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SOME FINITE GENERALIZATIONS OF GAUSS'S SQUARE EXPONENT IDENTITY

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ABSTRACT. We obtain three finite generalizations of Gauss's square exponent identity. For example, we prove that, for any non-negative integer m,

$$\sum_{k=-m}^{m}(-1)^k \begin{bmatrix} 3m-k+1 \\ m+k \end{bmatrix} (-q;q)_{m-k}q^{k^2}=1,$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \prod_{k=1}^{m}\frac{1-q^{n-k+1}}{1-q^k} \quad \text{and} \quad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k).$$

These identities reduce to Gauss's famous identity

$$\sum_{k=-\infty}^{\infty}(-1)^k q^{k^2}=\frac{(q;q)_\infty}{(-q;q)_\infty}$$

by letting $m\to\infty$.

1. Introduction. Euler's pentagonal number theorem [**1,** Corollary 1.7] plays an important role in the partition theory

$$\sum_{j=-\infty}^{\infty}(-1)^j q^{j(3j+1)/2}=(q;q)_\infty. \tag{1.1}$$

Here, and throughout the note, we use the following q-series notation:

$$(a;q)_0=1, \qquad (a;q)_n=\prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k),$$

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and

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}} & \text{if } 0 \leqslant m \leqslant n, \\ 0 & \text{otherwise.} \end{cases}$$

Berkovich and Garvan [**6**] found some finite generalizations of Euler's pentagonal number theorem. For example, they proved that

$$\sum_{j=-L}^{L} (-1)^j \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} q^{j(3j+1)/2} = 1. \tag{1.2}$$

Note that

$$\lim_{L\to\infty} \begin{bmatrix} 2L-j \\ L+j \end{bmatrix} = \frac{1}{(q;q)_\infty}.$$

Thus, letting $L \to \infty$ in (1.2) reduces to (1.1). By using a well-known cubic summation formula, Warnaar [**14**] obtained another interesting finite generalization of Euler's pentagonal number theorem.

Gauss's triangular exponent identity and square exponent identity [**1,** Corollary 2.10] are stated as follows

$$\sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.3}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}. \tag{1.4}$$

In particular, identity (1.4) can be used to prove Lagrange's four-square theorem, see [**7,** page 106]. Euler's pentagonal number theorem (1.1) and Gauss's identities (1.3)–(1.4) were historically spectacular achievements at the time of their discovery. However, with progress, the following Jacobi's triple product identity [**1,** Theorem 2.8] implies all of them. For $z \neq 0$ and $|q| < 1$,

$$\sum_{k=-\infty}^{\infty} z^k q^{k^2} = (q^2;q^2)_\infty(-zq;q^2)_\infty(-q/z;q^2)_\infty.$$

Shanks **[12]** proved that

$$\sum_{s=0}^{n-1} \frac{P_n}{P_s} q^{s(2n+1)} = \sum_{s=1}^{2n} q^{s(s-1)/2},$$

where $P_n = (q^2;q^2)_n/(q;q^2)_n$. Identity (1.3) directly follows from the above finite identity.

Some similar finite identities have been widely studied by several authors, see, for example, **[2, 5, 9, 10, 11, 13, 15]**.

Motivated by the work of Berkovich and Garvan **[6]** and Warnaar **[14]**, we shall prove three similar finite generalizations for Gauss's square exponent identity.

Theorem 1.1. *For any positive integer m and complex number q, we have*

$$\sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k+1 \\ m+k \end{bmatrix} (-q;q)_{m-k} q^{k^2} = 1, \tag{1.5}$$

$$(1-q^{2m}) \sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k \\ m+k \end{bmatrix} \frac{(-q;q)_{m-k}}{1-q^{3m-k}} q^{k^2} = 1, \tag{1.6}$$

$$\sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k-1 \\ m+k-1 \end{bmatrix} (-q;q)_{m-k} q^{k^2} = 1. \tag{1.7}$$

If $|q| < 1$, then (1.5)–(1.7) reduce to (1.4) by letting $m \to \infty$. Replacing k by $-m+k$ on the left-hand sides of (1.5)–(1.6), and k by $-m+k+1$ on the left-hand side of (1.7), we can rewrite Theorem 1.1 as:

$$\sum_{k=0}^{2m} (-1)^{m+k} \frac{(q;q)_{4m-k+1}(-q;q)_{2m-k} q^{(m-k)^2}}{(q;q)_{4m-2k+1}(q;q)_k} = 1,$$

$$(1-q^{2m}) \sum_{k=0}^{2m} (-1)^{m+k} \frac{(q;q)_{4m-k}(-q;q)_{2m-k} q^{(m-k)^2}}{(q;q)_{4m-2k}(q;q)_k (1-q^{4m-k})} = 1,$$

$$\sum_{k=0}^{2m-1} (-1)^{m+k+1} \frac{(q;q)_{4m-k-2}(-q;q)_{2m-k-1} q^{(m-k-1)^2}}{(q;q)_{4m-2k-2}(q;q)_k} = 1.$$

These basic hypergeometric forms of Theorem 1.1 may be helpful in finding the highest reasonable level of generality of such identities.

2. Proof of Theorem 1.1. In order to prove Theorem 1.1, we need the following lemma.

Lemma 2.1. ([**1,** page 35]). *Let* $0 \le m \le n$ *be integers. Then,*

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m-1 \end{bmatrix} + q^m \begin{bmatrix} n-1 \\ m, \end{bmatrix}$$
$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n-1 \\ m \end{bmatrix} + q^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}.$$

Proof of Theorem 1.1. In fact, (1.5)–(1.7) can be deduced from the following identities:

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix} (-q;q)_{n-k} q^{\binom{k}{2}} = \begin{cases} 0 & \text{if } n = 2m-1, \\ (-1)^m q^{m(3m+1)} & \text{if } n = 2m, \end{cases} \tag{2.1}$$

$$(1-q^n) \sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} \frac{(-q;q)_{n-k}}{1-q^{2n-k}} q^{\binom{k}{2}} = \begin{cases} 0 & \text{if } n = 2m-1, \\ (-1)^m q^{m(3m-1)} & \text{if } n = 2m, \end{cases} \tag{2.2}$$

$$\sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q)_{n-k} q^{\binom{k}{2}} = \begin{cases} (-1)^{m-1} q^{3m^2-3m+1} & \text{if } n = 2m-1, \\ (-1)^m q^{m(3m-1)} & \text{if } n = 2m. \end{cases} \tag{2.3}$$

Before proving these results, we shall draw conclusions from them.

Replacing n by $2m$ in (2.1) and (2.2), and n by $2m-1$ in (2.3), respectively, and then letting $k \to m+k$ in (2.1) and (2.2), and

$k \to m+k-1$ in (2.3), respectively, we obtain

$$\sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k+1 \\ m+k \end{bmatrix} (-q;q)_{m-k} q^{\binom{m+k}{2}} = q^{m(3m+1)}, \tag{2.4}$$

$$(1-q^{2m}) \sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k \\ m+k \end{bmatrix} \frac{(-q;q)_{m-k}}{1-q^{3m-k}} q^{\binom{m+k}{2}} = q^{m(3m-1)}, \tag{2.5}$$

$$\sum_{k=-m}^{m} (-1)^k \begin{bmatrix} 3m-k-1 \\ m+k-1 \end{bmatrix} (-q;q)_{m-k} q^{\binom{m+k-1}{2}} = q^{3m^2-3m+1}. \tag{2.6}$$

It is easy to verify that

$$\begin{bmatrix} n \\ m \end{bmatrix}_{q^{-1}} = \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)} \quad \text{and} \quad (-q^{-1};q^{-1})_n = (-q;q)_n q^{-\binom{n+1}{2}}. \tag{2.7}$$

Replacing q by q^{-1} in (2.4)–(2.6) and then noting (2.7), we immediately obtain (1.5)–(1.7).

Thus, it remains to prove (2.1)–(2.3). Denote the left-hand sides of (2.1), (2.2) and (2.3) by U_n, V_n and W_n, respectively. We shall prove these three identities by establishing the following relationships:

$$(1+q^n)V_n = W_n - q^{2n-1}W_{n-1}, \tag{2.8}$$

$$V_n = -q^{2n-2}U_{n-2}, \tag{2.9}$$

$$W_n = q^n U_{n-1} - q^{2n-2}U_{n-2}. \tag{2.10}$$

From (2.8)–(2.9), we obtain

$$U_n = -q^{3n-2}U_{n-2} \quad \text{for } n \geq 2.$$

We can deduce (2.1) by induction from the initial values $U_0 = 1$ and $U_1 = 0$. Substituting (2.1) into (2.9) and (2.10), we obtain (2.2) and (2.3) directly. Therefore, it suffices to prove (2.8)–(2.10).

Since $1-q^{2n} = 1-q^{2n-k} + q^{2n-k}(1-q^k)$, we have

$$\begin{aligned}(1+q^n)&V_n \\ &= \sum_{k=0}^{n} (-1)^k \begin{bmatrix} 2n-k \\ k \end{bmatrix} (-q;q)_{n-k} q^{\binom{k}{2}}\end{aligned}$$

$$
\begin{aligned}
&\quad - q^{2n-1}\sum_{k=1}^{n}(-1)^{k-1}\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}(-q;q)_{n-k}q^{\binom{k-1}{2}}\\
&= W_n - q^{2n-1}\sum_{k=0}^{n-1}(-1)^{k}\begin{bmatrix}2n-k-2\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k}{2}}\\
&= W_n - q^{2n-1}W_{n-1}.
\end{aligned}
$$

This concludes the proof of (2.8).

It is easy to verify that

$$
\begin{bmatrix}2n-k\\k\end{bmatrix}\frac{1-q^n}{1-q^{2n-k}} = \begin{bmatrix}2n-k-1\\k-1\end{bmatrix} + \begin{bmatrix}2n-k-1\\k\end{bmatrix}\frac{q^k}{1+q^{n-k}}. \tag{2.11}
$$

Substituting (2.11) into the left-hand side of (2.2), we obtain

$$
\begin{aligned}
V_n &= \sum_{k=1}^{n}(-1)^{k}\begin{bmatrix}2n-k-1\\k-1\end{bmatrix}(-q;q)_{n-k}q^{\binom{k}{2}}\\
&\quad + \sum_{k=0}^{n-1}(-1)^{k}\begin{bmatrix}2n-k-1\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}}\\
&= -\sum_{k=0}^{n-1}(-1)^{k}\begin{bmatrix}2n-k-2\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}}\\
&\quad + \sum_{k=0}^{n-1}(-1)^{k}\begin{bmatrix}2n-k-1\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}}.
\end{aligned}
$$

From Lemma 2.1, we have

$$
\begin{bmatrix}2n-k-1\\k\end{bmatrix} - \begin{bmatrix}2n-k-2\\k\end{bmatrix} = \begin{bmatrix}2n-k-2\\k-1\end{bmatrix}q^{2n-2k-1}. \tag{2.12}
$$

It follows that

$$
\begin{aligned}
V_n &= q^{2n-2}\sum_{k=1}^{n-1}(-1)^{k}\begin{bmatrix}2n-k-2\\k-1\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k-1}{2}}\\
&= -q^{2n-2}\sum_{k=0}^{n-2}(-1)^{k}\begin{bmatrix}2n-k-3\\k\end{bmatrix}(-q;q)_{n-k-2}q^{\binom{k}{2}}\\
&= -q^{2n-2}U_{n-2}.
\end{aligned}
$$

This proves (2.9).

From Lemma 2.1, we get

$$\begin{aligned}\begin{bmatrix}2n-k\\k\end{bmatrix}(1+q^{n-k}) - \begin{bmatrix}2n-k-1\\k\end{bmatrix}q^n \\ = \begin{bmatrix}2n-k-1\\k-1\end{bmatrix}(1+q^{n-k}) + \begin{bmatrix}2n-k-1\\k\end{bmatrix}q^k.\end{aligned}$$

Thus, we have

$$\begin{aligned}W_n - q^n U_{n-1} &= \sum_{k=1}^{n}(-1)^k \begin{bmatrix}2n-k-1\\k-1\end{bmatrix}(-q;q)_{n-k}q^{\binom{k}{2}} \\ &\quad + \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}2n-k-1\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}} \\ &= -\sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}2n-k-2\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}} \\ &\quad + \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix}2n-k-1\\k\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k+1}{2}}.\end{aligned}$$

From (2.12), we have

$$\begin{aligned}W_n - q^n U_{n-1} &= q^{2n-2}\sum_{k=1}^{n-1}(-1)^k \begin{bmatrix}2n-k-2\\k-1\end{bmatrix}(-q;q)_{n-k-1}q^{\binom{k-1}{2}} \\ &= -q^{2n-2}\sum_{k=0}^{n-2}(-1)^k \begin{bmatrix}2n-k-3\\k\end{bmatrix}(-q;q)_{n-k-2}q^{\binom{k}{2}} \\ &= -q^{2n-2}U_{n-2},\end{aligned}$$

which is (2.10). □

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