c,b]}$ and, using standard notation for base b, let

$$a = \sum_{i=0}^{n} a_i b^i.$$

First, assume that $a_0 \neq b-1$. Then

$$a = S_{[c,b]}(a) = c + \sum_{i=1}^{n} a_i^2 + a_0^2,$$

and thus,

$$a+1 = S_{[c,b]}(a+1) = c + \sum_{i=1}^{n} a_i^2 + (a_0+1)^2 = a + 2a_0 + 1.$$

Thus, $2a_0 = 0$, implying that $a_0 = 0$. Therefore, a is a multiple of b, as desired.

Next assume, for a contradiction, that $a_0 = b-1$. If every digit of a is equal to $b-1$, then $a+1 = b^{n+1}$, and thus, $a+1 = S(a+1) = c+1$. However, then $c = a = S(a) = c + (n+1)(b-1)^2$, a contradiction. Thus, we can let $j \in \mathbb{Z}^+$ be minimal such that $a_j \neq b-1$. Then,

$$a = S_{[c,b]}(a) = c + \sum_{i=j}^{n} a_i^2 + j(b-1)^2, \tag{2.1}$$

and since

$$a+1 = \sum_{i=j+1}^{n} b^i a_i + (a_j+1)b^j,$$

we have

$$a+1 = S_{[c,b]}(a+1) = c + \sum_{i=j+1}^{n} a_i^2 + (a_j+1)^2. \tag{2.2}$$

Combining equations (2.1) and (2.2) yields

$$a_j^2 + j(b-1)^2 + 1 = (a_j+1)^2.$$

Thus, $j(b-1)^2 = 2a_j$. Since $a_j < b-1$, $j(b-1) < 2$, and thus, $j = b-1 = 1$. However, then $2a_j = 1$, which is a contradiction.

Finally, Part 3 is immediate from Part 2. □

Lemma 2.2 provides another pairing of fixed points of $S_{[c,b]}$.

Lemma 2.2. *Fix $c \geq 0$, $b \geq 2$, and $a \in \mathbb{Z}^+$, where*

$$a = \sum_{i=0}^{n} a_i b^i,$$

in standard base b notation with $a_1 \neq 0$. Let

$$\widetilde{a} = \sum_{i=2}^{n} a_i b^i + (b - a_1)b + a_0.$$

Then, a is a fixed point of $S_{[c,b]}$ if and only if $\widetilde{a}$ is a fixed point of $S_{[c,b]}$.

Proof. Assume that a and $\widetilde{a}$ are as above, and that a is a fixed point of $S_{[c,b]}$. Then,

$$\begin{aligned} S_{[c,b]}(\widetilde{a}) &= S_{[c,b]}\left(\sum_{i=2}^{n} a_i b^i + (b - a_1)b + a_0 \right) \\ &= c + \sum_{i=2}^{n} a_i^2 + (b - a_1)^2 + a_0^2 \\ &= c + \sum_{i=0}^{n} a_i^2 + b^2 - 2a_1 b \\ &= S_{[c,b]}(a) + b^2 - 2a_1 b \\ &= a + b^2 - 2a_1 b \\ &= \sum_{i=0}^{n} a_i b^i + (b - 2a_1)b \\ &= \sum_{i=2}^{n} a_i b^i + (b - a_1)b + a_0 = \widetilde{a}. \end{aligned}$$

Therefore, $\widetilde{a}$ is also a fixed point of $S_{[c,b]}$. The converse is immediate by symmetry. □

Finally, we consider the parity of c that is required for $S_{[c,b]}$ to have a fixed point.

Lemma 2.3. *Fix $c \geq 0$ and $b \geq 2$, and let*

$$a = \sum_{i=0}^{n} a_i b^i$$

be a fixed point of $S_{[c,b]}$, in the usual base b notation.

1. *If b is odd, then c is even.*
2. *If b is even, then*

$$c \equiv \sum_{i=1}^{n} a_i \pmod 2.$$

Proof. For b odd, by [**1,** Lemma 2.3], $S_{[c,b]}(a) \equiv c + a \pmod 2$, which implies that c is even. For b even, since a is a fixed point of $S_{[c,b]}$,

$$a_0 \equiv a = S_{[c,b]}(a) = c + \sum_{i=0}^{n} a_i^2 \equiv c + \sum_{i=0}^{n} a_i \pmod 2.$$

Subtracting a_0 from both sides of the congruence yields the result. □

3. Counting the number of fixed points. In this section, we consider the *number* of fixed points of the function $S_{[c,b]}$ for fixed $c \geq 0$ and $b \geq 2$. In Corollary 3.5, we provide a formula for the number of fixed points of $S_{[c,b]}$ for all values of b and a range of values of c, depending on b.

We begin by determining the number of fixed points of $S_{[c,b]}$ of the form ub^n, where $0 < u < b$ and $n \geq 0$. To fix notation, for $c \geq 0$, $b \geq 2$, and $n \geq 0$, let

$$\mathcal{F}_{[c,b]}^{(n)} = \{a = ub^n \mid 0 < u < b \text{ and } S_{[c,b]}(a) = a\}.$$

In the following three lemmas, we provide conditions under which $\mathcal{F}_{[c,b]}^{(n)}$ assumes specified values.

Lemma 3.1. *Fix $b \geq 2$. For $c > 0$, $\mathcal{F}_{[c,b]}^{(0)}$ is empty, while $\mathcal{F}_{[0,b]}^{(0)} = \{1\}$.*

Proof. Let $0 < a < b$. Then, $a = S_{[c,b]}(a)$ implies that $c = a - a^2 \leq 0$. Hence, if $c = 0$, we have $a = 1$, and, if $c > 0$, we have a contradiction. □

Lemma 3.2. *Fix $c \geq 0$ and $b \geq 2$. The cardinality of $\mathcal{F}^{(1)}_{[c,b]}$ is*

$$\left|\mathcal{F}^{(1)}_{[c,b]}\right| = \begin{cases} 2 & \text{if } \alpha^2 - \alpha b + c = 0 \text{ for some integer } 1 \leq \alpha < b/2, \\ 1 & \text{if } b^2 = 4c, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From the definition of $\mathcal{F}^{(1)}_{[c,b]}$, given $a = ub$ with $0 < u < b$, we have $a \in \mathcal{F}^{(1)}_{[c,b]}$ if and only if $S_{[c,b]}(a) = a$ or, equivalently, $c + u^2 = ub$. Thus, $|\mathcal{F}^{(1)}_{[c,b]}| \neq 0$ if and only if there exists some integer u, $0 < u < b$, such that

$$u^2 - ub + c = 0. \tag{3.1}$$

Since this is a quadratic equation, there are at most two such values of u, and at most one if $b^2 = 4c$.

By Lemma 2.2, $a = ub$ is a fixed point of $S_{[c,b]}$ if and only if $\widetilde{a} = (b-u)b$ is a fixed point of $S_{[c,b]}$. Hence, $a = ub \in \mathcal{F}^{(1)}_{[c,b]}$ if and only if $(b-u)b \in \mathcal{F}^{(1)}_{[c,b]}$. Thus, $|\mathcal{F}^{(1)}_{[c,b]}| = 2$ if and only if there are two integer solutions to equation (3.1) with $0 < u < b$, in which case one of the solutions will satisfy $1 \leq u < b/2$. The lemma follows. □

Lemma 3.3. *For $c \geq 0$, $b \geq 2$, and $n \geq 2$, the cardinality of $\mathcal{F}^{(n)}_{[c,b]}$ is*

$$\left|\mathcal{F}^{(n)}_{[c,b]}\right| = \begin{cases} 1 & \text{if } b^{2n} - 4c > (b^n - 2b)^2 \text{ is a perfect square, and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix $n \geq 2$, and suppose that $a = ub^n$ is a fixed point of $S_{[c,b]}$ for some $0 < u < b$. Then, $ub^n = a = S_{[c,b]}(a) = c + u^2$, which implies that

$$u = \frac{b^n \pm \sqrt{b^{2n} - 4c}}{2}. \tag{3.2}$$

Since $u \in \mathbb{Z}^+$, $b^{2n} - 4c$ is a perfect square, and since $u < b$ and $n \geq 2$, $b^{2n} - 4c$ is nonzero. Conversely, if $b^{2n} - 4c$ is a nonzero perfect square, then $(b^n + \sqrt{b^{2n} - 4c})/2 > b$ and thus is not a candidate for u, while, letting $u = (b^n - \sqrt{b^{2n} - 4c})/2$, it is easily verified that $a = ub^n \in \mathcal{F}^{(n)}_{[c,b]}$. □

The following theorem and its proof were inspired by the work of Hargreaves and Siksek **[4]** on the number of fixed points of (unaugmented) generalized happy functions. As is standard, we let $r_2(n)$ denote the number of representations of $n \in \mathbb{Z}^+$ as the sum of two squares, that is,

$$r_2(n) = |\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\}|. \tag{3.3}$$

Theorem 3.4. *For $c > 0$ and $b \geq 2$, the number of two-digit fixed points of $S_{[c,b]}$ is given by*

$$\begin{cases} (1/2)r_2(b^2 - 4c + 1) + \left|\mathcal{F}^{(1)}_{[c,b]}\right| & \text{if } b \text{ is odd, and} \\ (1/4)r_2(b^2 - 4c + 1) + \left|\mathcal{F}^{(1)}_{[c,b]}\right| & \text{if } b \text{ is even.} \end{cases}$$

Proof. Note that $a = ub + v$ is a fixed point of $S_{[c,b]}$, with $0 < u < b$ and $0 \leq v < b$ if and only if $ub + v = S_{[c,b]}(ub + v) = c + u^2 + v^2$. Define

$$U = \{(u, v) \in \mathbb{Z}^2 \mid 0 < u, v < b \text{ and } ub + v = c + u^2 + v^2\}.$$

By the correspondence $(u, v) \leftrightarrow ub + v$, $|U|$ is equal to the number of two-digit fixed points of $S_{[c,b]}$ that are not multiples of b. Hence, the number of two-digit fixed points of $S_{[c,b]}$ is equal to $|U| + |\mathcal{F}^{(1)}_{[c,b]}|$.

Set

$$X = \{(x, y) \in \mathbb{Z}^2 \mid y \geq 1 \text{ odd, and } x^2 + y^2 = b^2 - 4c + 1\}.$$

In order to see that $|U| = |X|$, consider the functions $\phi : U \to X$ and $\psi : X \to U$, defined by

$$\phi(u, v) = (2u - b, 2v - 1)$$

and

$$\psi(x, y) = \left(\frac{x + b}{2}, \frac{y + 1}{2}\right).$$

Making a straightforward calculation and noting that $2v - 1 > 0$ and odd proves that the image of ϕ is contained in X. Let $(x, y) \in X$. To see that $\psi(x, y) \in U$, first note that y is odd and $x \equiv b \pmod 2$, and thus, $\psi(x, y) \in \mathbb{Z}^2$. Next, since

$$x^2 < x^2 + y^2 = b^2 - 4c + 1 < b^2,$$

we have $-b < x < b$, implying that $0 < x + b < 2b$, and thus $0 < (x + b)/2 < b$, as desired. Similarly, $1 \leq y < b$; thus, $1 \leq (y + 1)/2 < b$. Finally, a direct calculation verifies that the necessary equation is satisfied.

Since, as is easily checked, ϕ and ψ are inverses, it follows that $|U| = |X|$.

Now, note that X is a subset of

$$Z = \{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = b^2 - 4c + 1\},$$

and recall that, by equation (3.3), $|Z| = r_2(b^2 - 4c + 1)$.

If b is odd and $(x, y) \in Z$, then $b^2 - 4c + 1 \equiv 2 \pmod 4$, and thus, y must be odd. Therefore, $\varphi_{\text{odd}} : Z \to X$ defined by $(x, y) \mapsto (x, |y|)$ is a two-to-one surjective function. Hence, $|X| = |Z|/2 = (1/2)r_2(b^2 - 4c + 1)$.

If b is even and $(x, y) \in Z$, then $b^2 - 4c + 1$ is odd, and thus, exactly one of x and y is odd. Thus, $\varphi_{\text{even}} : Z \to X$ defined by

$$(x, y) \longmapsto \begin{cases} (x, |y|) & \text{if } y \text{ is odd,} \\ (y, |x|) & \text{if } x \text{ is odd,} \end{cases}$$

is a four-to-one surjective function. Hence, $|X| = |Z|/4 = (1/4)r_2(b^2 - 4c + 1)$.

Recalling that the number of two-digit fixed points of $S_{[c,b]}$ is $|U| + |\mathcal{F}^{(1)}_{[c,b]}| = |X| + |\mathcal{F}^{(1)}_{[c,b]}|$, the result follows. ☐

Corollary 3.5. *For $b \geq 2$ and $0 < c < 3b - 3$, the number of fixed points of $S_{[c,b]}$ is exactly*

$$\begin{cases} (1/2)r_2(b^2 - 4c + 1) + \left|\mathcal{F}^{(1)}_{[c,b]}\right| & \textit{if } b \textit{ is odd, and} \\ (1/4)r_2(b^2 - 4c + 1) + \left|\mathcal{F}^{(1)}_{[c,b]}\right| & \textit{if } b \textit{ is even.} \end{cases}$$

Proof. From [**1,** Lemma 2.2], since $c < 3b - 3$, for each $a > b^2$, $S(a) < a$ and, therefore, a is not a fixed point. Hence, each fixed point of $S_{[c,b]}$ has at most two digits. The corollary now follows directly from Lemma 3.1 and Theorem 3.4. ☐

4. Fixed point deserts. In this section, we fix the base $b \geq 2$ and consider consecutive values of c for which $S_{[c,b]}$ has no fixed points. Note that, for a fixed b, if

$$a = \sum_{i=0}^{n} a_i b^i$$

is a fixed point of $S_{[c,b]}$, with $0 \leq a_i < b$ for each i, then, solving for c, we have

$$c = \sum_{i=0}^{n} a_i (b^i - a_i). \tag{4.1}$$

Definition 4.1. For $b \geq 2$ and $k \in \mathbb{Z}^+$, a *k-desert base b* is a set of k consecutive non-negative integers c for each of which $S_{[c,b]}$ has no fixed points. A *desert base b* is a k-desert base b for some $k \geq 1$.

For example, for $28 \leq c \leq 35$, $S_{[c,10]}$ has no fixed points and, therefore, there is an 8-desert base 10 starting at $c = 28$.

We begin by determining bounds on the values of c such that $S_{[c,b]}$ has a fixed point of a given number of digits. For $b \geq 2$ and $n \geq 2$, define

$$m_{b,n} = b^n - b^2 + 3b - 3,$$

and

$$M_{b,n} = b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor)\lfloor b/2 \rfloor.$$

Theorem 4.2. *Let $b \geq 2$ and $n \geq 2$. If $S_{[c,b]}$ has an $n+1$-digit fixed point, then $m_{b,n} \leq c \leq M_{b,n}$. Further, these bounds are sharp.*

Proof. Let $b \geq 2$ and $n \geq 2$ be fixed. From equation (4.1), each fixed point a of $S_{[c,b]}$ determines the value of c. Treating the a_i in equation (4.1) as independent variables taking on integer values inclusively between 0 and $b-1$, we find the minimal possible value of c by minimizing each term. Observe that $a_0(b^0 - a_0)$ is minimal when $a_0 = b-1$; for $0 < i < n$, $a_i(b^i - a_i)$ is minimal when $a_i = 0$; and, since $a_n \neq 0$, $a_n(b^n - a_n)$ is minimal when $a_n = 1$. Hence, the minimal value of c is determined by

$$a = \sum_{i=0}^{n} a'_i b^i,$$

where

$$a'_i = \begin{cases} 1 & \text{for } i = n, \\ 0 & \text{for } \ 1 \leq i \leq n-1, \\ b-1 & \text{for } \ i = 0, \end{cases}$$

and thus, the minimal value of c is

$$c = (b-1)(b^0 - (b-1)) + 1 \cdot (b^n - 1) = b^n - b^2 + 3b - 3 = m_{b,n}.$$

Similarly, maximizing the terms of equation (4.1), we find that $a_0(b^0 - a_0)$ is maximal when $a_0 = 0$; $a_1(b^1 - a_1)$ is maximal when $a_1 = \lfloor b/2 \rfloor$; and, for $1 < i \leq n$, $a_i(b^i - a_i)$ is maximal when $a_i = b-1$. Hence, the maximal value of c is determined by

$$a = \sum_{i=0}^{n} a''_i b^i,$$

where

$$a''_i = \begin{cases} b-1 & \text{for } 2 \leq i \leq n, \\ \lfloor b/2 \rfloor & \text{for } \ i = 1, \\ 0 & \text{for } \ i = 0, \end{cases}$$

and, therefore, the maximal value of c is

$$\begin{aligned} c &= \lfloor b/2 \rfloor (b^1 - \lfloor b/2 \rfloor) + \sum_{i=2}^{n} (b^i - (b-1))(b-1) \\ &= b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor)\lfloor b/2 \rfloor \\ &= M_{b,n}. \end{aligned}$$

□

The next lemma is used to prove Theorem 4.4, which states that, for each $b \geq 2$, there exist arbitrarily long deserts base b.

Lemma 4.3. *Let $b \geq 2$ and $n \geq 2$. Then, between the numbers $M_{b,n}$ and $m_{b,n+1}$, there exists a k-desert base b, where*

$$k = m_{b,n+1} - M_{b,n} - 1 > (n - 5/4)(b-1)^2.$$

Proof. Let $b \geq 2$ and $n \geq 2$ be fixed. Note that

$$\begin{aligned} m_{b,n+1} - M_{b,n} - 1 &= (b^{n+1} - b^2 + 3b - 3) \\ &\quad - (b^{n+1} - b^2 - (n-1)(b-1)^2 + (b - \lfloor b/2 \rfloor)\lfloor b/2 \rfloor) - 1 \\ &\geq 3b - 3 + (n-1)(b-1)^2 - b^2/4 - 1 \\ &= (n-1)(b-1)^2 - b^2/4 + b/2 - 1/4 + 5b/2 - 15/4 \\ &> (n-1)(b-1)^2 - (b-1)^2/4 \\ &= (n - 5/4)(b-1)^2, \end{aligned}$$

since $b \geq 2$. Thus,

$$m_{b,n+1} > M_{b,n} + 1. \tag{4.2}$$

Recall that $M_{b,n}$ is an upper bound on values of c such that $S_{[c,b]}$ has an $(n+1)$-digit fixed point. Since $M_{b,x}$ increases as x increases, $M_{b,n}$ is an upper bound on values of c such that $S_{[c,b]}$ has a fixed point with less than or equal to $(n+1)$-digits. Similarly, $m_{b,n+1}$ is a lower bound on values of c such that $S_{[c,b]}$ has a fixed point with greater than or equal to $(n+2)$-digits. Thus, by equation (4.2), there is no value of c between $M_{b,n}$ and $m_{b,n+1}$ such that $S_{[c,b]}$ has a fixed point of any size. Hence, there exists a k-desert between these two numbers, where $k = m_{b,n+1} - M_{b,n} - 1$. □

Theorem 4.4. *For each $b \geq 2$ and $k \in \mathbb{Z}^+$, there exists a k-desert base b.*

Proof. Fix $b \geq 2$ and $k \in \mathbb{Z}^+$. Since $(n - 5/4)(b-1)^2$ is an increasing linear function of n, there exists some $n \geq 2$ such that $(n - 5/4)(b-1)^2 \geq k$. It follows from Lemma 4.3 that there exists a k-desert base b. □

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