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P-SPACES AND INTERMEDIATE RINGS OF CONTINUOUS FUNCTIONS

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ABSTRACT. A completely regular topological space X is called a P-space if every zero-set in X is open. An intermediate ring is a ring $A(X)$ of real-valued continuous functions on X containing all the bounded continuous functions. In this paper, we find new characterizations of P-spaces X in terms of properties of correspondences between ideals in $A(X)$ and z-filters on X. We also show that some characterizations of P-spaces that are described in terms of properties of $C(X)$ actually characterize $C(X)$ among intermediate rings on X.

1. Introduction. Throughout this paper, we let X denote a completely regular (Hausdorff) topological space, also known as a Tychonoff space. We say X is a *P-space* (pseudo-discrete space) if every zero-set in X is open. Such spaces were introduced by Gillman and Henriksen **[8]**, who used a different but equivalent definition. Their definition is based on an observation by Kaplansky **[11]** that the ring $C(X)$ of continuous functions on a discrete space X has a certain algebraic property. Further characterizations are given by Gillman and Jerison **[9]**. An intermediate ring of continuous functions $A(X)$ is a subring of $C(X)$ that contains $C^*(X)$ (the ring of bounded functions in $C(X)$). Intermediate rings have been extensively studied, for example, in **[2, 3, 5, 6, 7, 12, 13, 14]**. This paper examines relationships between P-spaces and intermediate rings of continuous functions.

For an intermediate ring $A(X)$ there are two natural correspondences, $\mathcal{Z}_A$ and $\mathfrak{Z}_A$, between the ideals of $A(X)$ and the z-filters on X (see **[12, 14]**). These correspondences extend to all intermediate rings the well-known correspondences, described in **[9,** subsections 2.3,

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2L], for $C^*(X)$ and $C(X)$, respectively. We give a new condition that determines whether X is a P-space in terms of the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$, namely, X is a P-space if and only if $\mathcal{Z}_A$ and $\mathfrak{Z}_A$ coincide for each intermediate ring $A(X)$ (Theorem 2.3). Other new characterizations are given: in terms of the ideals M_A^p and O_A^p for $p \in X$ (Theorems 2.5 and 2.8), by the property that $\mathcal{Z}_A$ maps maximal ideals to z-ultrafilters (Theorem 2.10), and by the property that every z-filter is a $\mathcal{Z}_A$-filter (Theorem 2.12). We note that the analogous characterization of P-spaces in terms of $\mathfrak{Z}_A$-filters does not hold (Example 2.13).

There are a number of alternative characterizations of P-spaces which are given in terms of algebraic properties of $C(X)$. For example, X is a P-space if and only if the ring $C(X)$ is (von Neumann) regular, equivalently, every prime ideal in $C(X)$ is maximal [**9,** Section 4J]. We show that some properties which characterize P-spaces X in terms of $C(X)$ actually characterize $C(X)$ among intermediate rings $A(X)$ when X is a given P-space. For example, the property that $A(X)$ is a regular ring characterizes $C(X)$ among intermediate rings $A(X)$ on a given P-space X (Theorem 3.3). Other characterizations of $C(X)$ when X is a P-space are given: by the property that every z-ideal is a $\mathcal{Z}_A$-ideal ($\mathfrak{Z}_A$-ideal) (Theorem 3.7), and by the property that $M_A^p = O_A^p$ for every $p \in \beta X$ (Theorem 3.10).

Although the property that every z-filter is a $\mathcal{Z}_A$-filter characterizes P-spaces, we show that this property does not in general characterize $C(X)$ among intermediate rings when X is a P-space (Example 3.8). Symmetrically, although the property that every ideal in $A(X)$ is a $\mathcal{Z}_A$-ideal ($\mathfrak{Z}_A$-ideal) characterizes $C(X)$ among intermediate rings when X is a P-space, we show that this property does not, for every intermediate ring $A(X)$, characterize P-spaces (Example 2.15). In the particular instance of $A(X) = C(X)$, we do know that the property that every ideal in $C(X)$ is a $\mathfrak{Z}_C$-ideal characterizes P-spaces (see [**9,** 4J] and [**12,** Corollary 2.4]). Furthermore, although our Theorem 2.5 tells us that the property that $M_A^p = O_A^p$ for every $p \in X$ characterizes P-spaces, we show that this property does not characterize $C(X)$ among intermediate rings when X is a P-space (Example 3.11), and, although the property $M_A^p = O_A^p$ for every $p \in \beta X$ characterizes $C(X)$ among intermediate rings when X is a P-space, we show that this property does not characterize P-spaces (Example 3.9). In the particular instance of $A(X) = C(X)$, we do know that the property

		Property		
To Characterize	F	I	X	B
P-spaces	yes	no	yes	no
$C(X)$ among $A(X)$ for X a P-space	no	yes	no	yes

that $M_C^p = O_C^p$ for every $p \in \beta X$ does characterize P-spaces [**9,** 7L]. In order to summarize, we provide the above chart, where we abbreviate by F the property that every z-filter is a $\mathcal{Z}_A$-filter, I the property that every ideal is a $\mathcal{Z}_A$-ideal, X the property that $M_A^p = O_A^p$ for each $p \in X$ and B the property that $M_A^p = O_A^p$ for each $p \in \beta X$. We mark by "no" the boxes where there is an appropriate space X and rings $A(X)$ in which the property corresponding to the column does not characterize the property corresponding to the row.

2. Characterizations of P-spaces. For any real-valued continuous function f on X, we define the *zero-set* of f to be

$$\boldsymbol{Z}(f) \stackrel{\text{def}}{=} \{x \in X \mid f(x) = 0\},$$

and

$$\boldsymbol{Z}[X] \stackrel{\text{def}}{=} \{\boldsymbol{Z}(f) \mid f \in C(X)\}$$

to be the set of all zero-sets. The complement of a zero-set is called a *cozero-set*. In this article, we use the following topological definition of a P-space.

Definition 2.1. A completely regular space X is a P*-space* if every zero-set in X is open.

An equivalent topological formulation of this definition is: X is a P-space if every cozero-set in X is C-embedded [**9,** Section 4J]. There are numerous characterizations of P-spaces in terms of properties of the ring of all real-valued continuous functions on the space. For example a P-space is defined in [**9**] to be a space X such that every prime ideal in $C(X)$ is maximal. We know of no previously given characterizations of P-spaces which are expressed in terms of intermediate rings $A(X)$. In this section, we introduce several new characterizations of P-spaces, all of which can be expressed in terms of intermediate rings $A(X)$.

2.1. The correspondences $\mathfrak{Z}_A$ and $\mathcal{Z}_A$. We give a characterization of P-spaces in terms of the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$.

Let $A(X)$ be an intermediate ring of continuous functions. If $f \in A(X)$ and E is a subset of X, we say that f is *E-regular* with respect to $A(X)$ if there exists $g \in A(X)$ such that $fg \equiv 1$ on E. We use the correspondences $\mathcal{Z}_A$ and $\mathfrak{Z}_A$, introduced in [**14, 12**] respectively, between ideals of $A(X)$ and z-filters on X, that are defined as follows. For $f \in A(X)$, we have

$$\mathcal{Z}_A(f) \stackrel{\text{def}}{=} \{E \in \boldsymbol{Z}[X] \mid f \text{ is } E^c\text{-regular}\},$$

$$\mathfrak{Z}_A(f) \stackrel{\text{def}}{=} \{E \in \boldsymbol{Z}[X] \mid f \text{ is } H\text{-regular for every zero-set } H \subseteq E^c\}.$$

For each ideal $I \subset A(X)$, it is known that

$$\mathcal{Z}_A[I] \stackrel{\text{def}}{=} \bigcup\{\mathcal{Z}_A(f) \mid f \in I\}$$

and

$$\mathfrak{Z}_A[I] \stackrel{\text{def}}{=} \bigcup\{\mathfrak{Z}_A(f) \mid f \in I\}$$

are z-filters on X ([**12,** Proposition 2.2] and [**14,** Theorem 1]). These correspondences extend the well-known correspondences $\boldsymbol{E}$ and $\boldsymbol{Z}$ for $C^*(X)$ and $C(X)$, respectively, which are discussed in [**9,** subsections 2.3, 2L], to any intermediate ring $A(X)$ ([**12,** Corollaries 1.3, 2.4]).

We begin with the following lemma, which clarifies the fourth and fifth lines of the proof of [**12,** Theorem 2.3].

Lemma 2.2. *Let $f \in C(X)$ be non-invertible, and let $E = \boldsymbol{Z}(f)$. Let $F \in \boldsymbol{Z}[X]$, such that $E \cap F = \emptyset$. Then, f is F-regular.*

Proof. From [**9,** subsection 1.15], disjoint zero-sets are completely separated. Let $h : X \to [0,1]$ be a separating function that is 0 on F and 1 on E. Let $k = f^2 + h$. Then, $\boldsymbol{Z}(k) = \emptyset$, and hence, k is invertible. Since $h(x) = 0$ for all $x \in F$, $k(x) = f^2(x)$ for all $x \in F$. Let $g = k^{-1} \cdot f$. Then, $f(x) \cdot g(x) = 1$ for all $x \in F$. □

Theorem 2.3. *A completely regular space X is a P-space if and only if for every intermediate ring $A(X)$ we have*

$$\mathfrak{Z}_A(f) = \mathcal{Z}_A(f)$$

for every non-invertible $f \in A(X)$.

Proof. We first observe that, if X is a P-space, then every zero-set is both open and closed. Thus, if E is a zero-set in X, then the characteristic function on E^c is continuous.

$\Rightarrow$. Let X be a P-space, and let $A(X)$ be an intermediate ring on X. Suppose $f \in A(X)$ and $E \in \mathfrak{Z}_A(f)$. Then, f is invertible on every zero-set $H \subseteq E^c$. However, since E^c itself is a zero-set, it follows that f is invertible on E^c. This precisely means that $E \in \mathcal{Z}_A(f)$, which shows that $\mathfrak{Z}_A(f) \subseteq \mathcal{Z}_A(f)$. Since the other containment always holds, it follows that $\mathfrak{Z}_A(f) = \mathcal{Z}_A(f)$.

$\Leftarrow$. Suppose that, for every intermediate ring $A(X)$ and for every non-invertible $f \in A(X)$, we have $\mathfrak{Z}_A(f) = \mathcal{Z}_A(f)$. In particular, for $C(X)$ and for every $f \in C(X)$, we have $\mathfrak{Z}_C(f) = \mathcal{Z}_C(f)$. Now, suppose that E is a zero-set in X, and let $f \in C(X)$ with $E = \boldsymbol{Z}(f)$. From Lemma 2.2, f is invertible in $C(X)$ on every zero-set H contained in E^c, and thus, $E \in \mathfrak{Z}_C(f)$. It follows (by our hypothesis) that $E \in \mathcal{Z}_C(f)$, which means that f is invertible on E^c. Therefore, there exists a $g \in C(X)$ such that $fg = 1$ on E^c, and of course, $fg = 0$ on $E = \boldsymbol{Z}(f)$. Since fg is continuous on X, it follows that E is an open set in X. This shows that every zero-set in X is open, and thus, X is a P-space. $\square$

Corollary 2.4. *A completely regular space X is a P-space if and only if, for every intermediate ring $A(X)$ and every ideal I in $A(X)$, we have $\mathfrak{Z}_A[I] = \mathcal{Z}_A[I]$.*

Proof.

$\Rightarrow$. From Theorem 2.3, $\mathcal{Z}_A(f) = \mathfrak{Z}_A(f)$ for every $f \in I$, hence $\mathcal{Z}_A[I] = \bigcup_{f\in I} \mathcal{Z}_A(f) = \bigcup_{f\in I} \mathfrak{Z}_A(f) = \mathfrak{Z}_A[I]$.

$\Leftarrow$. Suppose that $\mathfrak{Z}_A[I] = \mathcal{Z}_A[I]$ for every intermediate ring $A(X)$ and every ideal I in $A(X)$. Consider the principal ideals $I_f = \langle f \rangle$, for each non-invertible $f \in A(X)$. For any non-invertible $f \in A(X)$ and for any $g \in A(X)$, we have $\mathcal{Z}_A(fg) \subseteq \mathcal{Z}_A(f)$ (this follows from [**12,** Lemma 1.5 (a)], which states that $\mathcal{Z}_A(fg) = \mathcal{Z}_A(f) \wedge \mathcal{Z}_A(g)$) and

$\mathfrak{Z}_A(fg) \subseteq \mathfrak{Z}_A(f)$ (this similarly follows from [**16,** Corollary 13 (a)], which states that $\mathfrak{Z}_A(fg) = \mathfrak{Z}_A(f) \wedge \mathfrak{Z}_A(g)$). It follows that

$$\mathcal{Z}_A[I_f] = \mathcal{Z}_A(f)$$

and

$$\mathfrak{Z}_A[I_f] = \mathfrak{Z}_A(f).$$

Thus, by hypothesis, $\mathfrak{Z}_A(f) = \mathcal{Z}_A(f)$ for every non-invertible $f \in A(X)$. Then, by Theorem 2.3, X is a P-space. □

From [**12,** Theorem 3.1], we know that $\mathfrak{Z}_A(f) = kh\mathcal{Z}_A(f)$ for each non-invertible $f \in A(X)$, where, for any z-filter $\mathcal{F}$, the *hull* $h\mathcal{F}$ of $\mathcal{F}$ is the set of all z-ultrafilters containing $\mathcal{F}$, and, for every set $\mathfrak{U}$ of z-ultrafilters, the *kernel* $k\mathfrak{U}$ of $\mathfrak{U}$ is the intersection of all z-ultrafilters in $\mathfrak{U}$. Thus, Theorem 2.3 is equivalent to saying that X is a P space if and only if, for every intermediate ring $A(X)$ and non-invertible function $f \in A(X)$,

$$\mathcal{Z}_A(f) = kh\mathcal{Z}_A(f).$$

From Theorem 2.3, we know that, for any P-space X and any intermediate ring $A(X)$, $\mathcal{Z}_A = \mathfrak{Z}_A$. Conversely, we do not know that X is a P-space, given that $\mathcal{Z}_A = \mathfrak{Z}_A$ for some arbitrary $A(X)$. However, the proof of Theorem 2.3 shows that, if $\mathcal{Z}_C = \mathfrak{Z}_C$, then X must be a P-space.

2.2. The ideals M_A^p and O_A^p for $p \in X$. We consider, for each $p \in X$ and intermediate ring $A(X)$, the fixed maximal ideal M_A^p of functions that vanish at p, and the ideal O_A^p of functions that vanish on a neighborhood of p. (A *fixed ideal* is an ideal I for which $\bigcap\{\boldsymbol{Z}(f) \mid f \in I\} \neq \emptyset$.) In notation, for each $p \in X$, let

$$M_A^p \stackrel{\text{def}}{=} \{f \in A(X) : p \in \boldsymbol{Z}(f)\}$$
$$O_A^p \stackrel{\text{def}}{=} \{f \in A(X) : p \in \operatorname{int} \boldsymbol{Z}(f)\}.$$

In Section 3.3, we examine extensions of these to $p \in \beta X$. In the case where $A(X) = C(X)$ it is known that X is a P-space if and only if $M_A^p = O_A^p$ for all $p \in X$ [**9,** Section 4J]. We extend this result to all intermediate rings.

Theorem 2.5. *Let $A(X)$ be an intermediate ring. Then, X is a P-space if and only if $M_A^p = O_A^p$ for every $p \in X$.*

Proof.

$\Rightarrow$. Suppose that X is a P-space, and let $f \in M_A^p$, $p \in X$. So $f(p) = 0$. However, since X is a P-space, $\boldsymbol{Z}(f)$ is an open set containing p. Thus, $f \in O_A^p$. Therefore, $M_A^p \subseteq O_A^p$. Since the other containment is always true, it follows that $M_A^p = O_A^p$ for all $p \in X$.

$\Leftarrow$. Suppose that $M_A^p = O_A^p$ for all $p \in X$. Let E be a zero-set in X. Since E is a zero-set, there is an $f \in C(X)$ with $\boldsymbol{Z}(f) = E$; we may assume (by replacing f with $(f \wedge 1) \vee -1$, if necessary) that $f \in C^*(X) \subseteq A(X)$. Now, for every $p \in E$, we have $f \in M_A^p = O_A^p$, so E is a neighborhood of each of its points. Thus, E is open. Therefore, X is a P-space. $\square$

We will show that $\mathfrak{Z}_A$ preserves this characterization, that is, X is a P-space if and only if $\mathfrak{Z}_A(M_A^p) = \mathfrak{Z}_A(O_A^p)$. However, first we provide for $p \in X$ a lemma and general results regarding the images of M_A^p and O_A^p under the correspondences $\mathfrak{Z}_A$ and $\mathcal{Z}_A$.

Lemma 2.6. *If $p \in X$ and E is a zero-set neighborhood of p, then there exists a continuous function $h : X \to [0,1]$ such that $h = 1$ on E^c and $h = 0$ on some zero-set neighborhood of p.*

Proof. Let $H \overset{\text{def}}{=} cl_X E^c$. Since $p \notin H$, it follows by complete regularity that there is a function

$$f : X \longrightarrow [0,1], \quad f(p) = 0, \ f = 1 \text{ on } H.$$

The sets

$$F_1 = \{x \in X : f(x) \leq \tfrac{1}{2}\}$$

and

$$F_2 = \{x \in X : f(x) = 1\}$$

are disjoint zero-sets in X; thus, they are completely separated, that is, there exists an

$$h : X \longrightarrow [0,1]$$

such that $h = 0$ on F_1 and $h = 1$ on F_2. Clearly $E^c \subseteq F_2$, and F_1 is a zero-set neighborhood of p. □

The first part of the next lemma is the special case where $p \in X$ of [**5,** Theorem 4.1]; however, we give here a shorter and more direct proof of this case.

Proposition 2.7. *Let $A(X)$ be an intermediate ring of continuous functions. Then the following both hold for every $p \in X$:*

(a) $\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$.
(b) $\mathcal{Z}_A[O_A^p] = \mathfrak{Z}_A[O_A^p]$.

Proof.

(a) Since $O_A^p \subseteq M_A^p$, it is clear that

$$\mathcal{Z}_A[O_A^p] \subseteq \mathcal{Z}_A[M_A^p].$$

For the other containment, suppose that $E \in \mathcal{Z}_A[M_A^p]$. Then, there exists an $f \in M_A^p$ such that $E \in \mathcal{Z}_A(f)$. It follows that there is a $g \in A(X)$ such that $fg = 1$ on E^c. Now, the set

$$F = \{x \in X : |fg(x)| \leq \tfrac{1}{2}\}$$

is a zero-set neighborhood of p. Let

$$H = \{x \in X : |fg(x)| \geq 1\}.$$

Since F and H are disjoint zero-sets, they are completely separated [**9,** subsection 1.15]; thus, there is a function $h : X \to [0,1]$ such that $h = 0$ on F and $h = 1$ on H. Clearly, $h \in O_A^p$ and $E \in \mathcal{Z}_A(h)$; thus, $E \in \mathcal{Z}_A[O_A^p]$.

(b) For each $f \in A(X)$, we have

$$\mathcal{Z}_A(f) \subseteq \mathfrak{Z}_A(f);$$

thus,

$$\mathcal{Z}_A[O_A^p] \subseteq \mathfrak{Z}_A[O_A^p].$$

For the other containment, let $p \in X$, and suppose that $E \in \mathfrak{Z}_A[O_A^p]$. Then, $E \in \mathfrak{Z}_A(f)$ for some $f \in O_A^p$. Thus, $\boldsymbol{Z}(f)$ is a zero-set neighborhood of p, and, since E contains $\boldsymbol{Z}(f)$ by [**18,** Lemma 3.1] (which asserts that $\boldsymbol{Z}(f) = \bigcap\{E \mid E \in \mathfrak{Z}_A(f)\}$), it follows that E is a zero-set neighborhood of p. From Lemma 2.6, there exists an

$$h : X \longrightarrow [0,1]$$

such that $h = 0$ on some zero-set neighborhood of p and $h = 1$ on E^c. Since $h = 0$ on a zero-set neighborhood of p, and since h is bounded, it follows that $h \in O_A^p$. Further, since $h = 1$ on E^c, it is clear that h is E^c-regular. By definition, this means that $E \in \mathcal{Z}_A(h)$. Therefore, $E \in \mathcal{Z}_A[O_A^p]$. □

Theorem 2.8. *A completely regular space X is a P-space if and only if, for every intermediate ring $A(X)$ and every $p \in X$, we have $\mathfrak{Z}_A[M_A^p] = \mathfrak{Z}_A[O_A^p]$.*

Proof.

$\Rightarrow$. If X is a P-space, then, by Theorem 2.5, for each $p \in X$, $M_A^p = O_A^p$, and hence, $\mathfrak{Z}_A[M_A^p] = \mathfrak{Z}_A[O_A^p]$.

$\Leftarrow$. Let $A(X)$ be an intermediate ring. We claim that every E in $\mathfrak{Z}_A[M_A^p]$ is also a neighborhood of p. We first show that every E in $\mathfrak{Z}_A[O_A^p]$ is a neighborhood of p. Toward this end, let $E \in \mathfrak{Z}_A[O_A^p]$. Then, $E \in \mathfrak{Z}_A(f)$ for some $f \in O_A^p$. We always have $\boldsymbol{Z}(f) \subseteq E$. However, $f \in O_A^p$; thus, $\boldsymbol{Z}(f)$ is a neighborhood of p. Therefore, E is a neighborhood of p. It follows, by our hypothesis, that

$$\mathfrak{Z}_A[M_A^p] = \mathfrak{Z}_A[O_A^p],$$

and that every E in $\mathfrak{Z}_A[M_A^p]$ is also a neighborhood of p. This completes the proof of the claim. In particular, the claim holds for $A(X) = C(X)$.

Now, suppose that $g \in M_C^p$. Thus, $g(p) = 0$. From Lemma 2.2, g is invertible in $C(X)$ on every zero-set in the complement of $\boldsymbol{Z}(g)$; thus, it follows that $\boldsymbol{Z}(g) \in \mathfrak{Z}_A(g)$. Therefore, $\boldsymbol{Z}(g) \in \mathfrak{Z}_A[M_C^p]$, and thus, by the claim, $\boldsymbol{Z}(g)$ is a neighborhood of p. It follows that every zero-set in X is a neighborhood of each of its points. Therefore, every zero-set in X is open, and thus, X is a P-space. □

2.3. Mapping maximal ideals to z-ultrafilters. The next theorem characterizes P-spaces as those spaces X where, for any intermediate ring $A(X)$, the image under $\mathcal{Z}_A$ of a maximal ideal in $A(X)$ is a z-ultrafilter on X.

Lemma 2.9. *If X is a P-space and E a zero-set in X, there exists a function $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and $\mathcal{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.*

Proof. Let E be a zero-set in X, and let f be the characteristic function of E^c. By definition, $E \in \mathcal{Z}_A(f)$, and hence, $\mathcal{Z}_A(f) \supseteq \langle \boldsymbol{Z}(f) \rangle$. From [**15,** Proposition 2.2], which asserts that

$$\boldsymbol{Z}(f) = \bigcap \{E \mid E \in \mathcal{Z}_A(f)\},$$

we have that $\mathcal{Z}_A(f) \subseteq \langle \boldsymbol{Z}(f) \rangle$. $\square$

The proof of the next theorem uses the following definition. For any intermediate ring $A(X)$ and z-filter $\mathcal{F}$, let

$$\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] \stackrel{\text{def}}{=} \{f \in A(X) \mid \mathcal{Z}_A(f) \subseteq \mathcal{F}\}.$$

We define $\mathfrak{Z}_A^{\leftarrow}$ similarly. According to [**18,** Theorem 5.2], if X is a P-space and $A(X)$ is a C-ring (a ring $A(X)$ that is isomorphic to $C(Y)$ for some completely regular Y), then $\mathcal{Z}_A$ maps each maximal ideal in $A(X)$ to a z-ultrafilter on X. The next theorem strengthens this result not to depend upon $A(X)$ being a C-ring and to give a full characterization of X being a P-space. It also addresses [**18,** Problem 5.3].

Theorem 2.10. *Let $A(X)$ be an intermediate ring. Then, X is a P-space if and only if $\mathcal{Z}_A[M]$ is a z-ultrafilter whenever M is a maximal ideal in $A(X)$.*

Proof.

$\Rightarrow$. Let X be a P-space. From [**5,** Theorem 3.2(a)], there is a unique z-ultrafilter $\mathcal{U}$ such that $\mathcal{Z}_A[M] \subseteq \mathcal{U}$. Now, let $E \in \mathcal{U}$. From Lemma 2.9, there exists an $f \in A(X)$ such that $\mathcal{Z}_A(f) = \langle E \rangle \subseteq \mathcal{U}$. It is easy to see that

$$M \subseteq \mathcal{Z}_A^{\leftarrow}[\mathcal{Z}_A[M]] \subseteq \mathcal{Z}_A^{\leftarrow}[\mathcal{U}].$$

Since M is maximal, and from [**15,** Theorem 2.3], $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a proper ideal $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$. It follows that $f \in M$; thus, $E \in \mathcal{Z}_A[M]$. Therefore, $\mathcal{Z}_A[M] = \mathcal{U}$.

$\Leftarrow$. Suppose that $\mathcal{Z}_A[M]$ is a z-ultrafilter whenever M is a maximal ideal in $A(X)$. Let $p \in X$, and consider the maximal ideal M_A^p. By hypothesis, $\mathcal{Z}_A[M_A^p]$ is a z-ultrafilter; therefore, it must be that

$\mathcal{Z}_A[M_A^p] = \mathcal{U}_p$, where $\mathcal{U}_p$ is the z-ultrafilter consisting of all zero-sets containing p. From Proposition 2.7, it follows that $\mathcal{Z}_A[O_A^p] = \mathcal{U}_p$. However, $\mathcal{Z}_A[O_A^p]$ consists of all zero-set neighborhoods of p, except that, since $\mathcal{Z}_A[O_A^p] = \mathcal{U}_p$, it follows that $\mathcal{U}_p$ consists of zero-set neighborhoods of p. Thus, every zero-set containing p is a neighborhood of p. Therefore, X is a P-space. $\square$

Theorem 2.10 no longer holds if $\mathcal{Z}_A$ is replaced by $\mathfrak{Z}_A$. For example, by [**9,** subsection 2.5] and [**12,** Theorem 2.3], for any completely regular space X, $\mathfrak{Z}_C(M) = \boldsymbol{Z}(M)$ is a z-ultrafilter for any maximal ideal M of $C(X)$.

2.4. $\mathcal{Z}_A$- and $\mathfrak{Z}_A$-filters; $\mathcal{Z}_A$- and $\mathfrak{Z}_A$-ideals. By a $\mathcal{Z}_A$-*filter*, we mean a z-filter $\mathcal{F}$ with the property that $\mathcal{Z}_A\mathcal{Z}_A^{\leftarrow}[\mathcal{F}] = \mathcal{F}$. Similarly, $\mathcal{F}$ is a $\mathfrak{Z}_A$-*filter* if $\mathfrak{Z}_A\mathfrak{Z}_A^{\leftarrow}[\mathcal{F}] = \mathcal{F}$. The next proposition follows from the proof of $(a) \Leftrightarrow (b)$ of [**18,** Theorem 4.2] (although [**18,** Theorem 4.2] is stated for $A(X)$ a C-ring, the part $(a) \Leftrightarrow (b)$ does not require that $A(X)$ be a C-ring).

Proposition 2.11. *The following are equivalent for any intermediate ring $A(X)$:*

(a) *Every z-filter on X is a $\mathfrak{Z}_A$-filter.*
(b) *For every zero-set E in X, there exists an $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$.*

Note that, if $A(X) = C(X)$, then every z-filter is a $\mathfrak{Z}_A$-filter since, in this case, $\mathfrak{Z}_A = \boldsymbol{Z}$, and it is known that $\boldsymbol{Z}\boldsymbol{Z}^{\leftarrow}[\mathcal{F}] = \mathcal{F}$ for every z-filter $\mathcal{F}$ ([**9,** subsection 2.5]). In general, for intermediate rings, we have the following result.

Theorem 2.12. *Let $A(X)$ be an intermediate ring. Then, X is a P-space if and only if every z-filter on X is a $\mathcal{Z}_A$-filter.*

Proof.

$\Rightarrow$. Suppose that X is a P-space. From Lemma 2.9 and Theorem 2.3, for every zero-set E, there exists a function $f \in A(X)$ such that $E = \boldsymbol{Z}(f)$ and $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. Then, by Proposition 2.11, every

z-filter is a $\mathfrak{Z}_A$-filter. From Theorem 2.3, $\mathfrak{Z}_A = \mathcal{Z}_A$, and hence, every z-filter is also a $\mathcal{Z}_A$-filter.

$\Leftarrow$. Suppose that $A(X)$ is such that every z-filter on X is a $\mathcal{Z}_A$-filter. Let M be a maximal ideal, and let $\mathcal{U}$ be the unique z-ultrafilter containing $\mathcal{Z}_A[M]$, see [**5,** Theorem 3.2(a)]. From [**12,** Theorem 4.4], $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ is a maximal ideal. It is easy to see that $M = \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ (it is always the case that $M \subseteq \mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[M]$). Since $\mathcal{U}$ is a $\mathcal{Z}_A$-filter, we then have that

$$\mathcal{Z}_A[M] = \mathcal{Z}_A\mathcal{Z}_A^{\leftarrow}[\mathcal{U}] = \mathcal{U},$$

that is, $\mathcal{Z}_A$ maps maximal ideals to z-ultrafilters. Hence, it follows by Theorem 2.10 that X is a P-space. $\square$

The right-to-left direction of this theorem would not be true if we were to replace $\mathcal{Z}_A$ by $\mathfrak{Z}_A$. For $A(X) = C(X)$, every z-filter is a $\mathfrak{Z}_A$-filter, even if X is not a P-space. And, if $A(X) \neq C(X)$, the right-to-left direction does not hold for $\mathfrak{Z}_A$-filters, as the next example shows.

Example 2.13. Let $X = (0,1) \cup \{2,3,4,\ldots\}$, and note that a zero-set E in X is of the form $E = E_1 \cup E_2$ where E_1 is a zero-set in $(0,1)$ and E_2 is any subset of $\{2,3,4,\ldots\}$. Let $A(X)$ be the ring of all continuous functions on X that are bounded on $\{2,3,4,\ldots\}$. Then, for every zero-set $E = E_1 \cup E_2$, define a function $f : X \to \mathbb{R}$ as follows:

$$f(x) = \begin{cases} g(x) & \text{if } 0 < x < 1 \\ \chi_F(x) & \text{if } x \in \{2,3,4,\ldots\}, \end{cases}$$

where g is any continuous function on $(0,1)$ where $\boldsymbol{Z}(g) = E_1$ and χ_F is the characteristic function on $F = (E_2)^c$. Clearly, $f \in A(X)$. Moreover, $\boldsymbol{Z}(f) = E$ and $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. Then, from Proposition 2.11, every z-filter on X is a $\mathfrak{Z}_A$-filter. However, X is not a P-space.

An ideal I is a $\mathcal{Z}_A$-*ideal* if $\mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[I] = I$; equivalently, I is a $\mathcal{Z}_A$-ideal if $f \in I$ whenever $\mathcal{Z}_A(f) \subseteq \mathcal{Z}_A(I)$. We analogously define a $\mathfrak{Z}_A$-ideal.

Theorem 2.14. *Let $A(X)$ be an intermediate ring such that every ideal in $A(X)$ is a $\mathcal{Z}_A$-ideal* ($\mathfrak{Z}_A$*-ideal*)*. Then, X is a P-space.*

Proof. Suppose that every ideal is a $\mathcal{Z}_A$-ideal. Let $p \in X$. From Proposition 2.7, we have $\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$. Hence,

$$\mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p].$$

By hypothesis, O_A^p and M_A^p are $\mathcal{Z}_A$-ideals, which yields the first and third equalities of:

$$O_A^p = \mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[O_A^p] = \mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p] = M_A^p.$$

Thus, X is a P-space by Theorem 2.5.

Now, suppose that every ideal is a $\mathfrak{Z}_A$-ideal. Again, let $p \in X$, and consider the ideal O_A^p. By hypothesis, O_A^p is a $\mathfrak{Z}_A$-ideal. Thus,

$$\mathfrak{Z}_A^{\leftarrow}\mathfrak{Z}_A[O_A^p] = O_A^p. \tag{2.1}$$

From Proposition 2.7, we have $\mathfrak{Z}_A[O_A^p] = \mathcal{Z}_A[M_A^p]$; thus, we can write (2.1) as

$$\mathfrak{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p] = O_A^p. \tag{2.2}$$

However, $\mathfrak{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p] = M_A^p$ also since, by [**5,** Theorem 3.2(a)], $\mathcal{Z}_A[M_A^p]$ is the unique z-ultrafilter containing M_A^p, and thus, by [**12,** Proposition 4.4], $\mathfrak{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p]$ is a maximal ideal which must contain O_A^p (by (2.2)). Therefore, that maximal ideal must be M_A^p, that is,

$$\mathfrak{Z}_A^{\leftarrow}\mathcal{Z}_A[M_A^p] = M_A^p. \tag{2.3}$$

From (2.2) and (2.3), it follows that $O_A^p = M_A^p$ for every $p \in X$. Thus, X is a P-space by Theorem 2.5. $\square$

The converse of Theorem 2.14 is not true in general; the next example shows why.

Example 2.15. Let $X = \mathbb{N}$ be the set of positive integers, and let $A(X) = C^*(X)$. Note that X is discrete, and hence, a P-space. Let $I = \langle 1/n \rangle$ be the ideal generated by $f(n) = 1/n$. Note that $1/\sqrt{n} \notin I$, for otherwise, there would be a function g such that $gf = g/n = 1/\sqrt{n}$. However, then $g = \sqrt{n}$, is unbounded, and hence, not in $C^*(X)$. It is easy to see from the definition that $\mathfrak{Z}_A(f) = \mathfrak{Z}_A(f^2)$ for any $f \in A(X)$, and hence, we have that $\mathfrak{Z}_A(1/n) = \mathfrak{Z}_A(1/\sqrt{n})$. Thus, $\mathfrak{Z}_A(1/\sqrt{n}) \in \mathfrak{Z}_A(I)$. We conclude that I is not a $\mathfrak{Z}_A$-ideal. The same argument applies if $\mathfrak{Z}_A$ is replaced by $\mathcal{Z}_A$ (also recall by Corollary 2.4 that $\mathfrak{Z}_A(I) = \mathcal{Z}_A(I)$).

3. Characterizing $C(X)$ among intermediate rings on P-spaces. Several characterizations of $C(X)$ among its subrings are known (see [**4, 17, 18**]). In this section, we show that several of the characterizations of P-spaces in terms of the ring structure of $C(X)$ actually characterize $C(X)$ among intermediate rings on the P-space X.

3.1. Algebraic characterizations. A commutative ring R is (*von-Neumann*) *regular* if, for every $x \in R$, there exists a $y \in R$ such that $x = x^2y$. We first recall that it is well known that X is a P-space if and only if $C(X)$ is a regular ring [**9,** subsection 4J]. We show that any proper intermediate ring is never a regular ring.

The next lemma is immediate from [**19,** pages 293, 294, Problem 44C]; however, we give short proof of it here.

Lemma 3.1. *If $A(X) \neq C(X)$, then there exists an $f \in A(X)$ such that f is never zero and f is not invertible in $A(X)$.*

Proof. Let $g \in C(X) \setminus A(X)$. It can be assumed that $g \geq 0$, for, if not, g must be replaced by one of $g_1 \stackrel{\text{def}}{=} g \vee 0$ or $g_2 \stackrel{\text{def}}{=} -g \vee 0$. (Both g_1 and g_2 cannot be in $A(X)$, since then $g = g_1 - g_2$ would be in $A(X)$.) Now, $g+1 \notin A(X)$; thus, let $f = 1/(g+1)$. Then, $f \in C^*(X) \subseteq A(X)$, f never vanishes and f is not invertible in $A(X)$. $\square$

Proposition 3.2. *If $A(X) \neq C(X)$, then $A(X)$ is not a regular ring.*

Proof. Suppose that $A(X)$ is a regular ring. From Lemma 3.1, there exists an $f \in A(X)$ such that f is never zero and f is not invertible in $A(X)$. Since $A(X)$ is regular, there exists an $f_0 \in A(X)$ such that $f^2 f_0(x) = f(x)$ for all $x \in X$. Since $f(x)$ is never zero on X, we can divide by $f(x)$ to get $f f_0(x) = 1$. Hence, this means that f is invertible in $A(X)$, a contradiction. $\square$

Theorem 3.3. *Let X be a P-space and $A(X)$ an intermediate ring. Then, $A(X) = C(X)$ if and only if $A(X)$ is a regular ring.*

Proof. If $A(X) = C(X)$, then $A(X)$ is a regular ring [**9,** Section 4J]. If $A(X) \neq C(X)$, then $A(X)$ is not a regular ring by Proposition 3.2. $\square$

Remark 3.4. From [**10,** Theorem 1.16], any commutative ring R that has no non-zero nilpotents is regular if and only if every prime ideal of R is maximal. Since intermediate rings have no non-zero nilpotents, an intermediate ring $A(X)$ is regular if and only if every prime ideal in $A(X)$ is maximal. Thus, Theorem 3.3 is equivalent to the assertion that, when X is a P-space, then $A(X) = C(X)$ if and only if every prime ideal in $A(X)$ is maximal.

We now give an alternative proof that, if $A(X) \neq C(X)$, then there exists a prime ideal that is not maximal. This property was first proven in [**1**] using a different method than that used in this paper. In the following proof, we specify such a prime ideal. Let $A(X)$ be an intermediate ring of continuous functions, and let $\mathcal{F}$ be a z-filter on X. Define

$$I_0(\mathcal{F}) \overset{\text{def}}{=} \{f \in A(X) : \boldsymbol{Z}(f) \in \mathcal{F}\}. \tag{3.1}$$

Note that $I_0(\mathcal{F})$ is an ideal in $A(X)$ and, in general, $I_0(\mathcal{F}) \subseteq \mathfrak{Z}_A^{\leftarrow}(\mathcal{F})$. If $A(X) = C(X)$, then $I_0(\mathcal{F}) = \mathfrak{Z}_A^{\leftarrow}(\mathcal{F})$ since, in this case, for each $f \in C(X)$, we have $\mathfrak{Z}_C(f) = \langle \boldsymbol{Z}(f) \rangle$. In general, we have the following.

Proposition 3.5. *Let $A(X)$ be an intermediate ring, and let $\mathcal{G}$ be a prime z-filter on X. Then, $I_0(\mathcal{G})$ is a prime ideal in $A(X)$.*

Proof. Suppose that $f, g \in A(X)$ and $fg \in I_0(\mathcal{G})$. Then $\boldsymbol{Z}(fg) \in \mathcal{G}$. However, $\boldsymbol{Z}(fg) = \boldsymbol{Z}(f) \cup \boldsymbol{Z}(g)$ so $\boldsymbol{Z}(f) \cup \boldsymbol{Z}(g) \in \mathcal{G}$; and, since $\mathcal{G}$ is a prime z-filter, it follows that $\boldsymbol{Z}(f)$, say, belongs to $\mathcal{G}$. Then, $f \in I_0(\mathcal{G})$. Therefore, $I_0(\mathcal{G})$ is a prime ideal. $\square$

We use Proposition 3.5 to give an alternative proof for [**1,** Theorem 3.2].

Proposition 3.6. *If $A(X) \neq C(X)$, then $A(X)$ contains a nonmaximal prime ideal.*

Proof. If $A(X) \neq C(X)$, then $A(X)$ contains a non-invertible function f which never vanishes. Let $\mathcal{U}$ be any z-ultrafilter containing $\mathcal{Z}_A(f)$. Then, $I_0(\mathcal{U})$ is, by Proposition 3.5, a prime ideal, and it is not maximal since the ideal $\mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$ properly contains $I_0(\mathcal{U})$ (in particular, $f \in \mathcal{Z}_A^{\leftarrow}[\mathcal{U}]$, except that, as f never vanishes, $f \notin I_0(\mathcal{U})$). □

3.2. $\mathcal{Z}_A$- and $\mathfrak{Z}_A$-ideals; $\mathcal{Z}_A$- and $\mathfrak{Z}_A$-filters. It is known from [**9,** Section 4J] that X is a P-space if and only if every ideal in $C(X)$ is a z-ideal. Noting that the z-ideals coincide with $\mathfrak{Z}_C$-ideals, we see that the next theorem shows that $C(X)$ is the only intermediate ring for which this holds. In particular, we show that the property that every z-ideal is a $\mathcal{Z}_A$-ideal (which guarantees X to be a P-space by Theorem 2.14) also characterizes $C(X)$ among all intermediate rings when X is a P-space.

Theorem 3.7. *Let X be a P-space and $A(X)$ an intermediate ring. Then, $A(X) = C(X)$ if and only if every ideal in $A(X)$ is a $\mathcal{Z}_A$-ideal ($\mathfrak{Z}_A$-ideal).*

Proof. Suppose that $A(X) = C(X)$. Then, by [**12,** Corollary 2.4] (which states that, for any ideal I in $C(X)$, $\mathfrak{Z}_C[I] = \boldsymbol{Z}[I]$), any z-ideal is a $\mathfrak{Z}_C$-ideal. Since X is a P-space, every ideal is a z-ideal according to [**9,** page 211]. Thus, every ideal is a $\mathfrak{Z}_C$-ideal. From Theorem 2.3, $\mathcal{Z}_A(f) = \mathfrak{Z}_A(f)$ for all $f \in A(X)$. Hence, every ideal is also a $\mathcal{Z}_C$-ideal.

Conversely, suppose that $A(X) \neq C(X)$. Then, $A(X)$ contains a non-invertible function f which never vanishes. Let $\mathcal{F} = \mathcal{Z}_A(f)$, and let $I_0(\mathcal{F})$ be defined according to equation (3.1). Now, $\mathcal{F} \subseteq \mathcal{Z}_A[I_0(\mathcal{F})]$ since X is a P-space, and hence, for each $E \in \mathcal{F}$, the characteristic function χ_{E^c} of the complement of E is in $I_0(\mathcal{F})$ and

$$E \in \mathcal{Z}_A(\chi_{E^c}) \subseteq \mathcal{Z}_A[I_0(\mathcal{F})].$$

Thus, $f \in \mathcal{Z}_A^{\leftarrow}\mathcal{Z}_A[I_0(\mathcal{F})]$. However, $f \notin I_0(\mathcal{F})$ since f never vanishes. Hence, $I_0(\mathcal{F})$ is not a $\mathcal{Z}_A$-ideal. The same argument holds when $\mathcal{Z}_A$ is replaced by $\mathfrak{Z}_A$. □

Next, we note that the condition that every z-filter be a $\mathcal{Z}_A$-filter ($\mathfrak{Z}_A$-filter) does not characterize $C(X)$ among intermediate rings. The following example provides a reason.

Example 3.8. Consider $X = \mathbb{N}$, which is discrete, and hence, a P-space. Consider $A(X) = C^*(X)$. Let E be any subset of X (as X is discrete, E is a zero-set), and let $f = \chi_{E^c}$ be the binary-valued characteristic function on the complement of E. Then, $\boldsymbol{Z}(f) = E$, and clearly, $\mathfrak{Z}_A(f) = \langle \boldsymbol{Z}(f) \rangle$. Then, by Proposition 2.11, every z-filter is a $\mathfrak{Z}_A$-filter. However, clearly, $A(X) \neq C(X)$. Hence, the property that every z-filter be a $\mathfrak{Z}_A$-filter does not characterize $C(X)$ among intermediate rings when X is a P-space. From Theorem 2.3, the property that every z-filter be a $\mathcal{Z}_A$-filter does not characterize $C(X)$ among intermediate rings when X is a P-space either.

3.3. The ideals M_A^p and O_A^p for $p \in \beta X$. The ideals O_A^p defined for $p \in X$ in Section 2.3 can be defined for any $p \in \beta X$ by using the characterization for maximal ideals given in [**16**], as follows. For $p \in \beta X$, let

$$M_A^p = \{f \in A(X) \mid p \in h\mathcal{Z}_A(f)\}$$
$$O_A^p = \{f \in A(X) \mid p \in \operatorname{int} h\mathcal{Z}_A(f)\}$$

This coincides with the definition in [**5, 13**] and agrees with our definition in subsection 2.2 when $p \in X$.

We know from [**9,** Section 7L] that the property that X is a P-space can be characterized by the property that $M_C^p = O_C^p$ for all $p \in \beta X$, and we know, from [**9,** §4J], that the property that X is a P-space can also be characterized by $M_C^p = O_C^p$ for all $p \in X$. We showed in Theorem 2.5 that the characterization in terms of $p \in X$ can be extended from $C(X)$ to all intermediate rings. The next example, however, shows that the characterization in terms of $p \in \beta X$ does not extend to all intermediate rings.

Example 3.9. Let $X = \mathbb{N}$, which is discrete, and hence, a P-space. Let $A(X) = C^*(X)$. We show that a $p \in \beta X$ exists such that $M_A^p \neq O_A^p$, and hence, the property $M_A^p = O_A^p$ for all $p \in \beta X$ does not characterize P-spaces. It follows from [**9,** Section 4K1] that $C(\beta\mathbb{N})$ is not a regular ring, and hence, by [**9,** Section 4J], that $\beta\mathbb{N}$ is not a P-space. From Theorem 2.5, there is a point $p \in \beta X$ such that $M_{C(\beta\mathbb{N})}^p \neq O_{C(\beta\mathbb{N})}^p$. Then, however, as $A(X)$ (which is equal to $C^*(\mathbb{N})$) is isomorphic to $C(\beta\mathbb{N})$, it follows that $M_A^p \neq O_A^p$ for some $p \in \beta X$.

In Example 3.9, we could have used Theorem 3.3 instead of **[9,** Section 4K1] to show that $C(\beta\mathbb{N})$ is not a regular ring by observing that $C^*(\mathbb{N}) \neq C(\mathbb{N})$ (and hence by Theorem 3.3, $C^*(\mathbb{N})$ is not regular) and that $C^*(\mathbb{N})$ is isomorphic to $C(\beta\mathbb{N})$.

According to the next theorem, the condition $M_A^p = O_A^p$ for $p \in \beta X$ characterizes $C(X)$ among intermediate rings $A(X)$ when X is a P-space. This highlights, in the event that $A(X) \neq C(X)$, the significance of the two cases p ranging over X and p ranging over βX.

Theorem 3.10. *Let X be a P-space and $A(X)$ an intermediate ring. Then, $A(X) = C(X)$ if and only if, for all $p \in \beta X$, $M_A^p = O_A^p$.*

Proof. If $A(X) = C(X)$, then $M_A^p = O_A^p$ for every $p \in \beta X$ **[9,** Section 7L]. Suppose that $A(X) \neq C(X)$. Then, there exists a function $f \in A(X)$ that is not invertible in $A(X)$ but never vanishes. Let $\mathcal{U}_p$ be a z-ultrafilter such that $\mathcal{U}_p \supseteq \mathcal{Z}_A(f)$. Thus, $f \in M_A^p$. Note that, as $f(x) \neq 0$ for all $x \in X$, $\mathcal{Z}_A(f)$ (and any z-filter containing it) must be a free z-filter; hence, $h\mathcal{Z}_A(f) \subseteq \beta X \setminus X$. Then, since X is dense in βX, $h\mathcal{Z}_A(f)$ has empty interior. Thus, by definition, $f \notin O_A^p$. $\square$

We see that this characterization of $C(X)$ does not hold if the condition that $p \in \beta X$ is replaced by the condition that $p \in X$.

Example 3.11. Let $X = \mathbb{N}$, and let $A(X) = C^*(X)$. Recall that $\mathbb{N}$ is discrete, and hence, is a P-space. Furthermore, since $\mathbb{N}$ is discrete, for every subset $E \subseteq \mathbb{N}$, $E = \operatorname{int} E$. Hence, by definition, $M_A^p = O_A^p$ for all $p \in X$ and for any intermediate ring $A(X)$, in particular, where $A(X) = C^*(X)$. Clearly, however, $C(\mathbb{N}) \neq C^*(\mathbb{N})$. Therefore, the condition that $M_A^p = O_A^p$ for every $p \in X$ does not characterize $C(X)$ among intermediate rings when X is a P-space.

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