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MIXED CHORD-INTEGRALS OF INDEX i AND RADIAL BLASCHKE-MINKOWSKI HOMOMORPHISMS

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ABSTRACT. In this paper, two Brunn-Minkowski type inequalities for mixed chord-integrals of index i are established, which are related to the radial Blaschke-Minkowski homomorphisms of star bodies. Moreover, two inequalities similar to Giannopoulos, Hartzoulaki and Paouris's inequality are also considered.

1. Introduction and main results. Let $\mathcal{K}^n$ be the set of convex bodies, which are compact, convex subsets of $\mathbb{R}^n$ with nonempty interiors. S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Denote by $V(K)$ the n-dimensional volume of the body K. For the standard unit ball B in $\mathbb{R}^n$, we write $\omega_n = V(B)$ to denote its volume.

If K is a compact star-shaped (about the origin) set in $\mathbb{R}^n$, then its radial function

$$\rho_K = \rho(K, \cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow [0, \infty)$$

is defined by **[11, 32]**

$$\rho(K, u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

A set K in $\mathbb{R}^n$ is said to be a star body about the origin if the line segment from the origin to any point $x \in K$ is contained in K and K has a continuous and positive radial function ρ_K. The set of all star bodies (about the origin) in $\mathbb{R}^n$ is denoted by $\mathcal{S}^n$. Two star bodies $K, L \in \mathcal{S}^n$

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are said to be dilates of each other if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

Lutwak [**23**] introduced the notion of mixed width-integrals of convex bodies. Motivated by Lutwak's ideas, the notion of mixed chord-integrals of star bodies was recently defined in [**20**]. For $K_1, \ldots, K_n \in \mathcal{S}^n$, the mixed chord-integral, $C(K_1, \ldots, K_n)$, of $K_1, \ldots, K_n$ is defined by

$$C(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} c(K_1, u) \cdots c(K_n, u)\, dS(u), \tag{1.1}$$

where $dS(u)$ is the $(n-1)$-dimensional volume element on S^{n-1}, and $c(K, u)$ denotes the half chord of K in direction u, namely, $c(K, u) = \rho(K, u)/2 + \rho(K, -u)/2$.

When taking $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = B$ in (1.1), we write $C_i(K)$ for

$$C(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B \cdots B}_{i}),$$

which will be called the mixed chord-integrals of index i whose integral representation is

$$C_i(K) = \frac{1}{n} \int_{S^{n-1}} c(K, u)^{n-i} dS(u), \tag{1.2}$$

The map

$$C_i : \mathcal{S}^n \longrightarrow \mathbb{R}$$

is continuous, positive and homogeneous of degree $n - i$. Note that it is not invariant under any motion. In particular, it is not even invariant under translation. If there exists a constant $\lambda > 0$ such that $c(K, u) = \lambda c(L, u)$ for all $u \in S^{n-1}$, then we say that K and L have similar chords.

Schuster [**33**] introduced the definition of radial Blaschke-Minkowski homomorphisms as follows: a map

$$\Psi : \mathcal{S}^n \longrightarrow \mathcal{S}^n$$

is called a radial Blaschke-Minkowski homomorphism if it satisfies the following conditions.

(a) Ψ is continuous.

(b) For all $K, L \in \mathcal{S}^n$,

$$\Psi(K \breve{+} L) = \Psi K \tilde{+} \Psi L.$$

(c) For all $K \in \mathcal{S}^n$ and $\vartheta \in SO(n)$,

$$\Psi(\vartheta K) = \vartheta \Psi K.$$

Here, $\Psi K \tilde{+} \Psi L$ is the radial Minkowski sum, see (2.5), of ΨK and ΨL, and $K \breve{+} L$ denotes the radial Blaschke sum of $K, L \in \mathcal{S}^n$, see (2.6).

Radial Blaschke-Minkowski homomorphisms are an important notion in the theory of real-valued valuations. A systematic study was initiated by Blaschke in the 1930s and continued by Hadwiger, culminating in his famous classification of continuous and rigid motion invariant valuations on convex bodies. The surveys, **[26, 27]** and the book **[17]** are excellent sources for the classical theory of valuations. For some of the more recent results, see **[1, 2, 3, 4, 13, 14, 15, 16, 21, 22, 30, 31, 34, 35, 36]**.

The classical Brunn-Minkowski inequality states that, if $K, L \in \mathcal{K}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},$$

with equality if and only if K and L are homothetic. Here, $K + L$ denotes the Minkowski sum of K and L, see (2.3).

The Brunn-Minkowski inequality is one of the most powerful results in convex geometry. For extensive and beautiful surveys on it, the interested reader is referred to **[10]** in which the history of the Brunn-Minkowski inequality and some applications in other fields, such as probability and multivariate statistics, geometric tomography, elliptic partial differential equations, combinatorics, interacting gases, shapes of crystals and algebraic geometry, are summarized. Among many others, it has been key in the development of the Brunn-Minkowski theory, see **[7, 8, 18, 25, 37, 38, 39, 40, 41]**.

The aim of this paper is to establish the following Brunn-Minkowski type inequalities for the mixed chord-integrals of index i associated with the Blaschke-Minkowski homomorphisms.

Theorem 1.1. *If $K, L \in \mathcal{S}^n$ and $i, j \in \mathbb{R}$, then, for $i \le n-1 \le j \le n$ and $i \ne j$,*

(1.3)
$$\left(\frac{C_i(\Psi(K\breve{+}L))}{C_j(\Psi(K\breve{+}L))}\right)^{1/(j-i)} \le \left(\frac{C_i(\Psi K)}{C_j(\Psi K)}\right)^{1/(j-i)} + \left(\frac{C_i(\Psi L)}{C_j(\Psi L)}\right)^{1/(j-i)};$$

for $n-1 \le i \le n \le j$ and $i \ne j$,

(1.4)
$$\left(\frac{C_i(\Psi(K\breve{+}L))}{C_j(\Psi(K\breve{+}L))}\right)^{1/(j-i)} \ge \left(\frac{C_i(\Psi K)}{C_j(\Psi K)}\right)^{1/(j-i)} + \left(\frac{C_i(\Psi L)}{C_j(\Psi L)}\right)^{1/(j-i)},$$

with equality in every inequality if and only if ΨK and ΨL have similar chords.

Here, Ψ (and the Ψ in the following theorems) is a radial Blaschke-Minkowski homomorphism.

The next result of Giannopoulos, et al., **[12]** motivated the current work.

If K is a convex body in $\mathbb{R}^n$ and L is an n-ball in $\mathbb{R}^n$, then, for $k = 0, \ldots, n-1$,

$$\frac{W_k(K+L)}{W_{k+1}(K+L)} \ge \frac{W_k(K)}{W_{k+1}(K)} + \frac{W_k(L)}{W_{k+1}(L)}. \tag{1.5}$$

Inequality (1.5) does not hold for any arbitrary pair of nonempty compact convex sets K and L. However, (1.5) holds for arbitrary K and L if $k = n-2$ or $k = n-1$, see **[9]**. Here, $W_k(K)$ denotes the quermassintegrals of $K \in \mathcal{K}^n$, see (2.1).

We shall establish two analogous versions of the Giannopoulos, Hartzoulaki, Paouris inequality (1.5), which are related to the radial Blaschke-Minkowski homomorphisms for the mixed chord-integrals of index i.

Theorem 1.2. *If K and L are two star bodies in $\mathbb{R}^n$, then, for $k = n-2$ or $k = n-1$,*

$$\frac{C_k(\Psi(K\breve{+}L))}{C_{k+1}(\Psi(K\breve{+}L))} \le \frac{C_k(\Psi K)}{C_{k+1}(\Psi K)} + \frac{C_k(\Psi L)}{C_{k+1}(\Psi L))}. \tag{1.6}$$

Theorem 1.3. *Let K be a star body and L an n-ball in $\mathbb{R}^n$. Then, for all $k = 0, \ldots, n-1$,*

$$\frac{C_k(\Psi(K\breve{+}L))}{C_{k+1}(\Psi(K\breve{+}L))} \leq \frac{C_k(\Psi K)}{C_{k+1}(\Psi K)} + \frac{C_k(\Psi L)}{C_{k+1}(\Psi L)}. \tag{1.7}$$

The proofs of Theorems 1.1–1.3 will be given in Section 3 of this paper.

2. Preliminaries. We first collect some basic facts concerning the Brunn-Minkowski theory. For general references, we recommend the books by Gardner [**11**] and Schneider [**32**].

The support function

$$h_K = h(K, \cdot) : \mathbb{R}^n \longrightarrow (-\infty, \infty)$$

of a convex body $K \in \mathcal{K}^n$ is defined by

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad x \in \mathbb{R}^n,$$

where $x \cdot y$ denotes the standard inner product of x and y.

For $K \in \mathcal{K}^n$ and $i = 0, 1, \ldots, n-1$, the quermassintegrals $W_i(K)$ of K are

$$W_i(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)\, dS_i(K, u), \tag{2.1}$$

where $S_i(K, \cdot)$ denotes the mixed surface area measure of K. In addition, we know that

$$W_0(K) = \frac{1}{n} \int_{S^{n-1}} h(K, u)\, dS(K, u) = V(K). \tag{2.2}$$

For $K_1, K_2 \in \mathcal{K}^n$ and $\lambda_1, \lambda_2 \geq 0$ (only one of which is zero), the support function of the Minkowski linear combination $\lambda_1 K_1 + \lambda_2 K_2$ is

$$h(\lambda_1 K_1 + \lambda_2 K_2, \cdot) = \lambda_1 h(K_1, \cdot) + \lambda_2 h(K_2, \cdot). \tag{2.3}$$

The polar coordinate formula for the volume of a body $K \in \mathcal{S}^n$ is

$$V(K) = \frac{1}{n} \int_{S^{n-1}} \rho(K, u)^n dS(u). \tag{2.4}$$

For $L_1, L_2 \in \mathcal{S}^n$ and $\lambda_1, \lambda_2 \geq 0$ (only one of which is zero), the radial Minkowski linear combination $\lambda_1 L_1 \tilde{+} \lambda_2 L_2$ is the star body defined by

$$\rho(\lambda_1 L_1 \tilde{+} \lambda_2 L_2, \cdot) = \lambda_1 \rho(L_1, \cdot) + \lambda_2 \rho(L_2, \cdot). \tag{2.5}$$

If $\lambda_1, \lambda_2 \geq 0$ (only one of which is zero), then the radial Blaschke linear combination $\lambda_1 \circ L_1 \breve{+} \lambda_2 \circ L_2$ of $L_1, L_2 \in \mathcal{S}^n$ is the star body whose radial function satisfies

$$\rho^{n-1}(\lambda_1 \circ L_1 \breve{+} \lambda_2 \circ L_2, \cdot) = \lambda_1 \rho^{n-1}(L_1, \cdot) + \lambda_2 \rho^{n-1}(L_2, \cdot). \tag{2.6}$$

3. Proofs of main results. In order to prove Theorems 1.1, 1.2 and 1.3, we first introduce the following lemmas.

The Beckenbach-Dresher inequality **[6]** is an extension of Beckenbach's inequality **[5]** which was proved by Dresher through the method of moment-space techniques.

Lemma 3.1. (*The Beckenbach-Dresher inequality*). *If $p \geq 1 \geq r \geq 0$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{1/(p-r)} \leq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{1/(p-r)}, \tag{3.1}$$

with equality if and only if the functions f and g are positively proportional.

Here, $\mathbb{E}$ is a bounded measurable subset in $\mathbb{R}^n$.

The inverse Beckenbach-Dresher inequality was established in **[19]**.

Lemma 3.2. (*The inverse Beckenbach-Dresher inequality*). *If $1 \geq p \geq 0 \geq r$, $p \neq r$, $f, g \geq 0$, and ϕ is a distribution function, then*

$$\left(\frac{\int_{\mathbb{E}}(f+g)^p d\phi}{\int_{\mathbb{E}}(f+g)^r d\phi}\right)^{1/(p-r)} \geq \left(\frac{\int_{\mathbb{E}} f^p d\phi}{\int_{\mathbb{E}} f^r d\phi}\right)^{1/(p-r)} + \left(\frac{\int_{\mathbb{E}} g^p d\phi}{\int_{\mathbb{E}} g^r d\phi}\right)^{1/(p-r)}, \tag{3.2}$$

with equality if and only if the functions f and g are positively proportional.

Lemma 3.3 ([**33**])**.**

$$\Psi : \mathcal{S}^n \longrightarrow \mathcal{S}^n$$

is a radial Blaschke-Minkowski homomorphism if and only if there is a measure $\mu \in \mathcal{M}_+(S^{n-1}, \widehat{e})$ *such that*

$$\rho(\Psi K, \cdot) = \rho^{n-1}(K, \cdot) * \mu, \tag{3.3}$$

where $\mathcal{M}_+(S^{n-1}, \widehat{e})$ *denotes the set of nonnegative zonal measures on* S^{n-1}*.*

Proof of Theorem 1.1*.* Combining (2.6) with (3.3), we have

$$\Psi(K \breve{+} L) = \Psi K \tilde{+} \Psi L.$$

Thus, it follows from (1.2) that, for $p \geq 1 \geq r \geq 0$,

$$\begin{aligned} C_{n-p}(\Psi(K \breve{+} L)) &= \frac{1}{n} \int_{S^{n-1}} c(\Psi K \tilde{+} \Psi L, u)^p dS(u) \\ &= \frac{1}{n} \int_{S^{n-1}} (c(\Psi K, u) + c(\Psi L, u))^p dS(u). \end{aligned} \tag{3.4}$$

Similarly,

$$C_{n-r}(\Psi(K \breve{+} L)) = \frac{1}{n} \int_{S^{n-1}} (c(\Psi K, u) + c(\Psi L, u))^r dS(u). \tag{3.5}$$

From Lemma 3.1, (3.4) and (3.5), we get

$$\begin{aligned} &\left(\frac{C_{n-p}(\Psi(K \breve{+} L))}{C_{n-r}(\Psi(K \breve{+} L))} \right)^{1/(p-r)} \\ &\quad = \left(\frac{\int_{S^{n-1}} (c(\Psi K, u) + c(\Psi L, u))^p dS(u)}{\int_{S^{n-1}} (c(\Psi K, u) + c(\Psi L, u))^r dS(u)} \right)^{1/(p-r)} \\ &\quad \leq \left(\frac{\int_{S^{n-1}} c(\Psi K, u)^p dS(u)}{\int_{S^{n-1}} c(\Psi K, u)^r dS(u)} \right)^{1/(p-r)} + \left(\frac{\int_{S^{n-1}} c(\Psi L, u)^p dS(u)}{\int_{S^{n-1}} c(\Psi L, u)^r dS(u)} \right)^{1/(p-r)} \\ &\quad = \left(\frac{C_{n-p}(\Psi K)}{C_{n-r}(\Psi K)} \right)^{1/(p-r)} + \left(\frac{C_{n-p}(\Psi L)}{C_{n-r}(\Psi L)} \right)^{1/(p-r)}. \end{aligned} \tag{3.6}$$

Suppose that $p = n - i$ and $r = n - j$. From $0 \leq r \leq 1 \leq p$ and $p \neq r$, we have that $i \leq n - 1 \leq j \leq n$ and $i \neq j$. Let $p = n - i$ and $r = n - j$

in (3.6). Then, inequality (1.3) is given. Similar to the above method, the inequality (1.4) follows from Lemma 3.2.

The equality conditions of Lemmas 3.1 and 3.2 imply that equality holds in inequalities (1.3) and (1.4) if and only if $c(\Psi K, u)$ and $c(\Psi L, u)$ are positively proportional, namely, ΨK and ΨL have similar chords. Therefore, equality holds in every inequality if and only if ΨK and ΨL have similar chords. □

If $j = n$ in (1.3), then

$$C_n(\Psi(K\check{+}L)) = C_n(\Psi K) = C_n(\Psi L) = \frac{1}{n}\int_{S^{n-1}} dS(u) = \omega_n$$

is a constant. Thus, we have the following fact.

Corollary 3.4. *If $K, L \in \mathcal{S}^n$, then, for $i \leq n-1$,*

$$C_i(\Psi(K\check{+}L))^{1/(n-i)} \leq C_i(\Psi K)^{1/(n-i)} + C_i(\Psi L)^{1/(n-i)}, \tag{3.7}$$

with equality if and only if ΨK and ΨL have similar chords.

For $K \in \mathcal{S}^n$, the intersection body of K, IK, is the origin symmetric star body whose radial function on S^{n-1} is given by, see **[24]**,

$$\rho(IK, u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho(K, u)^{n-1} d\lambda_{n-2}(u), \tag{3.8}$$

where $d\lambda_{n-2}(u)$ is an $(n-2)$-dimensional spherical Lebesgue measure. For $u \in S^{n-1}$, $K \cap u^{\perp}$ denotes the intersection of K with the subspace $u^{\perp}$ that passes through the origin and is orthogonal to u.

Since the intersection body is a special example of the radial Blaschke-Minkowski homomorphisms, from Corollary 3.4 we obtain the following result.

Corollary 3.5. *If $K, L \in \mathcal{S}^n$, then, for $i \leq n-1$,*

$$C_i(I(K\check{+}L))^{1/(n-i)} \leq C_i(IK)^{1/(n-i)} + C_i(IL)^{1/(n-i)}, \tag{3.9}$$

with equality if and only if IK and IL have similar chords.

Lemma 3.6 ([20]). *If $L \in \mathcal{S}^n$, then, for $0 \le i \le n$,*

$$C_i^n(L) \le \omega_n^i V(L)^{n-i}, \tag{3.10}$$

with equality if and only if L is symmetric with respect to the origin.

Since the intersection body is with respect to the origin, the next corollary follows from Lemma 3.6 and Corollary 3.5.

Corollary 3.7. *If $K, L \in \mathcal{S}^n$, then*

$$V(I(K\check{+}L))^{1/n} \le V(IK)^{1/n} + V(IL)^{1/n},$$

with equality if and only if IK and IL have similar chords.

Lemma 3.8 ([20]). *If $K_1, \dots, K_n \in \mathcal{S}^n$ and $1 < m \le n$, then*

$$C(K_1, \dots, K_n)^m \le \prod_{i=1}^{m} C(K_1, \dots, K_{n-m}, K_{n-i+1}, \dots, K_{n-i+1}), \tag{3.11}$$

with equality if and only if $K_{n-m+1}, \dots, K_n$ are all of similar chords.

Proof of Theorem 1.2. Suppose that $k = n-2$, and $\mathfrak{B} = (B, \dots, B)$ is an $(n-2)$-tuple of the unit ball B. It follows from Lemma 3.8 that, for all $t, s \ge 0$,

$$C(\Psi K \tilde{+} sB, \Psi L \tilde{+} tB, \mathfrak{B})^2 - C_{n-2}(\Psi K \tilde{+} sB) C_{n-2}(\Psi L \tilde{+} tB) \le 0.$$

Since mixed chord-integrals are multilinear with respect to radial Minkowski linear combination, we obtain

$$\begin{aligned} s^2[C_{n-1}(\Psi L)^2 - \omega_n C_{n-2}(\Psi L)] \\ + 2st[\omega_n C(\Psi K, \Psi L, \mathfrak{B}) - C_{n-1}(\Psi K) C_{n-1}(\Psi L)] \\ + t^2[C_{n-1}(\Psi K)^2 - \omega_n C_{n-2}(\Psi K)] + g(s,t) \le 0, \end{aligned}$$

where $g(s,t)$ is a linear function of s and t. From Lemma 3.8, it follows that

$$C_{n-1}(\Psi K)^2 - \omega_n C_{n-2}(\Psi K) \le 0 \tag{3.12}$$

and

$$C_{n-1}(\Psi L)^2 - \omega_n C_{n-2}(\Psi L) \leq 0.$$

From (3.12), we have that either

$$\omega_n C(\Psi K, \Psi L, \mathfrak{B}) - C_{n-1}(\Psi K) C_{n-1}(\Psi L) \leq 0 \tag{3.13}$$

or

$$\begin{aligned}
&[\omega_n C(\Psi K, \Psi L, \mathfrak{B}) - C_{n-1}(\Psi K) C_{n-1}(\Psi L)]^2 \\
&\quad \leq [C_{n-1}(\Psi K)^2 - \omega_n C_{n-2}(\Psi K)] \\
&\quad \times [C_{n-1}(\Psi L)^2 - \omega_n C_{n-2}(\Psi L)].
\end{aligned} \tag{3.14}$$

From (3.12), the case (3.13) is included in (3.14). Thus, (3.14) always holds. Now, using (3.14), it follows from the arithmetic geometric means inequality that

$$\begin{aligned}
&\omega_n C(\Psi K, \Psi L, \mathfrak{B}) - C_{n-1}(\Psi K) C_{n-1}(\Psi L) \\
&\quad \leq [\omega_n C_{n-2}(\Psi K) - C_{n-1}(\Psi K)^2]^{1/2} [\omega_n C_{n-2}(\Psi L) - C_{n-1}(\Psi L)^2]^{1/2} \\
&\quad \leq \frac{1}{2} \frac{C_{n-1}(\Psi L)}{C_{n-1}(\Psi K)} [\omega_n C_{n-2}(\Psi K) - C_{n-1}(\Psi K)^2] \\
&\qquad + \frac{1}{2} \frac{C_{n-1}(\Psi K)}{C_{n-1}(\Psi L)} [\omega_n C_{n-2}(\Psi L) - C_{n-1}(\Psi L)^2] \\
&\quad = \frac{\omega_n}{2} \left[\frac{C_{n-1}(\Psi L)}{C_{n-1}(\Psi K)} C_{n-2}(\Psi K) + \frac{C_{n-1}(\Psi K)}{C_{n-1}(\Psi L)} C_{n-2}(\Psi L) \right] \\
&\qquad - C_{n-1}(\Psi K) C_{n-1}(\Psi L).
\end{aligned}$$

Hence,

$$2C(\Psi K, \Psi L, \mathfrak{B}) \leq \frac{C_{n-1}(\Psi L)}{C_{n-1}(\Psi K)} C_{n-2}(\Psi K) + \frac{C_{n-1}(\Psi K)}{C_{n-1}(\Psi L)} C_{n-2}(\Psi L). \tag{3.15}$$

From the multilinearity of mixed chord-integrals and inequality (3.15), this implies

$$\begin{aligned}
C_{n-2}(\Psi K \tilde{+} \Psi L) &= C_{n-2}(\Psi K) + C_{n-2}(\Psi L) + 2C(\Psi K, \Psi L, \mathfrak{B}) \\
&\leq C_{n-2}(\Psi K) + C_{n-2}(\Psi L)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C_{n-1}(\Psi L)}{C_{n-1}(\Psi K)}C_{n-2}(\Psi K)+\frac{C_{n-1}(\Psi K)}{C_{n-1}(\Psi L)}C_{n-2}(\Psi L)\\
&=\left[\frac{C_{n-2}(\Psi K)}{C_{n-1}(\Psi K)}+\frac{C_{n-2}(\Psi L)}{C_{n-1}(\Psi L)}\right](C_{n-1}(\Psi K)+C_{n-1}(\Psi L)).
\end{aligned}
$$

Thus,

$$\frac{C_{n-2}(\Psi K\tilde{+}\Psi L)}{C_{n-1}(\Psi K\tilde{+}\Psi L)}\le\frac{C_{n-2}(\Psi K)}{C_{n-1}(\Psi K)}+\frac{C_{n-2}(\Psi L)}{C_{n-1}(\Psi L)}.$$

It follows from $\Psi(K\breve{+}L)=\Psi K\tilde{+}\Psi L$ that

$$\frac{C_{n-2}(\Psi(K\breve{+}L))}{C_{n-1}(\Psi(K\breve{+}L))}\le\frac{C_{n-2}(\Psi K)}{C_{n-1}(\Psi K)}+\frac{C_{n-2}(\Psi L)}{C_{n-1}(\Psi L)}.$$

For the case $k=n-1$ in (1.6), note that

$$C_n(\Psi(K\breve{+}L))=C_n(\Psi K)=C_n(\Psi L)=\omega_n.$$

Hence, inequality (1.6) becomes

$$C_{n-1}(\Psi(K\breve{+}L))\le C_{n-1}(\Psi K)+C_{n-1}(\Psi L),$$

which holds for every pair of star bodies. $\square$

Proof of Theorem 1.3. Let $\Psi L=tB$ for $t\ge 0$, and define, for every $k=0,1,\dots,n-2$,

$$f_k(s)=C_k(\Psi K\tilde{+}sB).$$

From the multilinearity of mixed chord-integrals, it follows that

$$
\begin{aligned}
f_k(s+\varepsilon)&=C_k(\Psi K\tilde{+}sB\tilde{+}\varepsilon B)\\
&=C_k(\Psi K\tilde{+}sB)+\varepsilon(n-k)C_{k+1}(\Psi K\tilde{+}sB)+O(\varepsilon^2)\\
&=f_k(s)+\varepsilon(n-k)f_{k+1}(s)+O(\varepsilon^2).
\end{aligned}
$$

Thus,

$$f_k'(s)=(n-k)f_{k+1}(s).$$

From Lemma 3.8, it follows that, for $k=0,1,\dots,n-2$,

$$C_{k+1}(\Psi K\tilde{+}sB)^2\le C_k(\Psi K\tilde{+}sB)C_{k+2}(\Psi K\tilde{+}sB).$$

Hence,

$$f_{k+1}(s)^2\le f_k(s)f_{k+2}(s).$$

Define

$$F_k(s) = \frac{f_k(s)}{f_{k+1}(s)}, \quad k = 0, 1, \ldots, n-2. \tag{3.16}$$

This implies

$$\begin{aligned}
F_k^{'}(s) &= \frac{f_k^{'}(s)f_{k+1}(s) - f_k(s)f_{k+1}^{'}(s)}{f_{k+1}(s)^2} \\
&= \frac{f_{k+1}(s)^2 + (n-k-1)(f_{k+1}(s)^2 - f_k(s)f_{k+2}(s))}{f_{k+1}(s)^2} \\
&\leq 1.
\end{aligned}$$

Thus,

$$F_k(t) \leq F_k(0) + t. \tag{3.17}$$

Since $\Psi L = tB$,

$$\frac{C_k(\Psi L)}{C_{k+1}(\Psi L)} = \frac{C_k(tB)}{C_{k+1}(tB)} = t. \tag{3.18}$$

It follows from (3.16), (3.17) and (3.18) that, for $k = 0, 1, \ldots, n-2$,

$$\frac{C_k(\Psi(K \check{+} L))}{C_{k+1}(\Psi(K \check{+} L))} \leq \frac{C_k(\Psi K)}{C_{k+1}(\Psi K)} + \frac{C_k(\Psi L)}{C_{k+1}(\Psi L)}.$$

When $k = n-1$, inequality (1.7) becomes an equality. $\square$

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