ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 7, 2017

A WEAK CONVERGENCE THEOREM FOR MEAN NONEXPANSIVE MAPPINGS

TORREY M. GALLAGHER

ABSTRACT. In this paper, we first prove that the iterates of a mean nonexpansive map defined on a weakly compact, convex set converge weakly to a fixed point in the presence of Opial's property and asymptotic regularity at a point. Next, we prove the analogous result for closed, convex (not necessarily bounded) subsets of uniformly convex Opial spaces. These results generalize the classical theorems for nonexpansive maps of Browder and Petryshyn in Hilbert space and Opial in reflexive spaces, satisfying Opial's condition.

1. Introduction. Let $(X, \|\cdot\|)$ be a Banach space. Given $C \subseteq X$, we say that a function

$$T : C \longrightarrow X$$

is *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|$$

for all $x, y \in C$. It is a well-known application of Banach's contraction mapping theorem that every nonexpansive map

$$T : C \longrightarrow C$$

has an *approximate fixed point sequence*, that is, a sequence $(u_n)_n$ in C for which $\|Tu_n - u_n\| \to_n 0$. Also, we say that $T : C \to C$ is *asymptotically regular* at x if

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0.$$

If T is asymptotically regular at every $x \in C$, we simply say that T is asymptotically regular. Note that asymptotic regularity at x implies that $(T^n x)_n$ is an approximate fixed point sequence for T. Denote the set of all fixed points of T as $F(T)$, that is, $F(T) := \{x \in C : Tx = x\}$.

2010 AMS *Mathematics subject classification.* Primary 47H10.
Keywords and phrases. Mean nonexpansive, Opial's property, asymptotically regular, weak convergence.
Received by the editors on April 8, 2016.
DOI:10.1216/RMJ-2017-47-7-2167

In 1966, Browder and Petryshyn [**1,** Theorem 4] proved the following theorem for asymptotically regular nonexpansive mappings on Hilbert space.

Theorem 1.1. ([**1**]). *Suppose that H is a Hilbert space and*

$$T : H \longrightarrow H$$

is nonexpansive and asymptotically regular with $F(T) = \{x_0\}$. Then, $(T^n x)_n$ converges weakly to x_0.

In 1967, Opial [**6**] extended this theorem to spaces satisfying Opial's property. Recall that $C \subseteq X$ has the *Opial property* if, whenever $(u_n)_n$ is a sequence in C converging weakly to some $u \in X$, we have

$$\liminf_n \|u_n - u\| < \liminf_n \|u_n - v\|$$

for any $v \neq u$. All Hilbert spaces have the Opial property, as do ℓ^p spaces for all $p \in (1, \infty)$. The L^p spaces fail to have the Opial property for all $p \neq 2$, however.

We will further extend Opial's result to the class of mean nonexpansive maps, first defined in 2007 by Goebel and Japón Pineda [**4**]. We say that $T : C \to C$ is *mean nonexpansive* (or α-nonexpansive) if, for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1, \alpha_n > 0$, $\alpha_j \geq 0$ for all j, and $\alpha_1 + \cdots + \alpha_n = 1$, we have

$$\sum_{j=1}^{n} \alpha_j \|T^j x - T^j y\| \leq \|x - y\|$$

for all $x, y \in C$. Goebel and Japón Pineda further suggested the notion of (α, p)-nonexpansiveness, wherein T would satisfy

$$\sum_{j=1}^{n} \alpha_j \|T^j x - T^j y\|^p \leq \|x - y\|^p$$

for some $p \in [1, \infty)$. It is easy to verify that any (α, p)-nonexpansive map is mean nonexpansive, i.e., $(\alpha, 1)$-nonexpansive, but the converse does not necessarily hold [**7**].

In order to prove our theorem, we will need one further notion. We use "$\rightharpoonup$" to denote weak convergence and "$\to$" to denote strong convergence. We say that $T : C \to X$ is *demiclosed* at y if, whenever $x_n \rightharpoonup x$ in C and $Tx_n \to y$, it follows that $Tx = y$. The present author recently proved [**2,** Theorem 4.2] that, if $C \subseteq X$ is closed and convex and has the Opial property, then any mean nonexpansive map $T : C \to C$ is demiclosed at 0. We will use this demiclosedness principle to extend the theorems of [**1, 6**] stated above. First, we will present the results for the simple case of multi-indices of length 2 before proving the full theorem for multi-indices of arbitrary length.

2. Results for $\alpha = (\alpha_1, \alpha_2)$. The main theorem of this section follows. The proofs may be found in the next section.

Theorem 2.1. *Suppose that $(X, \|\cdot\|)$ is a Banach space and $C \subseteq X$ is weakly compact, convex, and has the Opial property. Suppose further that $T : C \to C$ is (α_1, α_2)-nonexpansive and asymptotically regular at some point $x \in C$. Then, $(T^n x)_n$ converges weakly to a fixed point of T.*

In order to ensure that this theorem is a genuine extension of the classical theorems for nonexpansive maps, we present an example of a $((1/2, 1/2), 2)$-nonexpansive (hence, mean nonexpansive) map defined on $(\ell^2, \|\cdot\|_2)$ for which none of its iterates are nonexpansive. The map below is based on an example given by Goebel and Sims [**5**] and may also be found in [**2**]; moreover, it is asymptotically regular.

Example 2.2. Let $(\ell^2, \|\cdot\|_2)$ be the Hilbert space of square-summable sequences endowed with its usual norm. Let

$$\tau : [-1, 1] \longrightarrow [-1, 1]$$

be given by

$$\tau(t) := \begin{cases} \sqrt{2}\,t + (\sqrt{2} - 1) & -1 \le t \le -(\sqrt{2} - 1)/\sqrt{2} \\ 0 & -(1 + \sqrt{2})/\sqrt{2} \le t \le (1 + \sqrt{2})/\sqrt{2} \\ \sqrt{2}\,t - (\sqrt{2} - 1) & (\sqrt{2} - 1)/\sqrt{2} \le t \le 1 \end{cases}$$

and note these facts regarding τ:

(1) τ is Lipschitz with $k(\tau) = \sqrt{2}$,
(2) $|\tau(t)| \le |t|$ for all $t \in [-1, 1]$.

Let B_{ℓ^2} denote the closed unit ball of $(\ell^2, \|\cdot\|_2)$ and, for any $x \in \ell^2$, define T by

$$T(x_1, x_2, \ldots) := \left(\tau(x_2), \sqrt{\frac{2}{3}}\, x_3, x_4, x_5, \ldots\right)$$

and

$$T^2(x_1, x_2, \ldots) = \left(\tau\left(\sqrt{\frac{2}{3}}\, x_3\right), \sqrt{\frac{2}{3}}\, x_4, x_5, \ldots\right).$$

Observe that $|\tau(t)| \le |t|$ implies that $T(B_{\ell^2}) \subseteq B_{\ell^2}$, $k(T) = \sqrt{2} > 1$ and $k(T^j) = 2/\sqrt{3} > 1$ for all $j \ge 2$. Now, for any $x, y \in B_{\ell^2}$ we find

$$\begin{aligned}
&\frac{1}{2}\|Tx - Ty\|_2^2 + \frac{1}{2}\|T^2x - T^2y\|_2^2 \\
&\quad = \frac{1}{2}\left(|\tau(x_2) - \tau(y_2)|^2 + \frac{2}{3}|x_3 - y_3|^2 + \sum_{j=4}^{\infty} |x_j - y_j|^2\right) \\
&\qquad + \frac{1}{2}\left(\left|\tau\left(\sqrt{\frac{2}{3}}\, x_3\right) - \tau\left(\sqrt{\frac{2}{3}}\, y_3\right)\right|^2 + \frac{2}{3}|x_4 - y_4|^2 + \sum_{j=5}^{\infty} |x_j - y_j|^2\right) \\
&\quad \le \frac{1}{2}\left(2|x_2 - y_2|^2 + \frac{4}{3}|x_3 - y_3|^2 + \frac{5}{3}|x_4 - y_4|^2 + 2\sum_{j=5}^{\infty} |x_j - y_j|^2\right) \\
&\le \|x - y\|_2^2.
\end{aligned}$$

Hence, $T : B_{\ell^2} \to B_{\ell^2}$ is a $((1/2, 1/2), 2)$-nonexpansive map for which each iterate T^j is not nonexpansive.

Before proving the theorem, we shall state some preliminary definitions and results. For any $x \in C$, let

$$\omega_w(x) := \{y \in C : y \text{ is a weak subsequential limit of } (T^n x)_n\},$$

and note that, if C is weakly compact, then $\omega_w(x) \neq \emptyset$. Further, note that, if $I - T$ is demiclosed at 0 and asymptotically regular at x, then

$$\emptyset \neq \omega_w(x) \subseteq F(T).$$

We have the following lemma.

Lemma 2.3. *Suppose that C is weakly compact and convex with the Opial property, and suppose that $T : C \to C$ is (α_1, α_2)-nonexpansive* *and asymptotically regular at some* $x \in C$. *Then, for all* $y \in \omega_w(x)$, $\lim_n \|T^n x - y\|$ *exists.*

Our theorem is proved if we can show that $\omega_w(x)$ is a singleton. This follows from the facts that C is Opial and that $(\|T^n x - y\|)_n$ converges for all $y \in \omega_w(x)$, as summarized in the next lemma.

Lemma 2.4. *If* $C \subseteq X$ *is Opial,* $T : C \to C$ *is a function, and, for some* $x \in C$, $\lim_n \|T^n x - y\|$ *exists for all* $y \in \omega_w(x)$. *Then* $\omega_w(x)$ *is empty or consists of a single point.*

3. Proofs.

Proof of Lemma 2.3. The fact that C is closed and convex with the Opial property implies that $I - T$ is demiclosed at 0, that is, whenever $(z_n)_n$ is a sequence in C converging weakly to some z (which is necessarily in C since closed and convex implies weakly closed) for which

$$\|(I - T)z_n\| \longrightarrow_n 0,$$

it follows that

$$(I - T)z = 0.$$

By the asymptotic regularity of T at x, we have that $(T^n x)_n$ is an approximate fixed point sequence for T.

Since $y \in \omega_w(x)$ and $I - T$ is demiclosed at 0, we have that y is a fixed point of T and we see that

$$\begin{aligned}\alpha_1\|Tx - y\| + \alpha_2\|T^2 x - y\| &= \alpha_1\|Tx - Ty\| + \alpha_2\|T^2 x - T^2 y\| \\ &\leq \|x - y\|.\end{aligned}$$

Hence, at least one of $\|Tx - y\|$ or $\|T^2 x - y\|$ must be $\leq \|x - y\|$. Let $k_1 \in \{1, 2\}$ be such that $\|T^{k_1} x - y\| \leq \|x - y\|$.

Next, we see that

$$\begin{aligned}&\alpha_1\|T^{k_1+1} x - y\| + \alpha_2\|T^{k_1+2} x - y\| \\ &\quad = \alpha_1\|T^{k_1+1} x - T^{k_1+1} y\| + \alpha_2\|T^{k_1+2} x - T^{k_1+2} y\| \\ &\quad \leq \|T^{k_1} x - T^{k_1} y\| = \|T^{k_1} x - y\|,\end{aligned}$$

and thus, one of $\|T^{k_1+1}x - y\|$ or $\|T^{k_1+2}x - y\|$ must be $\leq \|T^{k_1}x - y\|$. As above, let $k_2 \in \{k_1 + 1, k_1 + 2\}$ be such that $\|T^{k_2}x - y\| \leq \|T^{k_1}x - y\|$.

Inductively, build a sequence $(k_n)_n$ which satisfies

(1) $k_n + 1 \leq k_{n+1} \leq k_n + 2$, and
(2) $\|T^{k_{n+1}}x - y\| \leq \|T^{k_n}x - y\|$

for all $n \in \mathbb{N}$. Now $(\|T^{k_n}x - y\|)_n$ is a non-increasing sequence in $\mathbb{R}^+$ and is thus convergent to some $q \in \mathbb{R}^+$.

Consider the set $M := \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$. We have two cases. First, if M is a finite set, then the claim is proved. Second, if M is infinite, write $M = \{m_n : n \in \mathbb{N}\}$, where $(m_n)_n$ is strictly increasing. Note that, by property (1) of the sequence $(k_n)_n$, we must have that, for all $n \in \mathbb{N}$, there exists a $j_n \in \mathbb{N}$ for which

$$m_n = k_{j_n} + 1.$$

In addition, $(j_n)_n$ is strictly increasing. Asymptotic regularity of T at x and the fact that

$$\lim_n \|T^{k_n}x - y\| = q$$

gives us that, for any $\varepsilon > 0$, there is an n large enough such that

(1) $\|T^{m_n}x - T^{m_n-1}x\| < \varepsilon/2$, and
(2) $|\|T^{k_{j_n}}x - y\| - q| < \varepsilon/2$.

Thus,

$$\begin{aligned}
\|T^{m_n}x - y\| - q &\leq \|T^{m_n}x - T^{m_n-1}x\| + \|T^{m_n-1}x - y\| - q \\
&= \|T^{m_n}x - T^{m_n-1}x\| + \|T^{k_{j_n}}x - y\| - q \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Entirely similarly, we have that

$$\begin{aligned}
\|T^{m_n}x - y\| - q &\geq -\|T^{m_n}x - T^{m_n-1}x\| + \|T^{m_n-1}x - y\| - q \\
&= -\|T^{m_n}x - T^{m_n-1}x\| + \|T^{k_{j_n}}x - y\| - q \\
&> -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon.
\end{aligned}$$

Hence, $|\|T^{m_n}x - y\| - q| < \varepsilon$ for n large enough. Since

$$\{m_n : n \in \mathbb{N}\} \cup \{k_n : n \in \mathbb{N}\} = \mathbb{N},$$

we finally have that $\lim_n \|T^n x - y\|$ exists for any $y \in \omega_w(x)$. $\square$

Remark 3.1. The argument presented in the proof above actually works for any $y \in F(T)$, but, in particular, for $y \in \omega_w(x)$. This will be of use in Theorem 5.4.

Proof of Lemma 2.4. Suppose, for contradiction, that z and y are distinct elements of $\omega_w(x)$. Then, there exist $(n_k)_k$ and $(m_k)_k$ for which $T^{n_k}x \rightharpoonup_k z$ and $T^{m_k}x \rightharpoonup_k y$. Thus, using the fact that C is Opial, we have:

$$\begin{aligned}\lim_n \|T^n x - y\| &= \lim_n \|T^{m_k}x - y\| < \lim_n \|T^{m_k}x - z\| \\ &= \lim_n \|T^n x - z\| = \lim_n \|T^{n_k}x - z\| \\ &< \lim_n \|T^{n_k}x - y\| = \lim_n \|T^n x - y\|,\end{aligned}$$

which is a contradiction. Thus, $\omega_w(x)$ is a singleton. $\square$

Proof of Theorem 2.1. As stated above, let

$$\omega_w(x) := \{y \in C : y \text{ is a weak subsequential limit of } (T^n x)_n\},$$

and note that $\omega_w(x) \neq \emptyset$ since C is weakly compact, as well as that the demiclosedness of $I - T$ at 0 yields that $\omega_w(x) \subseteq F(T)$. By Lemma 2.4, we see that $\omega_w(x)$ consists of a single point, say y. Thus,

$$T^n x \rightharpoonup_n y,$$

and the theorem is proved. $\square$

4. Results for arbitrary α. A corresponding theorem for α of arbitrary length now follows.

Theorem 4.1. *If $C \subseteq X$ is weakly compact, convex, and has the Opial property, $T : C \to C$ is α-nonexpansive and asymptotically regular at some point $x \in C$, then $T^n x$ converges weakly to a fixed point of T.*

Theorem 4.1 immediately follows from the analogous lemma concerning convergence of the sequence $(\|T^n x - y\|)_n$ for any $y \in \omega_w(x)$.

Lemma 4.2. *Suppose that C is weakly compact and convex with the Opial property, and suppose that $T : C \to C$ is α-nonexpansive and asymptotically regular at some $x \in C$. Then, for all $y \in \omega_w(x)$, $\lim_n \|T^n x - y\|$ exists.*

Proof of Lemma 4.2. Let $\alpha = (\alpha_1, \ldots, \alpha_{n_0})$. In the same manner as above, a sequence $(k_n)_n$ is built for which

(1) $k_n + 1 \le k_{n+1} \le k_n + n_0$, and
(2) $\|T^{k_{n+1}} x - y\| \le \|T^{k_n} x - y\|$.

Again, as above, let $M = \mathbb{N} \setminus \{k_n : n \in \mathbb{N}\}$. If M is finite, we are finished. If M is infinite, then write the elements of M as $(m_n)_n$, strictly increasing. Note that, for all $n \in \mathbb{N}$, there exist $j_n \in \mathbb{N}$ and $i_n \in \{1, \ldots, n_0 - 1\}$ for which

$$m_n = k_{j_n} + i_n.$$

In addition, $(j_n)_n$ is strictly increasing. Now, for any $\varepsilon > 0$, we can find n large enough such that

$$\|T^{m_n - j+1} x - T^{m_n - j} x\| < \frac{\varepsilon}{n_0} \quad \text{for all } j = 1, \ldots, n_0 - 1,$$

and

$$\big|\|T^{k_{j_n}} x - y\| - q\big| < \frac{\varepsilon}{n_0} \quad \text{where } q = \lim_{n\to\infty} \|T^{k_n} x - y\|.$$

Thus, for n large,

$$\begin{aligned} \|T^{m_n} x - y\| - q &\le \|T^{m_n} x - T^{m_n - 1} x\| + \cdots \\ &\quad + \|T^{m_n - i_n + 1} x - T^{m_n - i_n} x\| + \|T^{m_n - i_n} x - y\| - q \\ &= \|T^{m_n} x - T^{m_n-1} x\| + \cdots + \|T^{m_n - i_n+1} x - T^{m_n - i_n} x\| \\ &\quad + \|T^{k_{j_n}} x - y\| - q \\ &< \underbrace{\frac{\varepsilon}{n_0} + \cdots + \frac{\varepsilon}{n_0}}_{i_n \text{ times}} + \frac{\varepsilon}{n_0} \le (n_0 - 1)\frac{\varepsilon}{n_0} + \frac{\varepsilon}{n_0} = \varepsilon. \end{aligned}$$

A similar argument proves that $|\|T^{m_n}x - y\| - q| < \varepsilon$ for n large, and Lemma 4.2 is proved. $\square$

5. Losing boundedness of C. Similar arguments show that, under appropriate circumstances, the assumption of boundedness of C may be dropped. Before stating the theorem, we need the notion of a duality mapping, a lemma due to Opial [**6,** Lemma 3] and a theorem of García and Piasecki [**3,** Theorem 4.2].

Definition 5.1. A mapping

$$J : X \longrightarrow X^*$$

is called a *duality mapping* of X into X^* with gauge function μ, that is, $\mu : [0,\infty) \to [0,\infty)$ is strictly increasing, continuous, and $\mu(0) = 0$, if, for every $x \in X$,

$$(Jx)(x) = \|Jx\|\|x\| = \mu(\|x\|)\|x\|.$$

Recall also that a Banach space $(X, \|\cdot\|)$ is called *uniformly convex* if, for every $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1)$ such that

$$\begin{cases} \|u\|, \|v\| \le 1 \\ \|u - v\| \ge \varepsilon \end{cases} \Longrightarrow \frac{1}{2}\|u+v\| \le 1 - \delta.$$

It is easy to see that this is equivalent to a sequential notion of uniform convexity, that is, X is uniformly convex if and only if, for every $R > 0$ and for any sequences $(u_n)_n$ and $(v_n)_n$ in X,

$$\begin{cases} \|u_n\|, \|v_n\| \le R \quad \text{for all n} \quad \text{and} \\ \frac{1}{2}\|u_n + v_n\| \longrightarrow R \end{cases} \Longrightarrow \|u_n - v_n\| \longrightarrow 0.$$

Next, we provide a lemma of Opial describing those uniformly convex spaces which have Opial's property.

Lemma 5.2. ([**6**]). *If $(X, \|\cdot\|)$ is uniformly convex and has a weakly continuous duality mapping, then $(X, \|\cdot\|)$ is Opial.*

Finally, we state a theorem of García and Piasecki regarding the structure of the set of fixed points for any mean nonexpansive mapping defined in a strictly convex space.

Theorem 5.3. ([**3**]). *Suppose that $C \subseteq X$ is closed and convex and $(X, \|\cdot\|)$ is strictly convex. Then, for any mean nonexpansive mapping*

$$T : C \longrightarrow C,$$

$F(T)$ is closed and convex.

We use the tools above to prove the following:

Theorem 5.4. *Suppose that $(X, \|\cdot\|)$ is uniformly convex with a weakly sequentially continuous duality map and $C \subseteq X$ is closed and convex. Assume further that $T : C \to C$ is α-nonexpansive, $F(T) \neq \emptyset$ and T is asymptotically regular at some $x \in C$. Then, $(T^n x)_n$ converges weakly to some $y_0 \in F(T)$.*

The proof follows largely from the work performed above and the original proof for nonexpansive mappings due to Opial [**6,** Theorem 1]; it is presented here for completeness.

Proof. By Opial's lemma, the fact that X is uniformly convex with a weakly continuous duality map implies that X is Opial. Thus, for every $y \in F(T)$, by the proof of Theorem 2.3 and Remark 3.1, we know that $\lim_n \|T^n x - y\|$ exists. In particular, this implies that $\{T^n x : n \in \mathbb{N}\}$ is bounded. Let

$$\varphi : F(T) \longrightarrow [0, \infty)$$

be given by

$$\varphi(y) := \lim_n \|T^n x - y\|.$$

For any $r \in [0, \infty)$, consider the set

$$F_r := \{y \in F(T) : \varphi(y) \leq r\} = \varphi^{-1}[0, r].$$

The relevant facts regarding F_r are summarized next.

Claim 5.5. *The sets F_r satisfy the following four properties*:

(1) *F_r is nonempty for r sufficiently large,*
(2) *F_r is closed, bounded, and convex for all $r \geq 0$,*
(3) *there is a minimal r_0 for which F_{r_0} is nonempty, and*
(4) *F_{r_0} is a singleton.*

Proof of Claim 5.5.

(1) and (2) are easy to verify.

(3) This follows from the fact that each F_r is weakly compact, since X is reflexive, and $\{F_r : r \geq 0\}$ forms a nested family. Thus, if each $F_r \neq \emptyset$ for $r > t$ for some $t \geq 0$, it follows that

$$F_t = \bigcap_{r>t} F_r \neq \emptyset.$$

(4) This follows from uniform convexity. Suppose that $u, v \in F_{r_0}$ with $u \neq v$, and let $z := (u+v)/2$. Note that $z \in F_{r_0}$ since F_{r_0} is convex. Since r_0 is minimal for which $F_{r_0} \neq \emptyset$, it follows that $\varphi(u) = r_0 = \varphi(v)$. We must show that $\varphi(z) < r_0$. Suppose, for contradiction, that $\varphi(z) = r_0$. Then,

$$\lim_n \frac{1}{2} \|(T^n x - u) + (T^n x - v)\| = \lim_n \|T^n x - z\| = r_0,$$

and uniform convexity implies that

$$\lim_n \|(T^n x - u) - (T^n x - v)\| = \|u - v\| = 0;$$

however, $\|u - v\| > 0$. This yields that $\varphi(z) < r_0$, which contradicts the minimality of r_0. Hence, F_{r_0} must be a singleton. This completes the proof of Claim 5.5. $\square$

Let $F_{r_0} = \{y_0\}$. We aim to show that $T^n x \rightharpoonup y_0$. For contradiction, suppose this is not the case. Since $\{T^n x : n \in \mathbb{N}\}$ is bounded and X is reflexive, there is some subsequence $(T^{n_k} x)_k$ converging weakly to some $y \neq y_0$. By asymptotic regularity of T and demiclosedness of $I - T$ at 0, we see that

$$\|(I - T)T^{n_k} x\| \longrightarrow 0$$

yields $Ty = y$, that is, $y \in F(T)$. Thus,

$$\begin{aligned} r_0 = \varphi(y_0) = \lim_n \|T^n x - y_0\| &= \lim_k \|T^{n_k} x - y_0\| \\ &> \lim_k \|T^{n_k} x - y\| = \lim_n \|T^n x - y\| = \varphi(y), \end{aligned}$$

which contradicts the minimality of r_0. Finally, we have that

$$T^n x \rightharpoonup y_0,$$

and the proof is complete. $\square$

Remark 5.6. We note here, just as did Opial, that the same result will hold in any reflexive Opial space where $F(T)$ is convex and F_{r_0} is a singleton. For example, to guarantee that $F(T)$ is convex for a mean nonexpansive map, we need only assume strict convexity of X as opposed to uniform convexity.

Acknowledgments. The author would like to thank Chris Lennard for his helpful suggestions regarding the preparation of this paper.

REFERENCES

1. F.E. Browder and W.V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 571–575.

2. T.M. Gallagher, *The demiclosedness principle for mean nonexpansive mappings*, J. Math. Anal. Appl. **439** (2016), 832–842.

3. V.P. García and Ł. Piasecki, *On mean nonexpansive mappings and the Lifshitz constant*, J. Math. Anal. Appl. **396** (2012), 448–454.

4. K. Goebel and M. Japón Pineda, *A new type of nonexpansiveness*, Proc. 8th Inter. Conf. Fixed Point Theory and Appl., Chiang Mai, 2007.

5. K. Goebel and B. Sims, *Mean Lipschitzian mappings*, Contemp. Math. **513** (2010), 157–167.

6. Z.O. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

7. Ł. Piasecki, *Classification of Lipschitz mappings*, CRC Press, Boca Raton, 2013.

University of Pittsburgh, Department of Mathematics, Pittsburgh, PA 15260
Email address: **tmg34@pitt.edu**