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THE FIRST MIXED PROBLEM FOR THE NONSTATIONARY LAMÉ SYSTEM

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ABSTRACT. We find an adequate interpretation of the stationary Lamé operator within the framework of elliptic complexes and study the first mixed problem for the nonstationary Lamé system.

1. Introduction. In his work on a systematic dynamical theory of elasticity, Gabriel Lamé in 1881 derived from Newtonian mechanics his basic equations which are also the conditions for equilibrium. From those, he derived what are now known as nonstationary Lamé equations in elastodynamics:

$$\rho u''_{tt} = -\mu \varDelta u + (\lambda + \mu)\, \nabla \operatorname{div} u + f, \tag{1.1}$$

where $u : \mathcal{X} \times (0,T) \to \mathbf{R}^3$ is a search-for displacement vector, ρ is the mass density, λ and μ are physical characteristics of the body under consideration, called Lamé constants, $\varDelta u = -u''_{x_1x_1} - u''_{x_2x_2} - u''_{x_3x_3}$ is the nonnegative Laplace operator in $\mathbf{R}^3$, and f is the density vector of outer forces, see **[3, 10, 11, 16]**, and elsewhere.

Here, $\mathcal{X}$ stands for a bounded domain in $\mathbf{R}^3$, whose boundary is assumed to be smooth enough. Hence, to specify a particular solution of nonstationary Lamé equations, we consider the first mixed problem for (1.1) in the cylinder $\mathcal{X} \times (O,T)$ by posing the initial conditions:

$$\begin{aligned} u(x,0) &= u_0(x) \quad \text{for } x \in \mathcal{X}, \\ u'_t(x,0) &= u_1(x) \quad \text{for } x \in \mathcal{X}, \end{aligned} \tag{1.2}$$

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on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) = u_l(x,t) \quad \text{for } (x,t) \in \partial\mathcal{X} \times (0,T), \tag{1.3}$$

on the lateral surface.

When working in adequate function spaces surviving under restriction to the lateral boundary, it may be assumed, without loss of generality, that $u_l \equiv 0$ for, if not, the Dirichlet problem must first be solved with data on $\partial\mathcal{X} \times [0,T]$ in the class of smooth functions.

To a certain extent, the theory of mixed problems for hyperbolic partial differential equations with variable coefficients is a completion of the classical area studying the Cauchy problem and mixed problem for the wave equation. The fundamental idea of Jean Leray in the early 1950s is that the energy form corresponding to a hyperbolic operator with simple real characteristics is an elliptic form with parameter, which allows one to obtain estimates in the case of variable coefficients. For a recent account of the theory we refer the interested reader to [**7,** Chapter 3]. The energy method for hyperbolic equations takes a considerable role in [**7**]. This method automatically extends to $2b$-parabolic differential equations with variable coefficients. Within the framework of energy method the theories of hyperbolic and parabolic equations can be combined into one theory of operators with the dominating principal the quasihomogeneous part.

In this paper, we apply the theory to the first mixed problem for a generalized Lamé system. While the classical Lamé system of (1.1) stems from dynamical theory of elasticity, the generalized Lamé system is well motivated by its origin in homological algebra. This work is intended to bring together two areas of mathematics, one applied and the other purely theoretical. This is a portion of our program for specifying the main equations of applied mathematics within the framework of differential geometry. Although the theory of mixed problems for equations with the dominating principal of quasihomogeneous part is well understood, see [**7**], the focus of the present paper is mainly on the study of a very specific and well motivated class of hyperbolic equations to which the general theory successfully applies.

Our approach makes no appeal to the theory of [**7**], and it is much more delicate than that of [**7**]. Using the geometric structure of the generalized Lamé system, we develop the Galerkin method, which enables us to construct an approximate solution of the mixed problem. We also prove the existence of a classical solution.

2. Generalized Lamé system. The stationary Lamé equations are easily specified within the framework of complexes on the underlying manifold $\mathcal{X}$. Upon introduction of the de Rham complex of $\mathcal{X}$

$$0 \longrightarrow \Omega^0(\mathcal{X}) \xrightarrow{d} \Omega^1(\mathcal{X}) \xrightarrow{d} \Omega^2(\mathcal{X}) \xrightarrow{d} \Omega^3(\mathcal{X}) \longrightarrow 0,$$

we can rewrite system (1.1) in the invariant form

$$\rho u''_{tt} = -\mu \Delta u - (\lambda + \mu)\, dd^*\, u + f \tag{2.1}$$

in the semicylinder $\mathcal{X} \times [0, \infty)$, where $\Delta = d^*d + dd^*$ is the Laplacian of the de Rham complex.

Example 2.1. When restricted to functions, i.e., differential forms of degree $i = 0$, equation (2.1) reads

$$\rho u''_{tt} = -\mu \Delta u + f,$$

which is precisely the wave equation in the cylinder $\mathcal{X} \times (0, T)$.

More generally, let $\mathcal{X}$ be a C^∞ compact manifold with boundary of dimension n. Consider a complex of first-order differential operators acting in sections of vector bundles over $\mathcal{X}$,

$$\begin{aligned} 0 \longrightarrow C^\infty(\mathcal{X}, F^0) \xrightarrow{A^0} C^\infty(\mathcal{X}, F^1) \xrightarrow{A^1} \cdots \\ \xrightarrow{A^{N-1}} C^\infty(\mathcal{X}, F^N) \longrightarrow 0, \end{aligned} \tag{2.2}$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^i, F^{i-1})$ satisfy the equality $A^{i+1}A^i = 0$ for all $i = 0, 1, \ldots, N-2$. Our basic assumption is that the complex (2.2) is elliptic, i.e., the corresponding complex of principal symbols is exact away from the zero section of the cotangent bundle $T^*\mathcal{X}$, see [**15,** 1.1.12]. We endow the manifold $\mathcal{X}$ and the vector bundles F^i by Riemannian metrics.

Set

$$F = \bigoplus_{i=0}^{N} F^i,$$

and consider two first-order differential operators A and A^* in $C^\infty(\mathcal{X}, F)$, given by the $((N+1) \times (N+1))$-matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ A^0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & A^1 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & A^{N-1} & 0 \end{pmatrix},$$

$$A^* = \begin{pmatrix} 0 & A^{0*} & 0 & \cdots & 0 & 0 \\ 0 & 0 & A^{1*} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 0 & A^{N-1*} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where $A^i \in \mathrm{Diff}^1(\mathcal{X}; F^{i+1}, F^i)$ stands for the formal adjoint of A^i. It is easily verified that $A \circ A = 0$ and $A^* \circ A^* = 0$, as well as

$$\Delta := A^*A + AA^* = \begin{pmatrix} \Delta^0 & 0 & \cdots & 0 \\ 0 & \Delta^1 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & \Delta^N \end{pmatrix}, \tag{2.3}$$

where $\Delta^i = A^{i*}A^i + A^{i-1}A^{i-1*}$ for $i = 0, 1, \ldots, N$ are the so-called Laplacians of complex (2.2). The ellipticity of complex (2.2) merely amounts to that of its Laplacians $\Delta^0, \Delta^1, \ldots, \Delta^N$.

Lemma 2.2. *Let r and s be real or complex numbers. Then, the operator $rA + sA^* \in \mathrm{Diff}^1(\mathcal{X}; F)$ is elliptic if and only if $rs \neq 0$.*

Proof. Necessity. If at least one of the scalars r and s vanishes, then the operator $rA + sA^*$ reduces to a scalar multiple of A or A^*, the operators of which cannot be elliptic due to their nilpotency.

Sufficiency. If both r and s are different from zero, then a trivial verification gives

$$(s^{-1}A + r^{-1}A^*)(rA + sA^*) = AA^* + A^*A,$$
$$(rA + sA^*)(s^{-1}A + r^{-1}A^*) = AA^* + A^*A,$$

showing the ellipticity of $rA + sA^*$. □

By generalized stationary Lamé operators related to the complex (2.2) is meant the product of two operators of the form $rA + sA^*$, where $rs \neq 0$. These are specifically operators $L \in \mathrm{Diff}^2(\mathcal{X}; F)$ of the form $L = rA^*A + sAA^*$, where $rs \neq 0$. They are elliptic and preserve the grading of complex (2.2) in the sense that if u is a section of F^i, then so is Lu.

Consider the Dirichlet problem for the elliptic operator $\Delta^2 = (A^*A)^2 + (AA^*)^2$ on $\mathcal{X}$ with data

$$\begin{aligned} u &= 0 \quad \text{at } \partial\mathcal{X}, \\ (A + A^*)u &= 0 \quad \text{at } \partial\mathcal{X}. \end{aligned} \tag{2.4}$$

This boundary value problem is elliptic and formally selfadjoint. As usual, it can be treated within the framework of densely defined unbounded operators in the Hilbert space $L^2(\mathcal{X}, F)$, cf., [**14**]. In particular, there is a bounded operator $G : L^2(\mathcal{X}, F) \to H^4(\mathcal{X}, F)$ called the *Green operator* such that $u = Gf$ satisfies (2.4) and

$$f = Hf + \Delta^2(Gf) \tag{2.5}$$

for all $f \in L^2(\mathcal{X}, F)$, where H is the orthogonal projection of $L^2(\mathcal{X}, F)$ onto the finite-dimensional subspace of $L^2(\mathcal{X}, F)$ consisting of all $h \in C^\infty(\mathcal{X}, F)$, which satisfy $(A + A^*)h = 0$ in $\mathcal{X}$ and $h = 0$ at $\partial\mathcal{X}$. The Green operator G is actually known to be a pseudodifferential operator of order -4 in Boutet de Monvel's algebra on $\mathcal{X}$, see [**2**].

If $A+A^*$ has the uniqueness property for the global Cauchy problem on $\mathcal{X}$, then $H = 0$. By the uniqueness property is meant that, if h is any solution to $(A+A^*)h = 0$ in a connected open set U in $\mathcal{X}$ and h vanishes in a nonempty open subset of U, then h is identically zero in U.

Lemma 2.3. *Suppose that $L = rA^*A + sAA^*$ is a stationary Lamé operator on $\mathcal{X}'$, the parameters r and s satisfying $rs \neq 0$. Then, $P = (\Delta^2/L)G$, with $\Delta^2/L = r^{-1}A^*A + s^{-1}AA^*$, is a paramatrix of L.*

Proof. From the above, we obtain

$$LP = L(\Delta^2/L)G = \Delta^2 G = I - H,$$

where $H \in \Psi^{-\infty}(\mathcal{X}; F)$. Hence, P is a left paramatrix of L. Since L is elliptic, P is also a right paramatrix of L in the interior of $\mathcal{X}$. $\square$

Write

$$L = r\Delta + (s-r)AA^* = -\mu\Delta - (\lambda+\mu)AA^*,$$

where $r = -\mu$ and $s = -\lambda - 2\mu$. Then, for the ellipticity of L, it is necessary and sufficient that $\mu \neq 0$ and $\lambda + 2\mu \neq 0$.

3. Wave equation. In the open cylinder $\mathcal{C}_T = \overset{\circ}{\mathcal{X}} \times (0,T)$, for some $T > 0$, we consider the hyperbolic system

$$\rho u''_{tt} = -\mu\Delta u - (\lambda+\mu)AA^*u + f \tag{3.1}$$

for a section u of the bundle

$$(x,t) \longmapsto F^i_x$$

over $\mathcal{X} \times [0,T]$, written F^i for short, cf., Figure 1. Assume $\rho = 1$ and $\mu > 0$.

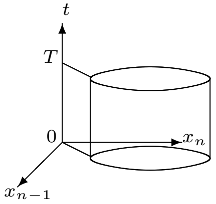

FIGURE 1. A cylinder $\mathcal{C}_T$.

A function $u \in C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0,T), F^i)$ satisfying equation (3.1) in $\mathcal{C}_T$, the initial conditions

$$\begin{aligned} u(x,0) &= u_0(x), \quad \text{for } x \in \overset{\circ}{\mathcal{X}}, \\ u'_t(x,0) &= u_1(x), \quad \text{for } x \in \overset{\circ}{\mathcal{X}}, \end{aligned} \tag{3.2}$$

on the lower basis of the cylinder and a Dirichlet condition

$$u(x,t) = u_l(x,t), \quad \text{for } (x,t) \in \partial\mathcal{X} \times (0,T), \tag{3.3}$$

on the lateral surface, is said to be a classical solution of the first mixed problem for the generalized Lamé equations. Since the case of inhomogeneous boundary conditions easily reduces to the case of homogeneous boundary conditions, we will assume $u_l \equiv 0$ in the sequel.

Let u be a classical solution of the first mixed problem for the generalized Lamé equations with $f \in L^2(\mathcal{C}_T, F^i)$. Given any $\varepsilon > 0$, we multiply both sides of (3.1) with g^*, where g is an arbitrary smooth function in the closure of $\mathcal{C}_{T-\varepsilon}$ vanishing at the lateral surface and the head of this cylinder, and integrate the resulting equality over $\mathcal{C}_{T-\varepsilon}$. We will write the inner product of the values of f and g at any point $(x,t) \in \mathcal{C}_{T-\varepsilon}$ simply (f,g) when no confusion can arise. Using the Stokes theorem, we get

$$\begin{aligned}\int_{\mathcal{C}_{T-\varepsilon}} (f,g)\,dx\,dt = &- \int_{\mathcal{X}} (u_1, g)\,dx \\ &+ \int_{\mathcal{C}_{T-\varepsilon}} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda{+}2\mu)(A^*u, A^*g))\,dx\,dt.\end{aligned}$$

We exploit this identity to introduce the concept of the weak solution of the first mixed problem for the generalized Lamé system. We assume that $f \in L^2(\mathcal{C}_T, F^i)$ and $u_1 \in L^2(\mathcal{X}, F^i)$.

A function $u \in H^1(\mathcal{C}_T, F^i)$ is called a *weak solution* of the first mixed problem for (3.1) in $\mathcal{C}_T$, if u satisfies

$$\begin{array}{ll} u(x,0) = u_0(x), & \text{for } x \in \overset{\circ}{\mathcal{X}}, \\ u(x,t) = 0, & \text{for } (x,t) \in \partial\mathcal{X} \times (0,T), \end{array}$$

and

$$\begin{aligned}\int_{\mathcal{X}} (u_1, g)\,dx &+ \int_{\mathcal{C}_T} (f,g)\,dx\,dt \\ &= \int_{\mathcal{C}_T} (-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda + 2\mu)(A^*u, A^*g))\;dx\,dt\end{aligned} \tag{3.4}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$, such that

$$\begin{array}{ll} g(x,T) = 0, & \text{for } x \in \overset{\circ}{\mathcal{X}}, \\ g(x,t) = 0, & \text{for } (x,t) \in \partial\mathcal{X} \times (0,T). \end{array} \tag{3.5}$$

Similarly to the classical solution, if u is a weak solution of the first mixed problem for the generalized Lamé system in $\mathcal{C}_T$, then u is a weak solution of the corresponding problem also in the cylinder $\mathcal{C}_{T'}$ with any $T' < T$. Indeed, u belongs to $H^1(\mathcal{C}_{T'}, F^i)$ for all $T' < T$, and it vanishes on the lateral boundary of $\mathcal{C}_{T'}$. Moreover, the identity (3.4) is fulfilled for all $g \in H^1(\mathcal{C}_T, F^i)$ with property (3.5). It may be readily verified that, if a function g belongs to $H^1(\mathcal{C}_{T'}, F^i)$, the trace of g at the cross-section $\{t = T'\}$ is zero and $g = 0$ in $\mathcal{C}_T \setminus \mathcal{C}_{T'}$, then $g \in H^1(\mathcal{C}_T)$ and $g(x, T) = 0$ for all x in the interior of $\mathcal{X}$. If, moreover, $g = 0$ at $\partial\mathcal{X} \times (0, T')$, then g vanishes at the lateral boundary of $\mathcal{C}_T$. Hence, it follows that the function u satisfies the integral identity by means of which the weak solution of the corresponding mixed problem in $\mathcal{C}_{T'}$ is defined.

Note that we introduced the concept of the weak solution of the first mixed problem as a natural generalization of the concept of the classical solution (with $f \in L^2(\mathcal{C}_T, F^i)$). We have actually proved that the classical solution of the first mixed problem in $\mathcal{C}_T$ with $f \in L^2(\mathcal{C}_T, F^i)$ is a weak solution of this problem in the smaller cylinder $\mathcal{C}_{T-\varepsilon}$ for any $\varepsilon \in (0, T)$.

Along with classical and weak solutions of the first mixed problem the notion of an 'almost everywhere' solution may be introduced. A function u is said to be an *almost everywhere* solution of the first mixed problem if $u \in H^2(\mathcal{C}_T, F^i)$ satisfies equation (3.1) for almost all $(x, t) \in \mathcal{C}_T$, initial conditions (3.2) for almost all x in the interior of $\mathcal{X}$ and the trace of u on the lateral surface vanishes almost everywhere. From the definition, it immediately follows that, if the classical solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$, then it is also an almost everywhere solution. Moreover, if an almost everywhere solution u of the first mixed problem belongs to the intersection $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0, T), F^i)$, then u is obviously a classical solution, too.

Every almost everywhere solution of the first mixed problem in $\mathcal{C}_T$ is a weak solution of this problem in $\mathcal{C}_T$. The converse assertion is also true.

Lemma 3.1. *If a weak solution of the first mixed problem belongs to the space $H^2(\mathcal{C}_T, F^i)$, then it is an almost everywhere solution of this problem. If a weak solution of the first mixed problem belongs to $C^2(\mathcal{C}_T, F^i) \cap C^1(\mathcal{X} \times [0, T), F^i)$, then it is a classical solution of this problem.*

Proof. This is a standard fact for functions with generalized derivatives, cf., [**12,** page 287, Lemma 1]. $\square$

We are now in a position to prove a uniqueness theorem for the weak solution of the first mixed problem.

Theorem 3.2. *Suppose that $\mu \geq 0$ and $\lambda + 2\mu \geq 0$. Then, the first mixed problem for the generalized Lamé system has at most one weak solution.*

Proof. Let $u \in H^1(\mathcal{C}_T, F^i)$ be a weak solution of the first mixed problem with $f = 0$ in $\mathcal{C}_T$ and $u_0 = u_1 = 0$ in the interior of $\mathcal{X}$.

Choose an arbitrary $s \in (0, T)$, and consider the function

$$g(x,t) = \begin{cases} \int_t^s u(x,\theta)\,d\theta & \text{if } 0 < t < s, \\ 0 & \text{if } s < t < T, \end{cases}$$

defined in $\mathcal{C}_T$. It is immediately verified that the function g has generalized derivatives

$$g'_{x^j}(x,t) = \begin{cases} \int_t^s u'_{x^j}(x,\theta)\,d\theta & \text{if } 0 < t < s, \\ 0 & \text{if } s < t < T, \end{cases}$$

and

$$g'_t(x,t) = \begin{cases} -u(x,t) & \text{if } 0 < t < s, \\ 0 & \text{if } s < t < T, \end{cases}$$

in $\mathcal{C}_T$. Therefore, we obtain $g \in H^1(\mathcal{C}_T)$. Moreover, g vanishes at the lateral boundary and the head of the cylinder $\mathcal{C}_T$.

Substituting the function g into identity (3.4) shows that the integral

$$\int_{\mathcal{C}_s} \Big((u'_t, u) + \mu\Big(Au, \int_t^s Au(\cdot,\theta)\,d\theta\Big) \\ + (\lambda + 2\mu)\Big(A^* u, \int_t^s A^* u(\cdot,\theta)\,d\theta\Big)\Big)\,dx\,dt$$

vanishes for all $s \in (0,T)$. It is obvious that

$$\Re \int_{\mathcal{C}_s} (u'_t, u)\,dx\,dt = \frac{1}{2}\int_{\mathcal{X}} |u(x,s)|^2 dx.$$

Since

$$\begin{aligned}
& \int_{\mathcal{C}_s} \Big(Au(x,t), \int_t^s Au(x,\theta)\, d\theta \Big)\, dx\, dt \\
& \quad = \int_{\mathcal{X}} \int_0^s \Big(Au(x,t), \int_t^s Au(x,\theta)\, d\theta \Big)\, dx\, dt \\
& \quad = \int_{\mathcal{X}} \int_0^s \Big(\int_0^\theta Au(x,t)\, dt, Au(x,\theta) \Big)\, dx\, d\theta,
\end{aligned}$$

which transforms to

$$\begin{aligned}
& \int_{\mathcal{X}} \Big(\int_0^s Au(x,t)\, dt, \int_0^s Au(x,\theta)\, d\theta \Big)\, dx \\
& \qquad - \int_{\mathcal{X}} \int_0^s \Big(\int_\theta^s Au(x,t)\, dt, Au(x,\theta) \Big)\, dx\, d\theta \\
& \quad = \int_{\mathcal{X}} \Big| \int_0^s Au(x,t)\, dt \Big|^2 dx - \int_{\mathcal{C}_s} \Big(\int_\theta^s Au(x,t)\, dt, Au(x,\theta) \Big) dx\, d\theta,
\end{aligned}$$

and yields

$$\Re \int_{\mathcal{C}_s} \Big(Au(x,t), \int_t^s Au(x,\theta)\, d\theta \Big)\, dx\, dt = \frac{1}{2} \int_{\mathcal{X}} \Big| \int_0^s Au(x,t)\, dt \Big|^2 dx.$$

Similarly, we obtain

$$\Re \int_{\mathcal{C}_s} \Big(A^* u(x,t), \int_t^s A^* u(x,\theta)\, d\theta \Big)\, dx\, dt = \frac{1}{2} \int_{\mathcal{X}} \Big| \int_0^s A^* u(x,t)\, dt \Big|^2 dx,$$

whence

$$\begin{aligned}
& \int_{\mathcal{X}} |u(x,s)|^2 dx + \mu \int_{\mathcal{X}} \Big| \int_0^s Au(x,t)\, dt \Big|^2 dx \\
& \qquad + (\lambda + 2\mu) \int_{\mathcal{X}} \Big| \int_0^s A^* u(x,t)\, dt \Big|^2 dx = 0
\end{aligned} \tag{3.6}$$

for all $s \in (0,T)$.

Since $\mu \geq 0$ and $\mu + 2\lambda \geq 0$, we conclude from (3.6) that

$$\int_{\mathcal{X}} |u(x,s)|^2 dx = 0$$

for all $s \in (0,T)$, and thus, $u = 0$ in $\mathcal{C}_T$, as desired. $\square$

As previously mentioned, a classical solution of the first mixed problem is also a weak solution of this problem in $\mathcal{C}_{T-\varepsilon}$ for each $\varepsilon \in (0,T)$. Hence, Theorem 3.2 implies the uniqueness of a classical solution as well. Furthermore, since almost everywhere solutions are weak solutions, we also deduce that, if $\mu \geq 0$ and $\mu + 2\lambda \geq 0$, then the first mixed problem for the generalized Lamé system has at most one almost everywhere solution.

4. Existence of a weak solution. We now turn to showing the existence of solutions of the first mixed problem for the generalized Lamé system. Toward this end, we use the Fourier method which consists of looking for the solution of the mixed problem in the form of a series over eigenfunctions of the corresponding elliptic boundary value problem.

Let v be a weak eigenfunction of the first boundary value problem for the generalized Lamé system:

$$\begin{array}{rll} -\mu\Delta v - (\lambda+\mu)AA^*v = \varkappa v & \text{in } \overset{\circ}{\mathcal{X}}, \\ v = 0 & \text{at } \partial\mathcal{X}, \end{array} \tag{4.1}$$

where $\varkappa$ is a corresponding eigenvalue. This amounts merely to saying that

$$\int_{\mathcal{X}} (-\mu(Av,Ag)_x - (\lambda+2\mu)(A^*v,A^*g)_x)\,dx - \varkappa \int_{\mathcal{X}} (v,g)_x\,dx = 0 \tag{4.2}$$

for all $g \in \overset{\circ}{H}{}^1(\mathcal{X},F^i)$.

Consider the orthonormal system $(v_k)_{k=1,2,\ldots}$ in $L^2(\mathcal{X},F^i)$ consisting of all weak eigenfunctions of problem (4.1). Let $(\varkappa_k)_{k=1,2,\ldots}$ be the sequence of corresponding eigenvalues. As is standard, we think of this sequence as nonincreasing with $\varkappa_1 < 0$, and each eigenvalue repeats itself in accordance with its multiplicity. The system $(v_k)_{k=1,2,\ldots}$ is known to be an orthonormal basis in $L^2(\mathcal{X},F^i)$ and $\varkappa_k \to -\infty$ when $k \to \infty$. Moreover, the first eigenvalue $\varkappa_1$ is strongly negative when $\mu > 0$ and $\lambda + 2\mu > 0$.

Suppose that the initial data u_0 and u_1 in (3.2) belong to $L^2(\mathcal{X},F^i)$, and f belongs to $L^2(\mathcal{C}_T,F^i)$. From the Fubini theorem, the section $f(\cdot,t)$ belongs to $L^2(\mathcal{X},F^i)$ for almost all $t \in (0,T)$. We represent the functions u_0 and u_1 as well as the function $f(\cdot,t)$ for almost all $t \in (0,T)$

as Fourier series over the system $(v_k)_{k=1,2,\ldots}$ of an eigenfunction of problem (4.1). To wit,

$$u_0(x) = \sum_{k=1}^{\infty} u_{0,k} v_k(x), \qquad u_1(x) = \sum_{k=1}^{\infty} u_{1,k} v_k(x),$$

where $u_{0,k} = (u_0, v_k)_{L^2(\mathcal{X},F^i)}$ and $u_{1,k} = (u_1, v_k)_{L^2(\mathcal{X},F^i)}$ for each $k = 1, 2, \ldots$. By the Parseval equality, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} |u_{0,k}|^2 &= \|u_0\|^2_{L^2(\mathcal{X},F^i)}, \\ \sum_{k=1}^{\infty} |u_{1,k}|^2 &= \|u_1\|^2_{L^2(\mathcal{X},F^i)}. \end{aligned} \tag{4.3}$$

Similarly, we get

$$f(x,t) = \sum_{k=1}^{\infty} f_k(t) v_k(x),$$

where

$$f_k(t) = \int_{\mathcal{X}} (f(\cdot,t), v_k)_x \, dx \quad \text{for } k = 1, 2, \ldots.$$

Since

$$|f_k(t)|^2 \leq \int_{\mathcal{X}} |f(\cdot,t)|^2 dx \int_{\mathcal{X}} |v_k|^2 dx = \int_{\mathcal{X}} |f(\cdot,t)|^2 dx,$$

it follows that $f_k \in L^2(0,T)$ for all $k = 1, 2, \ldots$. Moreover,

$$\sum_{k=1}^{\infty} |f_k(t)|^2 = \int_{\mathcal{X}} |f(\cdot,t)|^2 dx$$

holds for almost all $t \in (0,T)$, due to the Parseval equality. This readily yields

$$\sum_{k=1}^{\infty} \int_0^T |f_k(t)|^2 dt = \int_{\mathcal{C}_T} |f(x,t)|^2 dx \, dt. \tag{4.4}$$

First, take the kth harmonics $u_{0,k} v_k$ and $u_{1,k} v_k$ as initial data in (3.2), and the function $f_k(t) v_k(x)$ as the function on the right-hand side of (3.1), where $k = 1, 2, \ldots$. Consider the function

$$u_k(x,t) = w_k(t) v_k(x), \tag{4.5}$$

where

$$w_k(t) = u_{0,k} \cos \sqrt{-\varkappa_k} t + u_{1,k} \frac{\sin \sqrt{-\varkappa_k} t}{\sqrt{-\varkappa_k}} + \int_0^t f_k(t') \frac{\sin \sqrt{-\varkappa_k} (t-t')}{\sqrt{-\varkappa_k}} dt'.$$

Note that this formula still makes sense if $\varkappa_k = 0$, for the limit on the right-hand side exists as $\varkappa_k \to 0$. The function w_k belongs obviously to $H^2(0,T)$, satisfies the initial conditions $w_k(0) = u_{0,k}$ and $w'_k(0) = u_{1,k}$ and is a solution of the ordinary differential equation

$$w''_k - \varkappa_k w_k = f_k \tag{4.6}$$

for almost all $t \in (0,T)$.

Our next objective is to show that, if v_k is an eigenfunction of problem (4.1) corresponding to the eigenvalue $\varkappa_k$, then $u_k(x,t)$ is a weak solution of the first mixed problem for the equation

$$u''_{tt}(x,t) = -\mu \Delta u(x,t) - (\lambda + \mu) A A^* u(x,t) + f_k(t) v_k(x)$$

in $\mathcal{C}_T$, with initial data

$$\begin{aligned} u(x,0) &= u_{0,k} v_k(x) \quad \text{for } x \in \mathring{\mathcal{X}}, \\ u'_t(x,0) &= u_{1,k} v_k(x) \quad \text{for } x \in \mathring{\mathcal{X}}. \end{aligned}$$

Indeed, the function u_k given by (4.5) belongs to $H^1(\mathcal{C}_T, F^i)$, satisfies the initial conditions and vanishes at the lateral boundary of the cylinder. It remains to show that

$$\begin{aligned} \int_{\mathcal{C}_T} (-((u_k)'_t, g'_t) + \mu (A u_k, A g) + (\lambda + 2\mu)(A^* u_k, A^* g)) \, dx \, dt \\ = \int_{\mathcal{X}} u_{1,k} (v_k, g) \, dx + \int_{\mathcal{C}_T} f_k(t) (v_k, g) \, dx \, dt \end{aligned}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (3.5). It is sufficient to establish the above identity merely for functions $g \in C^1(\overline{\mathcal{C}}_T, F^i)$ satisfying (3.5).

From (4.5) and integration by parts,

$$\int_{\mathcal{C}_T} ((u_k)'_t, g'_t)\, dx\, dt = \int_{\mathcal{X}} \Big(v_k, \int_0^T w'_k(t) g'_t dt \Big)_x dx$$
$$= \int_{\mathcal{X}} \Big(v_k, -u_{1,k} g(x,0) - \int_0^T w''_k(t) g\, dt \Big)_x dx$$

which reduces, by (4.6), to

$$-\int_{\mathcal{X}} u_{1,k} (v_k, g(x,0))_x\, dx - \varkappa_k \int_{\mathcal{C}_T} (u_k, g)\, dx\, dt - \int_{\mathcal{C}_T} f_k(t) (v_k, g)\, dx\, dt.$$

Hence, the desired identity follows from (4.2), for

$$\int_{\mathcal{C}_T} (\mu (A u_k, A g) + (\lambda + 2\mu)(A^* u_k, A^* g))\, dx\, dt$$
$$= \int_0^T w_k(t) \Big(\int_{\mathcal{X}} (\mu (A v_k, A g)_x + (\lambda + 2\mu)(A^* v_k, A^* g)_x \Big) dx \Big) dt$$
$$= -\int_0^T w_k(t) \Big(\varkappa_k \int_{\mathcal{X}} (v_k, g)_x dx \Big) dt,$$

as desired.

It is possible to take the partial sums

$$\sum_{k=1}^{N} u_{0,k} v_k(x), \qquad \sum_{k=1}^{N} u_{1,k} v_k(x)$$

of the Fourier series for the functions u_0 and u_1, respectively, as initial data, and the partial sum

$$\sum_{k=1}^{N} f_k(t) v_k(x)$$

of the Fourier series for f as the right-hand side of the equation. Then, the weak solution of the first mixed problem is

$$s_N(x,t) = \sum_{k=1}^{N} u_k(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x),$$

in particular, the function s_N satisfies the identity

$$\begin{aligned}\int_{\mathcal{C}_T} (-((s_N)'_t, g'_t) + \mu(As_N, Ag) + (\lambda+2\mu)(A^*s_N, A^*g))\,dx\,dt \\ = \int_{\mathcal{X}} \Big(\sum_{k=1}^{N} u_{1,k}v_k, g\Big)\,dx + \int_{\mathcal{C}_T} \Big(\sum_{k=1}^{N} f_k(t)v_k, g\Big)\,dx\,dt\end{aligned} \tag{4.7}$$

for all $g \in H^1(\mathcal{C}_T, F^i)$ satisfying (3.5).

Thus, it is to be expected that, under certain assumptions on u_0, u_1 and f, the solution of the first mixed problem for the generalized Lamé system can be represented as the series

$$u(x,t) = \sum_{k=0}^{\infty} w_k(t)v_k(x), \tag{4.8}$$

where $(v_k)_{k=1,2,\ldots}$ are weak eigenfunctions of problem (4.1).

Theorem 4.1. *Let $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, $u_1 \in L^2(\mathcal{X}, F^i)$ and $f \in L^2(\mathcal{C}_T, F^i)$. Then, the first mixed problem possesses a weak solution given by series* (4.8) *which converges in $H^1(\mathcal{C}_T, F^i)$. Moreover,*

$$\|u\|_{H^1(\mathcal{C}_T,F^i)} \le C(\|f\|_{L^2(\mathcal{C}_T,F^i)} + \|u_0\|_{H^1(\mathcal{X},F^i)} + \|u_1\|_{L^2(\mathcal{X},F^i)}) \tag{4.9}$$

with C a constant independent of u_0, u_1 and f.

Proof. From the formula for w_k it follows that

$$|w_k(t)| \le |u_{0,k}| + \frac{1}{\sqrt{|\varkappa_k|}}\,|u_{1,k}| + \frac{1}{\sqrt{|\varkappa_k|}}\int_0^T |f_k(t')|\,dt'$$

for all $t \in [0,T]$ and $k = 1, 2, \ldots$. Hence,

$$\begin{aligned}|w_k(t)|^2 &\le 3|u_{0,k}|^2 + \frac{3}{|\varkappa_k|}|u_{1,k}|^2 + \frac{3}{|\varkappa_k|}\Big(\int_0^T |f_k(t')|\,dt'\Big)^2 \\ &\le c(T)\Big(|u_{0,k}|^2 + |\varkappa_k|^{-1}\,|u_{1,k}|^2 + |\varkappa_k|^{-1}\int_0^T |f_k(t')|^2\,dt'\Big).\end{aligned} \tag{4.10}$$

Furthermore, since

$$|w'_k(t)| \le \sqrt{|\varkappa_k|}\,|u_{0,k}| + |u_{1,k}| + \int_0^T |f_k(t')|\,dt'$$

for all $t \in [0, T]$, we obtain

$$|w_k'(t)|^2 \leq c(T)\Bigg(|\varkappa_k|\,|u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2\,dt' \Bigg). \tag{4.11}$$

Since the function u_0 belongs to $\overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, its Fourier series over the orthonormal system $(v_k)_{k=1,2,\ldots}$ actually converges to u_0 in the $H^1(\mathcal{X}, F^i)$-norm, see [**12,** page 181, Theorem 3] and elsewhere. Moreover, there is a constant $c > 0$ with the property that

$$\sum_{k=1}^{\infty} |\varkappa_k|\,|u_{0,k}|^2 \leq c\|u_0\|^2_{H^1(\mathcal{X},F^i)} \tag{4.12}$$

for all $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

Consider the partial sum $s_N(x,t)$ of the Fourier series (4.8). Since both w_k and w_k' are continuous on $[0,T]$, for each fixed $t \in [0,T]$, the function s_N and its derivative in t belong to $\overset{\circ}{H}{}^1(\mathcal{X}, F^i)$.

In order to study the values of

$$t \longmapsto s_N(\cdot, t) \quad \text{in } \overset{\circ}{H}{}^1(\mathcal{X}, F^i),$$

it is convenient to endow this space with the so-called Dirichlet scalar product

$$D(v,g) = \int_{\mathcal{X}} (\mu(Av, Ag)_x + (\lambda{+}2\mu)(A^*v, A^*g)_x)\,dx$$

and the Dirichlet norm $D(v) := \sqrt{D(v,v)}$. The system

$$\left(\frac{v_k}{\sqrt{-\varkappa_k}} \right)_{k=1,2,\ldots}$$

is obviously orthonormal with respect to the Dirichlet scalar product. From (4.10), if $1 \leq M < N$, then

$$\begin{aligned} D(s_N(\cdot,t) - s_M(\cdot,t))^2 &= D\Bigg(\sum_{k=M+1}^{N} w_k(t) v_k \Bigg)^2 = \sum_{k=M+1}^{N} |w_k(t)|^2\,|\varkappa_k| \\ &\leq c(T) \sum_{k=M+1}^{N} \Bigg(|\varkappa_k|\,|u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 dt' \Bigg) \end{aligned}$$

for all $t \in [0,T]$. Similarly, using (4.11), we obtain

$$\begin{aligned}
&\|(s_N)'_t(\cdot,t) - (s_M)'_t(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)} \\
&\quad = \Big\| \sum_{k=M+1}^{N} w'_k(t) v_k \Big\|^2_{L^2(\mathcal{X},F^i)} = \sum_{k=M+1}^{N} |w'_k(t)|^2 \\
&\quad \le c(T) \sum_{k=M+1}^{N} \left(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 dt' \right)
\end{aligned}$$

for $t \in [0,T]$. Here, $c(T)$ stands for a constant which depends upon T but not upon M and N, and which can be different in diverse applications.

Upon integrating these two inequalities in $t \in [0,T]$ and summing, we immediately obtain

(4.13)

$$\|s_N - s_M\|^2_{H^1(\mathcal{C}_T,F^i)} \le c(T) \sum_{k=M+1}^{N} \left(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 dt' \right)$$

for all $1 \le M < N$. Combining (4.13) with (4.3), (4.4) and (4.12), we conclude that $(s_N)_{N=1,2,\dots}$ is a Cauchy sequence in $H^1(\mathcal{C}_T, F^i)$. Therefore, series (4.8) converges in this space to a function $u(x,t)$ in $H^1(\mathcal{C}_T, F^i)$. Obviously, u satisfies the initial conditions (3.2) and vanishes at the lateral boundary of $\mathcal{C}_T$. Letting $N \to \infty$ in (4.7), we deduce that u is a weak solution of the first mixed problem for the generalized Lamé system.

In a similar manner, we derive the inequalities

$$\begin{aligned}
D(s_N(\cdot,t))^2 &= D\left(\sum_{k=1}^{N} w_k(t) v_k \right)^2 = \sum_{k=1}^{N} |w_k(t)|^2\, |\varkappa_k| \\
&\le c(T) \sum_{k=1}^{N} \left(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 dt' \right)
\end{aligned}$$

and

$$\begin{aligned}\|(s_N)'_t(\cdot,t)\|^2_{L^2(\mathcal{X},F^i)} &= \Big\| \sum_{k=1}^{N} w'_k(t) v_k \Big\|^2_{L^2(\mathcal{X},F^i)} = \sum_{k=1}^{N} |w'_k(t)|^2 \\ &\le c(T) \sum_{k=1}^{N} \Big(|\varkappa_k|\, |u_{0,k}|^2 + |u_{1,k}|^2 + \int_0^T |f_k(t')|^2 dt' \Big)\end{aligned}$$

for all $t \in [0,T]$ and $N \geq 1$. Integrating these inequalities in t over the interval $[0,T]$, summing up, and using (4.3), (4.4) and (4.12), we establish estimate (4.9), thus completing the proof. $\square$

5. Galerkin method. There are also other proofs of the existence of weak solutions to mixed problems which do not exploit eigenfunctions. In this section, we present the so-called Galerkin method which also allows the construction of an approximate solution of the mixed problem. In contrast to the Fourier method, the Galerkin method additionally applies in the case where the coefficients of A depend not only upon the space variables but also upon the time t.

As before, assume that $u_0 \in \overset{\circ}{H}{}^1(\mathcal{X}, F^i)$, $u_1 \in L^2(\mathcal{X}, F^i)$ and $f \in L^2(\mathcal{C}_T, F^i)$. Choose an arbitrary system $(v_k)_{k=1,2,\ldots}$ in $C^2(\mathcal{X}, F^i)$, which satisfies $v_k = 0$ at $\partial\mathcal{X}$ and is complete in

$$\overset{\circ}{H}{}^1(\mathcal{X}, F^i).$$

Given any integer $N \geq 1$, we solve problem (3.1), (3.2) and (3.3) with $u_l = 0$ in the finite-dimensional subspace V_N of $L^2(\mathcal{X}, F^i)$, spanned by the functions $v_1, \ldots, v_N$. More precisely, we look for a function u_N in $H^2(\mathcal{C}_T, F^i)$ such that $u_N(\cdot, t)$ belongs to the subspace V_N for any fixed $t \in [0,T]$; u_N satisfies conditions (3.2) with initial data

$$u_{0,N}(x) = \sum_{k=1}^{N} u_{0,k} v_k(x), \qquad u_{1,N}(x) = \sum_{k=1}^{N} u_{1,k} v_k(x)$$

being orthogonal projections of u_0 and u_1 onto V_N, respectively, and the orthogonal projections of $(u_N)''_{tt} + \mu \varDelta u_N + (\lambda + \mu) A A^* u_N$ and f onto V_N coincide for almost all $t \in [0,T]$. (Note that the orthogonality refers here to the inner product of $L^2(\mathcal{X}, F^i)$.)

Thus, we search for functions $w_1(t), \ldots, w_N(t)$ in $H^2(0,T)$ satisfying $w_k(0) = u_{0,k}$ and $w_k'(0) = u_{1,k}$ for all $k = 1, \ldots, N$, and such that

$$u_N(x,t) = \sum_{k=1}^{N} w_k(t) v_k(x)$$

fulfills

$$\int_{\mathcal{X}} ((u_N)''_{tt} + \mu \Delta u_N + (\lambda + \mu) A A^* u_N, v_k)_x \, dx = \int_{\mathcal{X}} (f, v_k)_x \, dx \tag{5.1}$$

for almost all $t \in [0,T]$ (for which $f(\cdot,t)$ belongs to $L^2(\mathcal{X}, F^i)$), where $k = 1, \ldots, N$. The Galerkin method consists of approximating the solution u of mixed problem (3.1), (3.2) and (3.3) with $u_l = 0$ by solutions u_N of the projected problems. In order to substantiate this method it needs to be shown that each projected problem has a unique solution u_N and that the sequence $(u_N)_{N=1,2,\ldots}$ converges in some sense (weakly in $H^1(\mathcal{C}_T, F^i)$) to u.

For simplicity, we restrict ourselves to the case of homogeneous initial conditions $u_0 = 0$ and $u_1 = 0$. Then, the coefficients $u_{0,k}$ and $u_{1,k}$ vanish, and we are led to the system

$$w_k(0) = 0, \qquad w_k'(0) = 0 \tag{5.2}$$

for all $k = 1, \ldots, N$.

Equations (5.1) constitute a system of second order linear ordinary differential equations with constant coefficients for the unknown functions $w_1(t), \ldots, w_N(t)$. To wit,

$$\sum_{j=1}^{N} \big(w_j''(t) (v_j, v_k)_{L^2(\mathcal{X}, F^i)} + w_j(t) D(v_j, v_k) \big) = f_k(t) \tag{5.3}$$

for $k = 1, \ldots, N$, where

$$f_k(t) = \int_{\mathcal{X}} (f(\cdot,t), v_k)_x \, dx$$

belongs to $L^2(\mathcal{X}, F^i)$.

Our task is to prove that system (5.3) has a unique solution $w_1, \ldots, w_N$ with components in $H^1(0,T)$ satisfying initial conditions (5.2). Since the system $v_1, \ldots, v_N$ is linearly independent for all integers $N \geq 1$, the (Gram-Schmidt) determinant of the $(N \times N)$-matrix with entries $(v_j, v_k)_{L^2(\mathcal{X},F^i)}$ is different from zero. Hence, system (5.3) can be resolved with respect to higher order derivatives. It follows that problem (5.2), (5.3) reduces to the initial problem of canonical form on $[0,T]$, namely,

$$\begin{aligned} W'(t) &= AW(t) + F(t) \quad \text{if } t \in (0,T), \\ W(0) &= 0, \end{aligned} \tag{5.4}$$

where $W = (w', w)^T$ and

$$A = -\begin{pmatrix} 0 & \left((v_j, v_k)_{L^2(\mathcal{X},F^i)}\right)^{-1} (D(v_j, v_k)) \\ E_N & 0 \end{pmatrix}$$

The components of the $2N$-column $F(t)$ belong to $L^2(0,T)$. We look for a solution W of problem (5.4) in $H^1((0,T), \mathbb{C}^{2N})$. As is standard, we replace this problem by the equivalent system of integral equations

$$W(t) = \int_0^t AW(t')\,dt' + \int_0^t F(t')\,dt', \tag{5.5}$$

the free term on the right-hand side belonging to $H^1((0,T), \mathbb{C}^{2N})$ and thus being continuous on $[0,T]$. If $W \in H^1((0,T), \mathbb{C}^{2N})$ is a solution of (5.4), then it is continuous on $[0,T]$ and satisfies equation (5.5). Conversely, if $W : [0,T] \to \mathbb{C}^{2N}$ is a continuous solution of equation (5.5), then it is actually of class $H^1((0,T), \mathbb{C}^{2N})$ and satisfies (5.4). In addition, the existence and uniqueness of a continuous solution to equation (5.4) is a direct consequence of the Banach fixed point theorem. We have thus proved that system (5.3) has a unique solution $w_1, \ldots, w_N$ in $H^1(0,T)$ satisfying (5.2).

Multiply equality (5.1) by $w_k'(t)$, integrate over $t \in (0,t')$, where t' is an arbitrary number of $[0,T]$, and sum up for $k = 1, \ldots, N$. Then, we obtain

$$\begin{aligned} \int_{\mathcal{C}_{t'}} ((u_N)''_{tt} + \mu \Delta u_N + (\lambda{+}\mu) AA^* u_N, (u_N)'_t)_x \,dx\,dt \\ = \int_{\mathcal{C}_{t'}} (f, (u_N)'_t)_x \,dx\,dt. \end{aligned} \tag{5.6}$$

Using the Stokes formula, the real part of the left-hand side of this equality can be transformed into

$$\frac{1}{2}\int_{\mathcal{X}} \left(|(u_N)'_t(x,t')|^2 + \mu|Au_N(x,t')|^2 + (\lambda+2\mu)\,|A^*u_N(x,t')|^2\right)\,dx$$

for all $t' \in [0,T]$. On the subspace $H^1_b(\mathcal{C}_T, F^i)$ of $H^1(\mathcal{C}_T, F^i)$ consisting of those functions which vanish on the lateral boundary of $\mathcal{C}_T$ and its base, the norm can be equivalently given by

$$\|u\|^2_{H^1_b(\mathcal{C}_T,F^i)} = \int_{\mathcal{C}_T} |u'_t|^2 dx\,dt + \int_0^T D(u(\cdot,t))^2 dt,$$

where $D(v)$ is the Dirichlet norm of $v \in \mathring{H}^1(\mathcal{X}, F^i)$. Hence,

$$\Re\int_0^T dt' \int_{\mathcal{C}_{t'}} ((u_N)''_{tt} + \mu\varDelta u_N + (\lambda+\mu)AA^*u_N, (u_N)'_t)_x\,dx\,dt$$

merely amounts to

$$\frac{1}{2}\|u_N\|^2_{H^1_b(\mathcal{C}_T,F^i)},$$

and equality (5.6) yields

$$\begin{aligned}\|u_N\|^2_{H^1_b(\mathcal{C}_T,F^i)} &= 2\Re\int_0^T dt'\int_{\mathcal{C}_{t'}} (f,(u_N)'_t)_x\,dx\,dt\\ &= 2\Re\int_{\mathcal{C}_T} (T-t)\,(f,(u_N)'_t)_x\,dx\,dt\\ &\le 2T\|f\|_{L^2(\mathcal{C}_T,F^i)}\|u_N\|_{H^1_b(\mathcal{C}_T,F^i)},\end{aligned}$$

whence

$$\|u_N\|_{H^1_b(\mathcal{C}_T,F^i)} \le 2T\|f\|_{L^2(\mathcal{C}_T,F^i)}.$$

We have thus proved that the set of functions u_N, where $N = 1,2,\ldots$, is bounded in the Hilbert space $H^1_b(\mathcal{C}_T,F^i)$. Therefore, this set is weakly compact in $H^1_b(\mathcal{C}_T,F^i)$, i.e., it has a subsequence which converges weakly in $H^1_b(\mathcal{C}_T,F^i)$ to a function $u \in H^1_b(\mathcal{C}_T,F^i)$. By abuse of notation, we continue to write u_N for this subsequence.

We claim that u is the desired weak solution of the first mixed problem for the generalized Lamé system. In order to show this it is sufficient to verify that the integral identity

$$\int_{\mathcal{C}_T} \left(-(u'_t, g'_t) + \mu(Au, Ag) + (\lambda + 2\mu)(A^*u, A^*g)\right) dx\,dt = \int_{\mathcal{C}_T} (f, g)\, dx\,dt \tag{5.7}$$

holds for all $g \in H^1(\mathcal{C}_T, F^i)$ which vanish at the lateral boundary of $\mathcal{C}_T$ and the cylinder head, cf., (3.4) with $u_1 = 0$. Let us introduce the temporary notation $H^1_c(\mathcal{C}_T, F^i)$ for the (obviously, closed) subspace of $H^1(\mathcal{C}_T, F^i)$ consisting of all such g. It is actually sufficient to establish (5.7) for all g in a complete subset Σ of $H^1_c(\mathcal{C}_T, F^i)$.

As Σ, we take the set of all functions of the form $z(t)v_k(x)$ where $k \geq 1$ is an integer and $z(t)$ a smooth function on $[0, T]$ satisfying $z(T) = 0$. First, we show that equality (5.7) is true for each function $g(x, t) = z(t)v_k(x)$ and then that the linear combinations of such functions are dense in $H^1_c(\mathcal{C}_T, F^i)$. Toward this end, we multiply equality (5.1) by $z(t)$, integrate it over $t \in (0, T)$ and apply the Stokes formula, obtaining

$$\int_{\mathcal{C}_T} \left(-((u_N)'_t, g'_t) + \mu(Au_N, Ag) + (\lambda + 2\mu)(A^*u_N, A^*g)\right)_x dx\,dt = \int_{\mathcal{C}_T} (f, g)_x\, dx\,dt$$

for all $N \geq k$, where $g = zv_k$. This readily implies (5.7) for $u_N \to u$ weakly in $H^1(\mathcal{C}_T, F^i)$.

Our next goal is to show that the linear hull of Σ is dense in $H^1_c(\mathcal{C}_T, F^i)$. In order to do this it is sufficient to prove that each function $g \in C^2(\overline{\mathcal{C}}_T, F^i)$ vanishing at the lateral boundary of the cylinder and its head (the set of such functions is dense in $H^1_c(\mathcal{C}_T, F^i)$) can be approximated in the $H^1(\mathcal{C}_T, F^i)$-norm by linear combinations of functions in Σ. This last assertion is actually well known within the framework of the theory of Sobolev spaces. For a proof, the interested reader is referred to [**12,** page 302].

Remark 5.1. Since the weak solution of the first mixed problem exists and is unique, not only a subsequence but also the sequence $(u_N)_{N=1,2,\ldots}$ itself converges weakly in $H^1(\mathcal{C}_T, F^i)$ to u.

6. Regularity of weak solutions. Assume that the boundary $\partial\mathcal{X}$ of $\mathcal{X}$ is of class C^s for some integer $s \geq 1$. Then the eigenfunctions $(v_k)_{k=1,2,\dots}$ of problem (4.1) belong to $H^s(\mathcal{X}, F^i)$ and satisfy the boundary conditions

$$(6.1) \qquad L^i v_k = 0 \quad \text{at} \quad \partial\mathcal{X} \quad \text{for } i = 0, 1, \dots, \left[\frac{s-1}{2}\right].$$

Let $H^s_{\mathcal{D}}(\mathcal{X}, F^i)$ stand for the subspace of $H^s(\mathcal{X}, F^i)$ consisting of all functions v satisfying (6.1). We place additional restrictions on the data of the problem to attain to a classical solution. More precisely, we require that $u_0 \in H^s_{\mathcal{D}}(\mathcal{X}, F^i)$, $u_1 \in H^{s-1}_{\mathcal{D}}(\mathcal{X}, F^i)$ and f belongs to the subspace of $H^{s-1}(\mathcal{C}_T, F^i)$ consisting of all functions satisfying

$$(6.2) \qquad L^i f = 0 \quad \text{at} \quad \partial\mathcal{X} \times (0, T) \quad \text{for } i = 0, 1, \dots, \left[\frac{s}{2}\right] - 1.$$

For $s = 1$, the latter equations are null, and we arrive at $f \in L^2(\mathcal{X}, F^i)$, as above.

Theorem 6.1. *Under the above hypotheses, series* (4.8) *converges to the weak solution $u(x,t)$ in $H^s(\mathcal{X}, F^i)$ uniformly in $t \in [0, T]$. Given any $j = 1, \dots, s$, the series obtained from (4.8) by the j-fold termwise differentiation in t converges in $H^{s-j}(\mathcal{X}, F^i)$ uniformly in $t \in [0, T]$. Moreover, there is a constant $c > 0$ independent of t, such that*

$$(6.3) \quad \sum_{j=0}^{s} \Big\| \sum_{k=1}^{\infty} w_k^{(j)}(t) v_k \Big\|^2_{H^{s-j}(\mathcal{X}, F^i)} \leq c\big(\|u_0\|^2_{H^s(\mathcal{X}, F^i)} + \|u_1\|^2_{H^{s-1}(\mathcal{X}, F^i)} + \|f\|^2_{H^{s-1}(\mathcal{C}_T, F^i)}\big)$$

for all $t \in [0, T]$.

Proof. The proof of this theorem runs similarly to the proof of **[12,** page 305, Theorem 3], the techniques developed earlier in Sections 3 and 4 are exploited. □

From (6.3), if $1 \leq M < N$, then

$$\sup_{t\in[0,T]} \Big\| \sum_{k=M+1}^{N} w_k^{(j)}(t) v_k \Big\|^2_{H^{s-j}(\mathcal{X}, F^i)} \longrightarrow 0,$$

as $M \to \infty$. Hence, the partial sums of series (4.8) converge in $H^s(\mathcal{C}_T, F^i)$ and, from (6.3), it follows that

(6.4)
$$\|u\|_{H^s(\mathcal{C}_T,F^i)} \le c' \left(\|u_0\|_{H^s(\mathcal{X},F^i)} + \|u_1\|_{H^{s-1}(\mathcal{X},F^i)} + \|f\|_{H^{s-1}(\mathcal{C}_T,F^i)}\right).$$

Corollary 6.2. *Under the above hypotheses, the weak solution of the first mixed problem for the generalized Lamé system belongs to $H^s(\mathcal{C}_T, F^i)$. Moreover, the series* (4.8) *converges to the weak solution in the $H^s(\mathcal{C}_T, F^i)$-norm, and inequality* (6.4) *holds true.*

From Corollary 6.2 with $s = 2$, it follows that the weak solution of the first mixed problem belongs to $H^2(\mathcal{C}_T, F^i)$, and thus, it is a solution almost everywhere. If, moreover, $s > n/2 + 2$, then the weak solution u belongs to the space $C^2(\overline{\mathcal{C}}_T, F^i)$, which is due to the Sobolev embedding theorem, and thus, u is a classical solution of the problem.

Note that, along with the smoothness of u_0, u_1 and f, Theorem 6.1 assumes that u_0 satisfies (6.1), u_1 satisfies (6.1) with s replaced by $s-1$ and f satisfies (6.2). The conditions are actually necessary. In order to show this, suppose that $s \ge 2$. Since $u_0(x) = u(x, 0)$ is represented by the series (4.8) which converges in $H^s(\mathcal{X}, F^i)$, and $u_1(x) = u'_t(x, 0)$ is represented by the series (4.8) which is differentiated termwise in t and converges in $H^{s-1}(\mathcal{X}, F^i)$, we readily conclude that u_0 satisfies (6.1) and u_1 satisfies (6.1) with s replaced by $s - 1$. Furthermore, since the series (4.8) converges to u in $H^s(\mathcal{C}_T, F^i)$, the series obtained from (4.8) by termwise applying the operators L and the second derivative in t converge in $H^{s-2}(\mathcal{C}_T, F^i)$ to Lu and u''_{tt}, respectively. Hence, if $s \ge 3$, then $f = u''_{tt} - Lu$ satisfies equalities (6.2) with s replaced by $s - 1$. In the case where s is even, the last condition of (6.2) is superfluous indeed, see [**12,** page 311, Corollary 2].

However, if the smoothness of the weak solution of the first mixed problem rather than the convergence of the Fourier series in the corresponding spaces is the focus of the proof, then conditions (6.1) and (6.2) can essentially be relaxed, see [**12,** page 323, Theorem 3′].

7. Reduction to the Schrödinger equation. There is a Lie algebraic connection between the wave equation and the Schrödinger equation. This allows for the construction of solutions of hyperbolic equations from solutions of the Schrödinger equation.

From the above, the unbounded operator $-L$ in $L^2(\mathcal{X}, F^i)$, whose domain is the set of all sections $v \in H^2(\mathcal{X}, F^i)$ vanishing at $\partial\mathcal{X}$, is closed, selfadjoint and positive, i.e., we have $-L \geq cI$ where c is a positive constant. Denote by $\sqrt{-L}$ the square root of $-L$, and impose upon the domain $\mathcal{D}_{\sqrt{-L}}$ of this operator a Hilbert space structure by identifying it with the range of $\sqrt{-L}$, i.e., the norm in $\mathcal{D}_{\sqrt{-L}}$ merely amounts to

$$D(v) = \|\sqrt{-L}v\|_{L^2(\mathcal{X},F^i)}.$$

Now, we split the solution of the first mixed problem (3.1), (3.2) and (3.3), with $u_l = 0$, into two parts. To wit, we are looking for two differentiable functions

$$F,\ U : [0,T] \longrightarrow L^2(\mathcal{X}, F^i)$$

with values in $\mathcal{D}_{\sqrt{-1}}$, i.e., curves in $L^2(\mathcal{X}, F^i)$, which satisfy

$$\begin{aligned} F'_t &= -\imath\sqrt{-L}F + f \quad \text{for } t \in (0,T), \\ F(0) &= u_1 - \imath\sqrt{-L}u_0, \end{aligned} \tag{7.1}$$

and

$$\begin{aligned} U'_t &= \imath\sqrt{-L}U + F \quad \text{for } t \in (0,T), \\ U(0) &= u_0. \end{aligned} \tag{7.2}$$

If $U : [0,T] \to L^2(\mathcal{X}, F^i)$ is twice differentiable in $t \in (0,T)$, then combining (7.1) and (7.2) yields

$$\begin{aligned} U''_{tt} &= \imath\sqrt{-L}U'_t + F'_t \\ &= \imath\sqrt{-L}\left(\imath\sqrt{-L}\,U + F\right) - \imath\sqrt{-L}F + f \\ &= LU + f \end{aligned}$$

in $(0,T)$ and

$$U(0) = u_0, \qquad U'(0) = \imath\sqrt{-L}\,u_0 + F(0) = u_1.$$

It follows that $u = U$ is a solution to the first mixed problem for the generalized Lamé system in $\mathcal{C}_T$.

It is worth pointing out that $\pm\imath\sqrt{-L}$ are skew-symmetric operators in $L^2(\mathcal{X}, F^i)$. For direct constructions along more classical lines, the interested reader is referred to **[1, 4, 5, 6]**.

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