

## $t$ -REDUCTIONS AND $t$ -INTEGRAL CLOSURE OF IDEALS

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**ABSTRACT.** Let  $R$  be an integral domain and  $I$  a non-zero ideal of  $R$ . An ideal  $J \subseteq I$  is a  $t$ -reduction of  $I$  if  $(JI^n)_t = (I^{n+1})_t$  for some integer  $n \geq 0$ . An element  $x \in R$  is  $t$ -integral over  $I$  if there is an equation  $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$  with  $a_i \in (I^i)_t$  for  $i = 1, \dots, n$ . The set of all elements that are  $t$ -integral over  $I$  is called the  $t$ -integral closure of  $I$ . This paper investigates the  $t$ -reductions and  $t$ -integral closure of ideals. Our objective is to establish satisfactory  $t$ -analogues of well known results in the literature, on the integral closure of ideals and its correlation with reductions, namely, Section 2 identifies basic properties of  $t$ -reductions of ideals and features explicit examples discriminating between the notions of reduction and  $t$ -reduction. Section 3 investigates the concept of  $t$ -integral closure of ideals, including its correlation with  $t$ -reductions. Section 4 studies the persistence and contraction of  $t$ -integral closure of ideals under ring homomorphisms. Throughout the paper, the main results are illustrated with original examples.

**1. Introduction.** All rings considered here are commutative with identity. Let  $R$  be a ring and  $I$  an ideal of  $R$ . An ideal  $J \subseteq I$  is a reduction of  $I$  if  $JI^n = I^{n+1}$  for some positive integer  $n$ . An ideal which has no reduction other than itself is called a *basic ideal* [15, 16, 26]. The notion of *reduction* was introduced by Northcott and Rees, and its usefulness resides mainly in two facts:

First, it defines a relationship between two ideals which is preserved under homomorphisms and ring exten-

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sions; secondly, what we may term the reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it [26].

Their main purpose was a contribution to the analytic theory of ideals in Noetherian (local) rings via minimal reductions.

Reductions became a very useful tool for the theory of integral dependence over ideals. Let  $I$  be an ideal in a ring  $R$ . An element  $x \in R$  is integral over  $I$  if there is an equation

$$x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$$

with  $a_i \in I^i$  for  $i = 1, \dots, n$ . The set of all elements that are integral over  $I$  is called the integral closure of  $I$  and is denoted by  $\bar{I}$ . If  $I = \bar{I}$ , then  $I$  is called integrally closed. It turns out that an element  $x \in R$  is integral over  $I$  if and only if  $I$  is a reduction of  $I + Rx$ , and, if  $I$  is finitely generated, then  $I \subseteq \bar{J}$  if and only if  $J$  is a reduction of  $I$  [20, Corollary 1.2.5]. This correlation allowed proving a number of crucial results in the theory, including the fact that the integral closure of an ideal is an ideal [20, Corollary 1.3.1]. For a full treatment of this topic, the reader is referred to Huneke and Swanson [20].

Let  $R$  be a domain with quotient field  $K$ ,  $I$  a nonzero fractional ideal of  $R$ , and let

$$I^{-1} := (R : I) = \{x \in K \mid xI \subseteq R\}.$$

The  $v$ - and  $t$ -closures of  $I$  are defined, respectively, by

$$I_v := (I^{-1})^{-1} \quad \text{and} \quad I_t := \cup J_v,$$

where  $J$  ranges over the set of finitely generated subideals of  $I$ . The ideal  $I$  is a  $v$ -ideal (or divisorial) if  $I_v = I$  and a  $t$ -ideal if  $I_t = I$ . Under the ideal  $t$ -multiplication

$$(I, J) \mapsto (IJ)_t,$$

the set  $F_t(R)$  of fractional  $t$ -ideals of  $R$  is a semigroup with unit  $R$ . Recall that factorial domains, Krull domains, GCDs and PvMDs can be regarded as  $t$ -analogues of the principal domains, Dedekind domains, Bézout domains and Prüfer domains, respectively. For instance, a domain is Prüfer, respectively, a PvMD, if every nonzero finitely generated

ideal is invertible, respectively,  $t$ -invertible. For relevant work on  $v$ - and  $t$ -operations, we refer the reader to [13, 19, 22, 23, 24, 27, 28, 29, 30].

This paper investigates the  $t$ -reductions and  $t$ -integral closures of ideals. Our objective is to establish satisfactory  $t$ -analogues of well-known results in the literature on the integral closure of ideals and correlations with reductions, namely, Section 2 identifies basic properties of  $t$ -reductions of ideals and features explicit examples discriminating between the notions of reduction and  $t$ -reduction. Section 3 investigates the concept of  $t$ -integral closure of ideals, including its correlation with  $t$ -reductions. Section 4 studies the persistence and contraction of  $t$ -integral closure of ideals under ring homomorphisms. Throughout the paper, the main results are illustrated with original examples.

**2.  $t$ -Reductions of ideals.** This section identifies basic ideal-theoretic properties of the notion of  $t$ -reduction including its behavior under localizations. As a prelude to this, we provide explicit examples discriminating between the notions of reduction and  $t$ -reduction.

Recall that in a ring  $R$  a subideal  $J$  of an ideal  $I$  is called a reduction of  $I$  if  $J I^n = I^{n+1}$  for some positive integer  $n$  [26]. An ideal which has no reduction other than itself is called a basic ideal [15, 16].

**Definition 2.1** (cf., [18, Definition 1.1]). Let  $R$  be a domain and  $I$  a nonzero ideal of  $R$ . An ideal  $J \subseteq I$  is a  $t$ -reduction of  $I$  if  $(J I^n)_t = (I^{n+1})_t$  for some integer  $n \geq 0$  (and, a fortiori, the relation holds for  $n \gg 0$ ). The ideal  $J$  is a *trivial  $t$ -reduction* of  $I$  if  $J_t = I_t$ . The ideal  $I$  is  *$t$ -basic* if it has no  $t$ -reduction other than the trivial  $t$ -reductions.

At this point, recall a basic property of the  $t$ -operation (which, in fact, holds for any star operation) that will be used throughout the paper. For any two nonzero ideals  $I$  and  $J$  of a domain, we have

$$(IJ)_t = (I_t J)_t = (I J_t)_t = (I_t J_t)_t.$$

Thus, obviously, for nonzero ideals  $J \subseteq I$ , we always have:

$$J \text{ is a } t\text{-reduction of } I \iff J \text{ is a } t\text{-reduction of } I_t \iff J_t \text{ is a } t\text{-reduction of } I_t.$$

Note also that a reduction is necessarily a  $t$ -reduction, and the converse is not true, in general, as shown by the next example which exhibits a domain  $R$  with two  $t$ -ideals  $J \subsetneq I$  such that  $J$  is a  $t$ -reduction but not a reduction of  $I$ .

**Example 2.2.** We use a construction from [21]. Let  $x$  be an indeterminate over  $\mathbb{Z}$ , and let

$$\begin{aligned} R &:= \mathbb{Z}[3x, x^2, x^3], \\ I &:= (3x, x^2, x^3), \end{aligned}$$

and

$$J := (3x, 3x^2, x^3, x^4).$$

Then  $J \subsetneq I$  are two finitely generated  $t$ -ideals of  $R$  such that:

$$JI^n \subsetneq I^{n+1} \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad (JI)_t = (I^2)_t.$$

*Proof.*  $I$  is a height-one prime ideal of  $R$  [21] and hence it is a  $t$ -ideal of  $R$ . Next, we prove that  $J$  is a  $t$ -ideal. We first claim that  $J^{-1} = (1/x)\mathbb{Z}[x]$ . Indeed, note that  $\mathbb{Q}(x)$  is the quotient field of  $R$  and, since  $3x \subseteq J$ , then  $J^{-1} \subseteq (1/3x)R$ . Thus, let  $f := g/3x \in J^{-1}$ , where

$$g = \sum_{i=0}^m a_i x^i \in \mathbb{Z}[x]$$

with  $a_1 \in 3\mathbb{Z}$ . Then, the fact that  $x^3 f \in R$  implies that  $a_i \in 3\mathbb{Z}$  for  $i = 0, 2, \dots, m$ , i.e.,  $g \in 3\mathbb{Z}[x]$ . Hence,  $f \in (1/x)\mathbb{Z}[x]$ , whence  $J^{-1} \subseteq (1/x)\mathbb{Z}[x]$ . The reverse inclusion holds since

$$\frac{1}{x}J\mathbb{Z}[x] = (3, 3x, x^2, x^3)\mathbb{Z}[x] \subseteq R,$$

proving the claim.

Next, let  $h \in (R : \mathbb{Z}[x]) \subseteq R$ . Then,  $xh \in R$  forces  $h(0) \in 3\mathbb{Z}$ , and thus,  $h \in (3, 3x, x^2, x^3)$ . Thus,  $(R : \mathbb{Z}[x]) \subseteq (3, 3x, x^2, x^3)$ ; hence,  $(R : \mathbb{Z}[x]) = (1/x)J$ . It follows that

$$J_t = J_v = \left( R : \frac{1}{x}\mathbb{Z}[x] \right) = x(R : \mathbb{Z}[x]) = J,$$

as desired.

Further, let  $n \in \mathbb{N}$ . It is to see that  $x^3x^{2n} = x^{2n+3}$  is the monic monomial with the smallest degree in  $JI^n$ . Therefore,  $x^{2(n+1)} = x^{2n+2} \in I^{n+1} \setminus JI^n$ , that is,  $J$  is not a reduction of  $I$ . It remains to prove  $(JI)_t = (I^2)_t$ . We first claim that  $(JI)^{-1} = (1/x^2)\mathbb{Z}[x]$ . Indeed,

$$(JI)^{-1} \subseteq (J^{-1})^2 = \frac{1}{x^2}\mathbb{Z}[x],$$

and the reverse inclusion holds since

$$\frac{1}{x^2}JI\mathbb{Z}[x] = (3, 3x, x^2, x^3)(3, x, x^2)\mathbb{Z}[x] \subseteq R,$$

proving the claim. Now, observe that

$$I^2 = (9x^2, 3x^3, x^4, x^5).$$

It follows that

$$(IJ)_t = (IJ)_v = \left( R : \frac{1}{x^2}\mathbb{Z}[x] \right) = x^2(R : \mathbb{Z}[x]) = xJ \supseteq I^2.$$

Thus,  $(IJ)_t \supseteq (I^2)_t$ , as desired. □

Observe that the domain  $R$  in the above example is not integrally closed. Next, we provide a class of integrally closed domains where the notions of reduction and  $t$ -reduction are always distinct.

**Example 2.3.** Let  $R$  be any integrally closed Mori domain that is not completely integrally closed, i.e., not Krull. Then, there always exist nonzero ideals  $J \subsetneq I$  in  $R$  such that  $J$  is a  $t$ -reduction but not a reduction of  $I$ .

*Proof.* These domains do exist; for instance, let  $k \subsetneq K$  be a field extension with  $k$  algebraically closed, and let  $x$  be an indeterminate over  $K$ . Then,  $R := k + xK[x]$  is an integrally closed Mori domain [12, Theorem 4.18] that is not completely integrally closed [14, Lemma 26.5], see [11, page 161]. Now, by [18, Proposition 1.5(1)], there exists a  $t$ -ideal  $A$  in  $R$  that is not  $t$ -basic, say,  $B \subseteq A$  is a  $t$ -reduction of  $A$  with  $B_t \subsetneq A_t$ . By [4, Theorem 2.1], there exist finitely generated ideals  $F \subseteq A$  and  $J \subseteq B$  such that  $A^{-1} = F^{-1}$  and  $B^{-1} = J^{-1}$ , yielding  $A_t = F_t$  and  $B_t = J_t$ . Let  $I := F + J$ . Then, it can easily be seen that  $J$  is a non-trivial  $t$ -reduction of  $I$ . Finally, we claim that  $J$  is not a reduction of  $I$ . Suppose this is not true.

Since  $I$  is finitely generated,  $I \subseteq \bar{J}$  by [20, Corollary 1.2.5]. However,  $\bar{J} \subseteq J_t$  by [25, Proposition 2.2]. It follows that  $J_t = I_t$ , the desired contradiction.  $\square$

Another crucial fact concerns reductions of  $t$ -ideals. Indeed, if  $J$  is a reduction of a  $t$ -ideal, then so is  $J_t$ ; the converse is not true, in general, as shown by the following example which features a domain  $R$  with a  $t$ -ideal  $I$  and an ideal  $J \subseteq I$  such that  $J_t$  is a reduction but  $J$  is not a reduction of  $I$ .

**Example 2.4.** Let  $k$  be a field, let  $x, y, z$  be indeterminates over  $k$ , let  $R := k[x] + M$ , where  $M := (y, z)k(x)[[y, z]]$  and let  $J := M^2$ . Note that  $R$  is a classical pullback issued from the local Noetherian and integrally closed domain  $T := k(x)[[y, z]]$ . Then,  $M$  is a divisorial ideal of  $R$  by [17, Corollary 5], and clearly, for all  $n \in \mathbb{N}$ ,

$$M^{n+2} \subsetneq M^{n+1},$$

that is,  $J$  is not a reduction of  $M$  in  $R$ . On the other hand, note that  $(M : M) = T$ , since  $T$  is integrally closed, and  $M$  is not principal in  $T$ . Therefore, by [17, Theorem 13], we have

$$\begin{aligned} (R : (R : M^2)) &= (R : (M^{-1} : M)) = (R : ((M : M) : M)) \\ &= (R : (T : M)) = (R : M^{-1}) = M, \end{aligned}$$

such that  $J_t = J_v = M$ . Hence,  $J_t$  is trivially a reduction of  $M$  in  $R$ .

In the sequel,  $R$  will denote a domain. For convenience, recall that, for any nonzero ideals  $I, J, H$  of  $R$ , the equality  $(IJ + H)_t = (I_tJ + H)_t$  always holds since

$$I_tJ \subseteq (I_tJ)_t = (IJ)_t \subseteq (IJ + H)_t.$$

This property will be used in the proof of the next basic result which examines the  $t$ -reduction of the sum and product of ideals.

**Lemma 2.5.** *Let  $J \subseteq I$  and  $J' \subseteq I'$  be nonzero ideals of  $R$ . If  $J$  and  $J'$  are  $t$ -reductions of  $I$  and  $I'$ , respectively, then  $J + J'$  is a  $t$ -reduction of  $I + I'$ , and  $JJ'$  is a  $t$ -reduction of  $II'$ .*

*Proof.* Let  $n$  be a positive integer. Then the following implication always holds:

$$(2.1) \quad (JI^n)_t = (I^{n+1})_t \implies (JJ^m)_t = (I^{m+1})_t \quad \text{for all } m \geq n.$$

Indeed, multiply the first equation through by  $I^{m-n}$ , and apply the  $t$ -closure to both sides. By (2.1), let  $m$  be a positive integer such that

$$(2.2) \quad (JI^m)_t = (I^{m+1})_t \quad \text{and} \quad (J'I^m)_t = (I'^{m+1})_t.$$

By (2.2), we obtain

$$\begin{aligned} ((I + I')^{2m+1})_t &\subseteq (I^{m+1}(I + I')^m + I'^{m+1}(I + I')^m)_t \\ &= ((I^{m+1})_t(I + I')^m + (I'^{m+1})_t(I + I')^m)_t \\ &= ((JI^m)_t(I + I')^m + (J'I^m)_t(I + I')^m)_t \\ &= (JI^m(I + I')^m + J'I^m(I + I')^m)_t \\ &\subseteq ((J + J')(I + I')^{2m})_t \\ &\subseteq ((I + I')^{2m+1})_t, \end{aligned}$$

and then equality holds throughout, proving the first statement. The proof of the second statement is straightforward via (2.2).  $\square$

The next basic result examines the transitivity for  $t$ -reduction.

**Lemma 2.6.** *Let  $K \subseteq J \subseteq I$  be nonzero ideals of  $R$ . Then:*

- (a) *If  $K$  is a  $t$ -reduction of  $J$  and  $J$  is a  $t$ -reduction of  $I$ , then  $K$  is a  $t$ -reduction of  $I$ .*
- (b) *If  $K$  is a  $t$ -reduction of  $I$ , then  $J$  is a  $t$ -reduction of  $I$ .*

*Proof.* For any positive integer  $m$ , we always have

$$(2.3) \quad (JI^m)_t = (I^{m+1})_t \implies (J^n I^m)_t = (I^{m+n})_t \quad \text{for all } n \geq 1.$$

Indeed, multiply the first equation through by  $J^{n-1}$ , apply the  $t$ -closure to both sides, and conclude by induction on  $n$ . Let  $(KJ^n)_t = (J^{n+1})_t$  and  $(JI^m)_t = (I^{m+1})_t$ , for some positive integers  $n$  and  $m$ . By (2.3), we obtain

$$(I^{m+n+1})_t = (J^{n+1}I^m)_t = ((J^{n+1})_t I^m)_t = ((KJ^n)_t I^m)_t = (KI^{m+n})_t,$$

proving (a). The proof of (b) is straightforward.  $\square$

The next basic result examines the  $t$ -reduction of the power of an ideal.

**Lemma 2.7.** *Let  $J \subseteq I$  be nonzero ideals of  $R$  and let  $n$  be a positive integer. Then:*

- (a)  $J$  is a  $t$ -reduction of  $I \Leftrightarrow J^n$  is a  $t$ -reduction of  $I^n$ .
- (b) If  $J = (a_1, \dots, a_k)$ , then,  $J$  is a  $t$ -reduction of  $I \Leftrightarrow (a_1^n, \dots, a_k^n)$  is a  $t$ -reduction of  $I^n$ .

*Proof.*

(a) The “only if” implication holds by Lemma 2.5. For the converse, suppose that  $(J^n I^{nm})_t = (I^{nm+n})_t$  for some positive integer  $m$ . Then,

$$(I^{nm+n})_t = (J J^{n-1} I^{nm})_t \subseteq (J I^{nm+n-1})_t \subseteq (I^{nm+n})_t,$$

and thus, equality holds throughout, as desired.

(b) Assume that  $J$  is a  $t$ -reduction of  $I$ . From [20, (8.1.6)], we always have the following equality

$$(2.4) \quad (a_1^n, \dots, a_k^n)(a_1, \dots, a_k)^{(k-1)(n-1)} = (a_1, \dots, a_k)^{(n-1)k+1}$$

and, multiplying (2.4) through by  $J^{k-1}$ , we obtain  $(a_1^n, \dots, a_k^n)J^{nk-n} = J^{nk}$ . Therefore,  $(a_1^n, \dots, a_k^n)$  is a  $t$ -reduction of  $J^n$  and a fortiori of  $I^n$  by (a) and Proposition 2.6. The converse holds by (a) and Proposition 2.6. □

The next basic result examines the  $t$ -reduction of localizations.

**Lemma 2.8.** *Let  $J \subseteq I$  be nonzero ideals of  $R$ , and let  $S$  be a multiplicatively closed subset of  $R$ . If  $J$  is a  $t$ -reduction of  $I$ , then  $S^{-1}J$  is a  $t$ -reduction of  $S^{-1}I$ .*

*Proof.* Assume that  $(JI^n)_t = (I^{n+1})_t$  for some positive integer  $n$ . Let  $t_1$  denote the  $t$ -operation with respect to  $S^{-1}R$ . By [24, Lemma 3.4], we have:

$$\begin{aligned} ((S^{-1}I)^{n+1})_{t_1} &= (S^{-1}(I^{n+1}))_{t_1} = (S^{-1}((I^{n+1})_t))_{t_1} \\ &= (S^{-1}((JI^n)_t))_{t_1} = (S^{-1}(JI^n))_{t_1} \\ &= ((S^{-1}J)(S^{-1}I)^n)_{t_1}. \end{aligned} \quad \square$$

It is worth noting here that, in a PvMD,  $J$  is a  $t$ -reduction of  $I$  if and only if  $J$  is  $t$ -locally a reduction of  $I$ , i.e.,  $JR_M$  is a reduction of  $IR_M$  for every maximal  $t$ -ideal  $M$  of  $R$  [18, Lemma 2.2].



**3.  $t$ -Integral closure of ideals.** This section investigates the concept of  $t$ -integral closure of ideals and its correlation with  $t$ -reductions. Our objective is to establish satisfactory  $t$ -analogues of (and in some cases generalize) well-known results in the literature on the integral closure of ideals and its correlation with reductions.

**Definition 3.1.** Let  $R$  be a domain and  $I$  a nonzero ideal of  $R$ . An element  $x \in R$  is  $t$ -integral over  $I$ , if there is an equation

$$x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$$

with  $a_i \in (I^i)_t$  for all  $i = 1, \dots, n$ .

The set of all elements that are  $t$ -integral over  $I$  is called the  $t$ -integral closure of  $I$  and is denoted by  $\tilde{I}$ . If  $I = \tilde{I}$ , then  $I$  is called  $t$ -integrally closed.

Note that the  $t$ -integral closure of the ideal  $R$  is always  $R$ , whereas the  $t$ -integral closure of the ring  $R$ , also called the *pseudo-integral closure*, may be larger than  $R$ , e.g., consider any non  $v$ -domain [3, 11]. In addition, we have

$$J \subseteq I \implies \tilde{J} \subseteq \tilde{I}.$$

More ideal-theoretic properties are provided in Remark 3.8.

It is well known that the integral closure of an ideal is an ideal which is integrally closed [20, Corollary 1.3.1]. Next, we establish a  $t$ -analogue for this result.

**Theorem 3.2.** *The  $t$ -integral closure of an ideal is an integrally closed ideal. In general, it is not  $t$ -closed and, a fortiori, not  $t$ -integrally closed.*

The proof of Theorem 3.2 relies on the next lemma which sets a  $t$ -analogue for the notion of Rees algebra of an ideal [20, Chapter 5]. Recall, for convenience, that the Rees algebra of an ideal  $I$ , in a ring  $R$ , is the graded subring of  $R[x]$  given by

$$R[Ix] := \bigoplus_{n \geq 0} I^n x^n,$$

[20, Definition 5.1.1] and whose integral closure in  $R[x]$  is the graded ring

$$\bigoplus_{n \geq 0} \overline{I^n} x^n,$$

[20, Proposition 5.2.1].

**Lemma 3.3.** *Let  $R$  be a domain,  $I$  a  $t$ -ideal of  $R$  and  $x$  an indeterminate over  $R$ . Let*

$$R_t[Ix] := \bigoplus_{n \geq 0} (I^n)_t x^n.$$

*Then  $R_t[Ix]$  is a graded subring of  $R[x]$ , and its integral closure in  $R[x]$  is the graded ring*

$$\bigoplus_{n \geq 0} \widetilde{I^n} x^n.$$

*Proof.* That  $R_t[Ix]$  is  $\mathbb{N}$ -graded follows from the fact that

$$(I^i)_t \cdot (I^j)_t \subseteq (I^{i+j})_t \quad \text{for all } i, j \in \mathbb{N}.$$

Let  $\overline{R_t[Ix]}$  denote its integral closure in  $R[x]$ . By [20, Theorem 2.3.2],  $\overline{R_t[Ix]}$  is an  $\mathbb{N}$ -graded ring. Let  $k \in \mathbb{N}$ , and let  $S_k$  denote the homogeneous component of  $\overline{R_t[Ix]}$  of degree  $k$ . We shall prove that  $S_k = \widetilde{I^k} x^k$ . Let  $s := s_k x^k \in S_k$  for some  $s_k \in R$ . Then,

$$s^n + a_1 s^{n-1} + \dots + a_n = 0$$

for some positive integer  $n$  and  $a_i \in R_t[Ix]$ ,  $i = 1, \dots, n$ . Expanding each

$$a_i = \sum_{j=0}^{k_i} a_{i,j} x^j$$

with  $a_{i,j} \in (I^j)_t$ , the coefficient of the monomial of degree  $kn$  in the above equation is

$$s_k^n + \sum_{i=1}^n a_{i,ik} s_k^{n-i} = 0,$$

with  $a_{i,ik} \in (I^{ik})_t$ . It follows that  $s_k \in \widetilde{I^k}$ , and thus,  $S_k \subseteq \widetilde{I^k} x^k$ .

For the reverse inclusion, let  $z_k := y_k x^k \in \widetilde{I^k} x^k$  for  $y_k \in \widetilde{I^k}$ . Then,  $y_k^n + a_1 y_k^{n-1} + \dots + a_n = 0$  for positive integer  $n$  and  $a_j \in (I^{kj})_t$ ,  $j = 1, \dots, n$ . Multiplying through by  $x^{kn}$  yields

$$z_k^n + a_1 x^k z_k^{n-1} + \dots + a_n x^{kn} = 0$$

with

$$a_j x^{kj} \in (I^{kj})_t x^{kj} \subseteq R_t[Ix], \quad j = 1, \dots, n,$$

that is,  $z_k \in \overline{R_t[Ix]}$ . However,  $z_k$  is homogeneous of degree  $k$  in  $\overline{R_t[Ix]}$ . Therefore,  $z_k \in S_k$ , and hence,  $\widetilde{I^k} x^k \subseteq S_k$ , completing the proof of Lemma 3.3.  $\square$

**Definition 3.4.** The  $t$ -Rees algebra of an ideal  $I$  (in a domain  $R$ ) is the graded subring of  $R[x]$  given by

$$R_t[Ix] := \bigoplus_{n \geq 0} (I^n)_t x^n.$$

*Proof of Theorem 3.2.* Let  $R$  be a domain and  $I$  a nonzero ideal of  $R$ . Since  $\widetilde{I} = \widetilde{I}_t$ , we assume  $I$  to be a  $t$ -ideal. We first prove that  $\widetilde{I}$  is an ideal. Clearly,  $\widetilde{I}$  is closed under multiplication. Next, we show that  $\widetilde{I}$  is closed under addition. Let  $a, b \in \widetilde{I}$ . Then, by Lemma 3.3,  $ax$  and  $bx \in \overline{R_t[Ix]}$ . Hence,

$$ax + bx = (a + b)x \in \overline{R_t[Ix]}.$$

Again, by Lemma 3.3,  $a + b \in \widetilde{I}$ , as desired.

Next, we prove that  $\widetilde{I}$  is integrally closed. For this purpose, observe that, for all  $n \in \mathbb{N}$ ,  $(S_1)^n \subseteq S_n$  forces

$$(3.1) \quad (\widetilde{I})^n \subseteq \widetilde{I^n} \quad \text{for all } n \in \mathbb{N}.$$

Consider the Rees algebra of the ideal  $\widetilde{I}$ ,

$$R[\widetilde{I}x] = \bigoplus_{n \geq 0} (\widetilde{I})^n x^n.$$

Therefore,  $R[\widetilde{I}x] \subseteq \overline{R_t[Ix]}$ , and hence,  $\overline{R[\widetilde{I}x]} \subseteq \overline{R_t[Ix]}$ . Now, a combination of Lemma 3.3 and [20, Proposition 5.2.1] yields

$$\bigoplus_{n \geq 0} \overline{(\widetilde{I})^n x^n} \subseteq \bigoplus_{n \geq 0} \widetilde{I}^n x^n.$$

In particular,  $\overline{\widetilde{I}} \subseteq \widetilde{I}$ , that is,  $\widetilde{I}$  is integrally closed. The proof of the last statement of Theorem 3.2 is handled by Example 3.10 (b), where we provide a domain with an ideal  $I$  such that  $\widetilde{I} \subsetneq (\widetilde{I})_t$ , that is,  $\widetilde{I}$  is not a  $t$ -ideal, and hence, not  $t$ -integrally closed since  $(\widetilde{I})_t \subseteq \widetilde{\widetilde{I}}$  always holds. □

The next result shows that the  $t$ -integral closure collapses to the  $t$ -closure in the class of integrally closed domains. It also completes two existing results in the literature on the integral closure of ideals, [14, 25].

**Theorem 3.5.** *Let  $R$  be a domain. The following assertions are equivalent:*

- (a)  $R$  is integrally closed;
- (b) Every principal ideal of  $R$  is integrally closed;
- (c) Every  $t$ -ideal of  $R$  is integrally closed;
- (d)  $\overline{I} \subseteq I_t$  for each nonzero ideal  $I$  of  $R$ ;
- (e) Every principal ideal of  $R$  is  $t$ -integrally closed;
- (f) Every  $t$ -ideal of  $R$  is  $t$ -integrally closed;
- (g)  $\widetilde{I} = I_t$  for each nonzero ideal  $I$  of  $R$ .

*Proof.*

(a)  $\Leftrightarrow$  (b) and (a)  $\Leftrightarrow$  (c)  $\Leftrightarrow$  (d) are handled by [14, Lemma 24.6] and [25, Proposition 2.2], respectively. Also, (g)  $\Leftrightarrow$  (f)  $\Rightarrow$  (e)  $\Rightarrow$  (b) are straightforward. Thus, it remains to prove (a)  $\Rightarrow$  (g). Assume that  $R$  is integrally closed, and let  $I$  be a nonzero ideal of  $R$ . The inclusion  $I_t \subseteq \widetilde{I}$  holds in any domain. Next, let  $\alpha \in \widetilde{I}$ .

**Claim 1.** *There exists a finitely generated ideal  $J \subseteq I$  such that  $\alpha \in \widetilde{J}$ .*

Indeed,  $\alpha$  satisfies an equation of the form

$$\alpha^n + a_1\alpha^{n-1} + \dots + a_n = 0$$

with  $a_i \in (I^i)_t$  for all  $i = 1, \dots, n$ . Now, let  $i \in \{1, \dots, n\}$ . Hence, there exists a finitely generated ideal  $F_i \subseteq I^i$  such that  $a_i \in F_{i_v}$ . Further, each generator of  $F_i$  is a finite combination of elements of the form

$$\prod_{1 \leq j \leq i} c_j \in I^i.$$

Let  $J$  denote the subideal of  $I$  generated by all  $c_j$ s emanating from all  $F_i$ s. Clearly,  $a_i \in (J^i)_t$  for all  $i = 1, \dots, n$ , that is,  $\alpha \in \tilde{J}$ , proving the claim.

**Claim 2.**  $\tilde{J} \subseteq J_t$ .

Indeed, we first prove that  $J^{-1} = (\tilde{J})^{-1}$ . Clearly,  $(\tilde{J})^{-1} \subseteq J^{-1}$ .

For the reverse inclusion, let  $x \in J^{-1}$  and  $y \in \tilde{J}$ . Then,  $y$  satisfies an equation of the form

$$y^n + a_1y^{n-1} + \dots + a_n = 0$$

with  $a_i \in (J^i)_t$  for all  $i = 1, \dots, n$ . It follows that  $(yx)^n + a_1x(yx)^{n-1} + \dots + a_nx^n = 0$  with

$$a_ix^i \in (J^i)_t(J^{-1})^i \subseteq (J^i)_t(J^i)^{-1} = (J^i)_t((J^i)_t)^{-1} \subseteq R.$$

Hence,  $yx \in R$ . Thus,  $x \in (\tilde{J})^{-1}$ , as desired. Therefore,

$$\tilde{J} \subseteq (\tilde{J})_v = J_v = J_t,$$

proving the claim.

Now, by the above claims, we have  $\alpha \in \tilde{J} \subseteq J_t \subseteq I_t$ . Consequently,  $\tilde{I} = I_t$ , which completes the proof of the theorem. □

In the case where all ideals of a domain are  $t$ -integrally closed, it must then be Prüfer. This is a well-known result in the literature:

**Corollary 3.6** ([14, Theorem 24.7]). *A domain  $R$  is Prüfer if and only if every ideal of  $R$  is ( $t$ -)integrally closed.*

Now, we examine the correlation between the  $t$ -integral closure and  $t$ -reductions of ideals. In this vein, recall that, for the trivial operation, two crucial results assert that

$$x \in \bar{I} \iff I \text{ is a reduction of } I + Rx$$

[20, Corollary 1.2.2], and if  $I$  is finitely generated and  $J \subseteq I$ , then

$$I \subseteq \bar{J} \iff J \text{ is a reduction of } I$$

[20, Corollary 1.2.5].

Next, we establish  $t$ -analogues of these two results.

**Proposition 3.7.** *Let  $R$  be a domain, and let  $J \subseteq I$  be nonzero ideals of  $R$ .*

- (a)  $x \in \tilde{I} \Rightarrow I$  is a  $t$ -reduction of  $I + Rx$ .
- (b) Assume that  $I$  is finitely generated. Then,  $I \subseteq \tilde{J} \Rightarrow J$  is a  $t$ -reduction of  $I$ .

Moreover, both implications are irreversible in general.

*Proof.*

(a) Let  $x \in \tilde{I}$ . Then,  $x^n + a_1x^{n-1} + \dots + a_n = 0$  for some  $a_i \in (I^i)_t$  for each  $i \in \{1, \dots, n\}$ . Hence,

$$x^n \in I_t x^{n-1} + \dots + (I^n)_t \subseteq (I_t x^{n-1} + \dots + (I^n)_t)_t \subseteq (I(I + Rx)^{n-1})_t.$$

It follows that

$$(I + Rx)^n \subseteq (I(I + Rx)^{n-1})_t.$$

Hence,

$$((I + Rx)^n)_t = (I(I + Rx)^{n-1})_t.$$

Thus,  $I$  is a  $t$ -reduction of  $I + Rx$ .

(b) Assume that  $I = (a_1, \dots, a_n)$  for some integer  $n \geq 1$  and  $a_i \in R$  for all  $i = 1, \dots, n$ . Suppose that  $I \subseteq \tilde{J}$ . By (a),  $J$  is a  $t$ -reduction of  $J + Ra_i$ , for each  $i \in \{1, \dots, n\}$ . By Lemma 2.5,  $J$  is a  $t$ -reduction of  $J + (a_1, \dots, a_n) = I$ , as desired.

The converse of (a) is not true, in general, as shown by Example 3.10 (a). Also, (b) can be irreversible even with both  $I$  and  $J$  being finitely generated. For instance, consider the integrally domain  $R$  of

Example 2.3 with two ideals  $J \subsetneq I$ , where  $J$  is a non-trivial  $t$ -reduction of  $I$ , i.e.,  $J_t \subsetneq I_t$ . By Theorem 3.5,  $\widetilde{J} = J_t \not\subseteq I$ . □

Next, we collect some ideal-theoretic properties of the integral closure of ideals.

**Remark 3.8.** Let  $R$  be a domain, and let  $I, J$  be nonzero ideals of  $R$ . Then,

(1)  $I \subseteq \bar{I} \subseteq \widetilde{I} \subseteq \sqrt{I}$ . Example 3.9 (a) features a  $t$ -ideal for which these three containments are strict. However, note that radical, and, a fortiori, prime,  $t$ -ideals are necessarily  $t$ -integrally closed.

(2)  $\widetilde{I \cap J} \subseteq \widetilde{I} \cap \widetilde{J}$ . The inclusion can be strict, for instance, in any integrally closed domain that is not a PvMD by [1, Theorem 6] and Theorem 3.5. Another example is provided in the non-integrally closed case by Example 3.9 (c).

(3)  $\widetilde{I + J} \subseteq \widetilde{I + J}$ . The inclusion can be strict. For instance, in  $\mathbb{Z}[x]$ , we have

$$\widetilde{(2)} + \widetilde{(x)} = (2, x) \quad \text{and} \quad (2, x)^{-1} = \mathbb{Z}[x]$$

so that

$$\widetilde{(2, x)} = (2, x)_t = \mathbb{Z}[x]$$

via Theorem 3.5.

(4) By (3.1), for all  $n \geq 1$ ,  $(\widetilde{I})^n \subseteq \widetilde{I^n}$ . The inclusion can be strict, as shown by Example 3.9 (b).

(5) For all  $x \in R$ ,  $x\widetilde{I} \subseteq \widetilde{xI}$ . Indeed, let  $y \in x\widetilde{I}$ . Then, there is an equation of the form

$$y^n + (xa_1)y^{n-1} + \dots + x^n a_n = 0$$

with

$$x^i a_i \in x^i (I^i)_t = ((xI)^i)_t, \quad i = 1, \dots, n.$$

Hence,  $y \in \widetilde{xI}$ . Note that  $x\widetilde{I} = \widetilde{xI}$  for all  $x \in R$  and for all  $I$  ideal  $\Leftrightarrow R$  is integrally closed, Theorem 3.5.

We close this section with the next examples.

**Example 3.9.** Let

$$R := \mathbb{Z}[\sqrt{-3}][2x, x^2, x^3],$$

$$I := (2x^2, 2x^3, x^4, x^5)$$

and

$$J := (x^3),$$

where  $x$  is an indeterminate over  $\mathbb{Z}$ . Then  $I$  is a  $t$ -ideal of  $R$  such that

- (a)  $I \subsetneq \bar{I} \subsetneq \tilde{I} \subsetneq \sqrt{I}$ .
- (b)  $(\tilde{I})^2 \subsetneq \tilde{I}^2$ .
- (c)  $\widetilde{J \cap I} \subsetneq \tilde{J} \cap \tilde{I}$ .

*Proof.* We first show that  $I$  is a  $t$ -ideal. Clearly,  $(1/x^2)\mathbb{Z}[\sqrt{-3}][x] \subseteq I^{-1}$ . For the reverse inclusion, let  $f \in I^{-1} \subseteq x^{-4}R$ . Then  $f = (a_0 + a_1x + \dots + a_nx^n)/x^4$  for some  $n \in \mathbb{N}$ ,  $a_0 \in \mathbb{Z}[\sqrt{-3}]$ ,  $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$  and  $a_i \in \mathbb{Z}[\sqrt{-3}]$  for  $i \geq 2$ . Since  $2x^2f \in R$ , then  $a_0 = a_1 = 0$ . It follows that  $f \in (1/x^2)\mathbb{Z}[\sqrt{-3}][x]$ . Therefore,  $I^{-1} = (1/x^2)\mathbb{Z}[\sqrt{-3}][x]$ . Next, let  $g \in (R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq R$ . Then,  $xg \in R$  forces  $g(0) \in 2\mathbb{Z}[\sqrt{-3}]$ , and hence,  $g \in (2, 2x, x^2, x^3)$ . Thus,  $(R : \mathbb{Z}[\sqrt{-3}][x]) \subseteq (2, 2x, x^2, x^3)$ . The reverse inclusion is obvious. Therefore,  $(R : \mathbb{Z}[\sqrt{-3}][x]) = (2, 2x, x^2, x^3)$ . Consequently, we obtain

$$I_t = I_v = \left( R : \frac{1}{x^2}\mathbb{Z}[\sqrt{-3}][x] \right) = x^2(R : \mathbb{Z}[\sqrt{-3}][x]) = I.$$

(a) Next, we prove the strict inclusions  $I \subsetneq \bar{I} \subsetneq \tilde{I} \subsetneq \sqrt{I}$ . For  $I \subsetneq \bar{I}$ , note that  $(1 + \sqrt{-3})x^2 \in \bar{I} \setminus I$  as  $((1 + \sqrt{-3})x^2)^3 = -8x^6 \in I^3$  and  $1 + \sqrt{-3} \notin 2\mathbb{Z}[\sqrt{-3}]$ .

For  $\bar{I} \subsetneq \tilde{I}$ , we claim that  $x^3 \in \tilde{I} \setminus \bar{I}$ . Indeed, let  $f \in (I^2)^{-1} \subseteq x^{-8}R$ . Then, there are  $n \in \mathbb{N}$ ,  $a_i \in \mathbb{Z}[\sqrt{-3}]$  for  $i \in \{0, 2, \dots, n\}$ , and  $a_1 \in 2\mathbb{Z}[\sqrt{-3}]$  such that  $f = (a_0 + a_1x + \dots + a_nx^n)/x^8$ . Since  $4x^4f \in R$ ,  $a_0 = a_1 = a_2 = a_3 = 0$ . Therefore,  $(I^2)^{-1} \subseteq (1/x^4)\mathbb{Z}[\sqrt{-3}][x]$ . The reverse inclusion is obvious. Hence,  $(I^2)^{-1} = (1/x^4)\mathbb{Z}[\sqrt{-3}][x]$ . It follows that

$$(I^2)_t = (I^2)_v = \left( R : \frac{1}{x^4}\mathbb{Z}[\sqrt{-3}][x] \right) = x^4(R : \mathbb{Z}[\sqrt{-3}][x]) = x^2I.$$

Hence  $x^6 \in (I^2)_t$ , and thus,  $x^3 \in \tilde{I}$ .



It remains to show that  $x^3 \notin \bar{I}$ . By [20, Corollary 1.2.2], it suffices to show that  $I$  is not a reduction of  $I + (x^3)$ . Let  $n \in \mathbb{N}$ . It is easy to see that  $x^4x^{3n}$  is the monic monomial with the smallest degree in  $I(I + (x^3))^n$ . Therefore,

$$x^{3(n+1)} = x^{3n+3} \in (I + (x^3))^{n+1} \setminus I(I + (x^3))^n.$$

Hence,  $I$  is not a reduction of  $I + (x^3)$ , as desired.

For  $\tilde{I} \subsetneq \sqrt{I}$ , we claim that  $x^2 \in \sqrt{I} \setminus \tilde{I}$ . Obviously,  $x^2 \in \sqrt{I}$ . In order to prove that  $x^2 \notin \tilde{I}$ , it suffices by Proposition 3.7 to show that  $I$  is not a  $t$ -reduction of  $I + (x^2)$ . Towards this purpose, note that  $I + (x^2) = (x^2)$ . Suppose, by way of contradiction, that  $(I(I + (x^2))^n)_t = ((I + (x^2))^{n+1})_t$  for some  $n \in \mathbb{N}$ . Then,

$$(x^2)^{n+1} = x^{2n+2} \in (I(I + (x^2))^n)_t = x^{2n}I.$$

Consequently,  $x^2 \in I$ , which is absurd.

(b) We first prove that  $\tilde{I} = (2x^2, (1 + \sqrt{-3})x^2, x^3, x^4)$ . In view of (a) and its proof, we have  $(2x^2, (1 + \sqrt{-3})x^2, x^3, x^4) \subseteq \tilde{I}$ . Next, let  $\alpha := (a + b\sqrt{-3})x^2 \in \tilde{I}$ , where  $a, b \in \mathbb{Z}$ . If  $b = 0$ , then  $a \neq 1$  as  $x^2 \notin \tilde{I}$ . Moreover, since  $2x^2 \in \tilde{I}$ ,  $a$  must be even, that is,  $\alpha \in (2x^2)$ . Now, assume that  $b \neq 0$ . If  $a = 0$ , then  $b \neq 1$  as  $\sqrt{-3}x^2 \notin \tilde{I}$ . Moreover, since  $2\sqrt{-3}x^2 \in \tilde{I}$ ,  $b$  must be even, that is,  $\alpha \in (2x^2)$ . Therefore, suppose that  $a \neq 0$ . Then, similar arguments force  $a$  and  $b$  to be of the same parity. Further, if  $a$  and  $b$  are even, then  $\alpha \in (2x^2)$ , and, if  $a$  and  $b$  are odd, then  $\alpha \in (2x^2, (1 + \sqrt{-3})x^2)$ . Finally, we claim that  $\tilde{I}$  contains no monomials of degree 1.

Suppose this is not true. Let  $ax \in \tilde{I}$ , for some nonzero  $a \in 2\mathbb{Z}[\sqrt{-3}]$ . Then, by [20, Remark 1.1.3 (7)],

$$ax \in \tilde{I} \subseteq \widetilde{(x^2)} = \overline{(x^2)} \subseteq \overline{x^2\mathbb{Z}[\sqrt{-3}][x]}.$$

By [20, Corollary 1.2.2],  $(x^2)$  is a reduction of  $(ax, x^2)$  in  $\mathbb{Z}[\sqrt{-3}][x]$ , which is absurd. Consequently,

$$\tilde{I} = (2x^2, (1 + \sqrt{-3})x^2, x^3, x^4).$$

Now, we are ready to verify that  $(\tilde{I})^2 \subsetneq \tilde{I}^2$ . For this purpose, recall that  $(I^2)_t = x^2I$ . Thus,  $2x^4 \in \tilde{I}^2$ . We claim that  $2x^4 \notin (\tilde{I})^2$ .

Suppose this is not true. Then,  $2x^4 \in (4x^4, 2(1 + \sqrt{-3})x^4)$  yields  $x^2 \in (2x^2, (1 + \sqrt{-3})x^2) \subseteq \tilde{I}$ , which is absurd.

(c) We claim that  $x^3 \in \tilde{I} \cap \tilde{J} \setminus \widetilde{I \cap J}$ . In (a), it was proven that  $x^3 \in \tilde{I}$ . Thus,  $x^3 \in \tilde{I} \cap \tilde{J}$ . Now, observe that  $I \cap J = xI$  and assume, by way of contradiction, that  $x^3 \in \widetilde{I \cap J} = \widetilde{xI}$ . Then,  $x^3$  satisfies an equation of the form

$$(x^3)^n + a_1(x^3)^{n-1} + \dots + a_n = 0$$

with

$$a_i \in ((xI)^i)_t = x^i(I^i)_t, \quad i = 1, \dots, n.$$

For each  $i$ , let  $a_i = x^i b_i$ , for some  $b_i \in (I^i)_t$ . Therefore,

$$(x^2)^n + b_1(x^2)^{n-1} + \dots + b_n = 0.$$

It follows that  $x^2 \in \tilde{I}$ , the desired contradiction. □

**Example 3.10.** Let  $R := \mathbb{Z} + x\mathbb{Q}(\sqrt{2})[x]$ ,  $I := (x/\sqrt{2})$  and  $a := x/2$ , where  $x$  is an indeterminate over  $\mathbb{Q}$ . Then:

- (a)  $I$  is a  $t$ -reduction of  $I + aR$  and  $a \notin \tilde{I}$ .
- (b)  $\tilde{I} \subsetneq (\tilde{I})_t$ , and  $\tilde{I} \subsetneq \tilde{\tilde{I}}$ .

*Proof.*

(a) First, we prove that  $(I(I + aR))_t = ((I + aR)^2)_t$ . It suffices to show that  $a^2 \in (I(I + aR))_t$ . For this purpose, let

$$f \in (I(I + aR))^{-1} = \left(\frac{x^2}{2}, \frac{x^2}{2\sqrt{2}}\right)^{-1} \subseteq \left(\frac{x^2}{2}\right)^{-1} = \frac{2}{x^2}R.$$

Then,  $f = 2(a_0 + a_1x + \dots + a_nx^n)/x^2$ , for some  $n \geq 0$ ,  $a_0 \in \mathbb{Z}$ , and  $a_i \in \mathbb{Q}(\sqrt{2})$  for  $i \geq 1$ . Since  $(x^2/2\sqrt{2})f \in R$ ,  $a_0 = 0$ , it follows that

$$(I(I + aR))^{-1} \subseteq \frac{1}{x}\mathbb{Q}(\sqrt{2})[x].$$

On the other hand,

$$(I(I + aR))\left(\frac{1}{x}\mathbb{Q}(\sqrt{2})[x]\right) \subseteq R.$$

Thus, we have

$$(3.2) \quad (I(I + aR))^{-1} = \left( \frac{x^2}{2}, \frac{x^2}{2\sqrt{2}} \right)^{-1} = \frac{1}{x} \mathbb{Q}(\sqrt{2})[x].$$

Now, clearly,  $a^2(I(I + aR))^{-1} \subseteq R$ . Therefore,  $a^2 \in (I(I + aR))_v = (I(I + aR))_t$ , as desired.

Next, we prove that  $a \notin \tilde{I} = \bar{I}$ . By [20, Corollary 1.2.2], it suffices to show that  $I$  is not a reduction of  $I + aR$ . Suppose this is not true, and that  $I(I + aR)^n = (I + aR)^{n+1}$  for some positive integer  $n$ . Then

$$a^{n+1} = \left( \frac{x}{2} \right)^{n+1} \in I(I + aR)^n = \frac{x}{\sqrt{2}} \left( \frac{x}{\sqrt{2}}, \frac{x}{2} \right)^n.$$

It can be verified that this yields

$$1 \in \sqrt{2}(\sqrt{2}, 1)^n \subseteq (\sqrt{2}) \quad \text{in } \mathbb{Z}[\sqrt{2}],$$

the desired contradiction.

(b) We claim that  $a \in (\tilde{I})_t$ . Note first that  $x \in \tilde{I}$  as  $x^2 \in I^2 = (I^2)_t$ . Therefore,  $A := (x, (x/\sqrt{2})) \subseteq \tilde{I}$ . Clearly,

$$A = \frac{2}{x} \left( \frac{x^2}{2}, \frac{x^2}{2\sqrt{2}} \right).$$

Hence, by (3.2),

$$A^{-1} = \mathbb{Q}(\sqrt{2})[x].$$

However,  $aA^{-1} \subseteq R$ , whence,

$$a \in A_v = A_t \subseteq (\tilde{I})_t.$$

Consequently,  $a \in (\tilde{I})_t \setminus \tilde{I}$ . □

**4. Persistence and contraction of the  $t$ -integral closure.** Recall that the persistence and contraction of the integral closure describe, respectively, the facts that, for any ring homomorphism  $\varphi : R \rightarrow T$ ,  $\varphi(\bar{I}) \subseteq \overline{\varphi(I)T}$  for every ideal  $I$  of  $R$  and  $\overline{\varphi^{-1}(J)} = \varphi^{-1}(J)$  for every integrally closed ideal  $J$  of  $T$ .

This section studies the persistence and contraction of the  $t$ -integral closure. For this purpose, we first introduce the concept of  $t$ -compatible homomorphism which extends the well-known notion of  $t$ -compatible

extension [2]. Throughout, we denote by  $t$ , respectively  $t_1$ , and  $v$ , respectively  $v_1$ , the  $t$ - and  $v$ - closures in  $R$ , respectively  $T$ .

**Lemma 4.1.** *Let  $\varphi : R \rightarrow T$  be a homomorphism of domains. Then, the following statements are equivalent:*

- (a)  $\varphi(I_v)T \subseteq (\varphi(I)T)_{v_1}$  for each nonzero finitely generated ideal  $I$  of  $R$ ;
- (b)  $\varphi(I_t)T \subseteq (\varphi(I)T)_{t_1}$  for each nonzero ideal  $I$  of  $R$ ;
- (c)  $\varphi^{-1}(J)$  is a  $t$ -ideal of  $R$  for each  $t_1$ -ideal  $J$  of  $T$  such that  $\varphi^{-1}(J) \neq 0$ .

*Proof.*

(a)  $\Rightarrow$  (c). Let  $J$  be a  $t_1$ -ideal of  $T$ , and let  $A$  be any finitely generated ideal of  $R$  contained in  $\varphi^{-1}(J)$ . Then,  $\varphi(A)T \subseteq J = J_{t_1}$ . Further,  $\varphi(A)T$  is finitely generated. Hence,  $(\varphi(A)T)_{v_1} \subseteq J$ . It follows, via (a), that

$$\varphi(A_v)T \subseteq (\varphi(A)T)_{v_1} \subseteq J.$$

Therefore,  $A_v \subseteq \varphi^{-1}(J)$ , and thus,  $\varphi^{-1}(J)$  is a  $t$ -ideal.

(c)  $\Rightarrow$  (b). Let  $I$  be a nonzero ideal of  $R$ . The ideal  $J := (\varphi(I)T)_{t_1}$  is clearly a  $t_1$ -ideal of  $T$  with  $\varphi^{-1}(J) \neq 0$ . By (c),  $\varphi^{-1}(J)$  is a  $t$ -ideal of  $R$ . Consequently, we obtain

$$I_t \subseteq (\varphi^{-1}(\varphi(I)T))_t \subseteq (\varphi^{-1}(\varphi(I)T)_{t_1})_t = (\varphi^{-1}(J))_t = \varphi^{-1}(J)$$

such that  $\varphi(I_t)T \subseteq J = (\varphi(I)T)_{t_1}$ , as desired.

(b)  $\Rightarrow$  (a). Trivial. □

**Definition 4.2.** A homomorphism of domains  $\varphi : R \rightarrow T$  is called  $t$ -compatible if it satisfies the equivalent conditions of Lemma 4.1.

When  $\varphi$  denotes the natural embedding  $R \subseteq T$ , this definition matches the notion of  $t$ -compatible extension, i.e.,  $I_t T \subseteq (IT)_{t_1}$  for every ideal  $I$  of  $R$ , previously profusely studied [2, 5, 6, 10].

Next, we detail the main results of this section which establish persistence and contraction of  $t$ -integral closure under  $t$ -compatible homomorphisms.

**Proposition 4.3.** *Let  $\varphi : R \rightarrow T$  be a  $t$ -compatible homomorphism of domains,  $I$  an ideal of  $R$  and  $J$  an ideal of  $T$ . Then*

- (a)  $\varphi(\widetilde{I})T \subseteq \widetilde{\varphi(I)T}$ .
- (b)  $\varphi^{-1}(J) \subseteq \varphi^{-1}(\widetilde{J})$ .

Moreover, if  $J$  is  $t$ -integrally closed, then  $\widetilde{\varphi^{-1}(J)} = \varphi^{-1}(J)$ .

*Proof.*

(a) Let  $x \in \widetilde{I}$ ,  $y := \varphi(x)$  and  $z \in T$ . We prove that  $yz \in \widetilde{\varphi(I)T}$ . Suppose that  $x$  satisfies the equation

$$x^n + a_1x^{n-1} + \dots + a_n = 0$$

with  $a_i \in (I^i)_t$  for  $i = 1, \dots, n$ . Then, apply  $\varphi$  and multiply through by  $z^n$ , to obtain

$$(yz)^n + b_1z(yz)^{n-1} + \dots + b_{n-1}z^{n-1}(yz) + b_nz^n = 0,$$

where

$$b_i := \varphi(a_i) \in \varphi((I^i)_t)T \subseteq (\varphi(I^i)T)_{t_1} = ((\varphi(I)T)^i)_{t_1}$$

by  $t$ -compatibility. Hence,  $b_iz^i \in ((\varphi(I)T)^i)_{t_1}$  for  $i = 1, \dots, n$ . Consequently,  $yz \in \widetilde{\varphi(I)T}$ .

(b) Let  $H := \varphi(\varphi^{-1}(J))T$ . Then, by (a), we have

$$\varphi(\widetilde{\varphi^{-1}(J)})T \subseteq \widetilde{H} \subseteq \widetilde{J}.$$

It follows that  $\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J})$ , as desired. Now, if  $J$  is  $t$ -integrally closed, then

$$\widetilde{\varphi^{-1}(J)} \subseteq \varphi^{-1}(\widetilde{J}) = \varphi^{-1}(J) \subseteq \widetilde{\varphi^{-1}(J)},$$

and hence, the equality holds. □

In the special case where both  $R$  and  $T$  are integrally closed, persistence of  $t$ -integral closure coincides with  $t$ -compatibility by Theorem 3.5. This shows that the  $t$ -compatibility assumption in Proposition 4.3 is imperative.

**Corollary 4.4.** *Let  $R \subseteq T$  be a  $t$ -compatible extension of domains and  $I$  an ideal of  $R$ . Then*

- (a)  $\widetilde{IT} \subseteq \widetilde{IT}$ .
- (b)  $\widetilde{I} \subseteq \widetilde{IT} \cap R \subseteq \widetilde{IT} \cap R$ .

*Moreover, the above inclusions are strict in general.*

*Proof.* (a) and (b) are direct consequences of Proposition 4.3. The inclusion in (a) and second inclusion in (b) can be strict as shown by Example 4.6. The first inclusion in (b) can also be strict. For instance, let  $R$  be an integrally closed domain and let  $P \subsetneq Q$  be prime ideals of  $R$  with  $x \in Q \setminus P$ . Then  $\widetilde{(x)} = (x)$  by Theorem 3.5, while

$$x\widetilde{R_P} \cap R = \widetilde{R_P} \cap R = R,$$

that is,

$$\widetilde{(x)} \subsetneq (x)\widetilde{R_P} \cap R. \quad \square$$

**Corollary 4.5.** *Let  $R$  be a domain,  $I$  an ideal of  $R$  and  $S$  a multiplicatively closed subset of  $R$ . Then,  $S^{-1}\widetilde{I} \subseteq \widetilde{S^{-1}I}$ .*

*Proof.* It is well known that flatness implies  $t$ -compatibility [10, Proposition 0.6]. Hence, Corollary 4.4 leads to the conclusion.  $\square$

For the integral closure, we always have  $S^{-1}\overline{I} = \overline{S^{-1}I}$  [20, Proposition 1.1.4]. However, in Corollary 4.5, the inclusion can be strict, as shown in the following example.

**Example 4.6.** We use a construction due to Zafrullah [28]. Let  $E$  be the ring of entire functions and  $x$  an indeterminate over  $E$ . Let  $S$  denote the set generated by the principal primes of  $E$ . Then, we claim that  $R := E + xS^{-1}E[x]$  contains a prime ideal  $P$  such that  $S^{-1}\widetilde{P} \subsetneq \widetilde{S^{-1}P}$ . Indeed,  $R$  is a  $P$ -domain which is not a PvMD [28, Example 2.6]. By [29, Proposition 3.3], there exists a prime  $t$ -ideal  $P$  in  $R$  such that  $PR_P$  is not a  $t$ -ideal of  $R_P$ . By Theorem 3.5, we have

$$\widetilde{P}R_P = PR_P \subsetneq R_p = (PR_P)_t = \widetilde{P}R_P$$

since  $R$  is integrally closed. Also note that

$$P = \widetilde{PR_P} \cap R \subsetneq \widetilde{PR_P} \cap R = R.$$

**Corollary 4.7.** *Let  $R$  be a domain and  $I$  a  $t$ -ideal that is  $t$ -locally  $t$ -integrally closed, i.e.,  $I_M$  is  $t$ -integrally closed in  $R_M$  for every maximal  $t$ -ideal  $M$  of  $R$ . Then,  $I$  is  $t$ -integrally closed.*

*Proof.* Let  $\text{Max}_t(R)$  denote the set of maximal  $t$ -ideals of  $R$ . By Corollary 4.5, we have

$$\begin{aligned} \widetilde{I} &\subseteq \bigcap_{M \in \text{Max}_t(R)} (\widetilde{I})_{M_i} \\ &\subseteq \bigcap_{M \in \text{Max}_t(R)} \widetilde{I}_{M_i} \\ &= \bigcap_{M \in \text{Max}_t(R)} I_{M_i} \\ &= I. \end{aligned}$$

Consequently,  $I$  is  $t$ -integrally closed. □

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